The Convergence of the Series in Mathieu's Functions.

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Periodic solutions of Mathieu's equation*

$$\frac{d^2y}{dz^2} + \{a + 16q\cos 2z\}y = 0,$$

where a is a suitable function of q have recently been discussed in several papers in these *Proceedings*. An elegant method of determining these solutions, which are written

$$ce_0(z) \;\; ce_1(z) \ldots \;\; ce_m(z), \;\ldots$$
$$se_1(z) \ldots \;\; se_m(z), \;\ldots$$

was given by Whittaker,† who obtained the integral equation

$$f(z) = \lambda \int_0^{2\pi} e^{\sqrt{(32q)}\cos z \cos\theta} f(\theta) d\theta$$

which is satisfied by periodic solutions of Mathieu's equation.

In particular, he shewed that

$$ce_0(z) = 1 + \sum_{n=1}^{\infty} \left\{ \frac{2^{n+1}q^n}{(n\,!)^2} - \frac{2^{n+3}n(3n+4)q^{n+2}}{(n+1\,!)^2} + \ldots \right\} \cos 2nz = 1 + \sum_{n=1}^{\infty} q^n A_n(q) \cos 2nz$$

satisfies the equation

$$\frac{d^2y}{dz^2} + \{-32q^2 + 224q^4 - \ldots + 16q\cos 2z\}y = 0.$$

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* *Liouville's Journal*, sér. 2, t. XIII., pp. 137–203.

† *Proceedings of the Mathematical Congress*, 1912, vol. 1.

Subsequent terms in the expression for $A_n(q)$ are very complicated. For instance, I have calculated the coefficient of q^4 to be $2^{n+1}n\{189n^3+982n^2+1653n+888\}\div(n+2\,!)^2$.

There is no very simple relation connecting coefficients, and consequently the only direct method of investigating the convergence of $A_n(q)$ is by constructing a "fonction majorante" and investigating its convergence. In this way I shall shew that the series for $A_n(q)$ certainly converges when $32\,|\,q\,|^2<1$, and that if this condition is satisfied, the series for $ce_0(z)$ converges for all values of z (real or complex). The method does not seem to be applicable to the function of the general order m, as the coefficients in the corresponding series do not seem to alternate in sign according to any obvious law.

To determine the coefficients $A_n(q)$ we substitute

$$1+\sum_{n=1}^{\infty} q^n A_n(q)\cos 2nz$$

in the equation

$$\frac{d^2y}{dz^2}+\{a+16q\cos 2z\}y=0,$$

re-write $2