A NOTE ON THE TARRY-ESCOTT PROBLEM

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The Tarry-Escott problem in Diophantine analysis is the following: consider the system of equations

$$\Sigma_{i=1}^{k} a_i^j = \Sigma_{i=1}^{k} b_i^j, \quad j = 1, \ldots n, \tag{1}$$

for the unknowns $a_1, \ldots, a_k, b_1, \ldots, b_k$, what is the smallest integer $K = K(n)$ in the set of all k's for which the system (1) possesses a non-trivial solution in integers ? By an integer we mean a rational integer and a solution is called trivial if the sets $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ are permutations of each other. By the Tarry-Escott problem one may also mean the problem of finding an actual solution of (1) or finding all such solutions but we shall be concerned with the bound $K(n)$ only. However, the method given here is constructive, that is to say, with each estimate K_1 for K it leads to finding actual non-trivial solutions of (1) with $k = K_1$.

Suppose that $k \leq n$ in (1), then from elementary properties of the symmetric functions it follows that $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are roots of the same equation of degree k. Consequently the solutions are trivial. Therefore

$$K(n) \geq n + 1. \tag{2}$$

It has been conjectured in [1] that in fact $K(n) = n + 1$; this is easily proved to be true for the first few values of n, [1]. A simple combinatorial proof has been given in [1] of the estimate

$$K(n) \leq [n(n + 1)/2] + 1; \tag{3}$$

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the above bound has been slightly improved in [2] to

$$K(n) \leq (n^2 + 4)/2,$$

and this would appear to be the best bound known so far.

In this note we shall prove a theorem giving an exact expression for $K(n)$. Our expression is of non-constructive nature, that is, it does not allow one to compute $K(n)$, but it leads to estimates for $K(n)$ which are better that (3) for certain values of n. Let $\mathfrak{A}$ be the class of all polynomials whose coefficients are integers, not all 0. We put

$$S[P] = \Sigma_{i=0}^{N} |a_i| \quad \text{for} \quad P = P(x) = \Sigma_{i=0}^{N} a_i x^i.$$

THEOREM 1.

$$K(n) = \tfrac{1}{2} \min_{P \in \mathfrak{A}} S[P(x)(1 - x)^{n+1}]. \tag{4}$$

Proof. By the factorial binomial theorem, or otherwise, it is easy to show that

$$\Sigma_{j=0}^{n+1} (-1)^j \binom{n+1}{j} (x - j)^s \equiv 0, \quad s = 1, \ldots, n. \tag{5}$$

Rearranging this by taking all the powers $(x - j)^s$ with negative coefficients to the other side, one gets

$$\Sigma_{j=0,\, 2|j}^{n+1} \binom{n+1}{j} (x - j)^s \equiv \Sigma_{j=0,\, 2\nmid j}^{n+1} \binom{n+1}{j} (x - j)^s, \quad s = 1, \ldots, n$$

Letting in the above x be any integer we obtain a non-trivial solution of (1) with

$$k = \tfrac{1}{2} \Sigma_{j=0}^{n+1} \binom{n+1}{j} = 2^n.$$

Write (5) symbolically as

$$(1 - x)^{n+1} = 0 \tag{6}$$

and call (5) the expanded form of (6) and (6) the contracted form of (5). It is clear what operation on (5) leads to (6), and vice versa. Further, for every polynomial $Q \in \mathfrak{A}$ there is an expanded form of $Q(x) = 0$. Let $P \in \mathfrak{A}$, then the expansion of the contracted form $P(x)(1 - x)^{n+1}$ leads, as above, to a non-trivial solution of (1), with

$$k = \tfrac{1}{2} S[P(x)(1 - x)^{n+1}]. \tag{7}$$

Hence

$$K(n) \leq \tfrac{1}{2} \min_{P \in \mathfrak{A}} S[P(x)(1 - x)^{n+1}]. \tag{8}$$

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Suppose next that $K(n) = K$; let $\alpha_1, \ldots, \alpha_K, \beta_1, \ldots, \beta_K$ be a non-trivial solution of (1). From

(9) $$\Sigma_{i=1}^{K} \alpha_i^j = \Sigma_{i=1}^{K} \beta_i^j, \quad j = 1, \ldots, n,$$

one obtains immediately

(10) $$\Sigma_{i=1}^{K} (x - \alpha_i)^j \equiv \Sigma_{i=1}^{K} (x - \beta_i)^j, \quad j = 1, \ldots, n.$$

It is clear from the definition of K that no equality $\alpha_i = \beta_j$ holds. On the other hand, some of the α' s, and β' s, may be equal. Suppose that the distinct values among the numbers $\alpha_1, \ldots, \alpha_K$ are $p_1, \ldots, p_r$ in the order of increasing magnitude, and let the multiplicity of the occurence of p_i be m_i. Define similarly the numbers $q_1 \ldots q_s$ and the multiplicities $n_1, \ldots, n_s$ for the set $\beta_1, \ldots, \beta_K$. Now (10) may be written as $\Sigma_{i=1}^{r} m_i (x - p_i)^j \equiv \Sigma_{i=1}^{s} n_i (x - q_i)^j$, $j = 1, \ldots, n$, with $m_i \geq 1$, $n_i \geq 1$ and $p_{i_1} \neq q_{i_2}$. We have

(11) $$\Sigma_{i=1}^{r} m_i = \Sigma_{i=1}^{s} n_i = K.$$

Let

$$F(x) = \Sigma_{i=1}^{r} m_i x^{p_i} - \Sigma_{i=1}^{s} n_i x^{q_i},$$

then (10) is the expanded form of the contraction $F(x) = 0$ and we have by (11)

(12) $$K = \tfrac{1}{2} S[F(x)].$$

We next verify that the equations (9) can also be written as

$$F^{(j)}(x)|_{x=1} = 0, \quad j = 1, \ldots, n.$$

Since also clearly $F(1) = 0$ it follows that $x = 1$ is an $(n+1)$-tuple root of the equation $F(x) = 0$ and hence

$$(1 - x)^{n+1} | F(x).$$

We have therefore $F(x) = (1 - x)^{n+1} T(x)$, where $T(x)$ is a polynomial. From the definition of K it follows that

$$\text{g.c.d.}(m_1, \ldots, m_r, n_1, \ldots, n_s) = 1,$$

that is, the g.c.d. of the coefficients of $F(x)$ is 1. This implies that the coefficients of T are integers, and so $T \in \mathfrak{A}$.

Therefore by (12)

$$K = \frac{1}{2} S[T(x)(1 - x)^{n+1}], \quad T(x) \in \mathfrak{A},$$

which together with (8) proves (4).

While the equation (4) does not allow an explicit calculation of K(n), it is possible to obtain bounds on K(n) by taking in (4) suitable polynomials P(x). By trial and error it has been found that relatively low bounds result from taking P(x) of the form

$$P(x) = [\prod_{j=1}^{s} (1 - x^j)^{\varepsilon_j}] [\prod_{k=1}^{n} \Sigma_{j=0}^{k} x^j]$$

where s is a small positive integer and $\varepsilon_j = 0$ or 1. The bounds on K(n) are then of the form

$$(13) \qquad \frac{1}{2} S[Q(x) \prod_{j=1}^{n+1} (1 - x^j)]$$

with $Q(x) = \prod_{j=1}^{s} (1 - x^j)^{\varepsilon_j}$. Since the calculation of expressions like (13) is rather lengthy for $n \geq 10$ say, and since no easy analytical procedure seems to apply, the problem was programmed for an automatic calculator (the IBM 704). The results are given in the enclosed Table 1 where b_n is the upper bound on K(n) obtained from (13), the corresponding polynomial Q is listed, and the bound $[n(n + 1)]/2 + 1$ is given for comparison. In constructing the table only four multipliers Q were considered: 1, $1 - x$, $1 - x^2$ and $(1 - x)(1 - x^2)$.

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2. E. M. Wright, Quart. J. Math., Oxford Ser. 6 (1935), 261 - 267.

Table 1

n	b_n	Q	$[n(n+1)]/2+1$	n	b_n	Q	$[n(n+1)]/2+1$
2	3	1	4	16	58	$1-x$	137
3	4	1	7	17	75	$1-x$	154
4	6	1	11	18	74	$1-x$	172
5	8	1	16	19	92	$(1-x)(1-x^2)$	191
6	10	1	22	20	100	$1-x$	211
7	14	1	29	21	124	$(1-x)(1-x^2)$	232
8	18	1	37	22	118	$(1-x)(1-x^2)$	254
9	22	1	46	23	146	$(1-x)(1-x^2)$	277
10	22	$1-x$	56	24	159	$(1-x)(1-x^2)$	301
11	34	$1-x$	67	25	170	$(1-x)(1-x^2)$	326
12	32	$1-x$	79	26	196	$(1-x)(1-x^2)$	352
13	41	$1-x$	92	27	216	$(1-x)(1-x^2)$	379
14	46	$1-x$	106	28	207	$(1-x)(1-x^2)$	407
15	58	$1-x$	121	29	266	$(1-x)(1-x^2)$	436

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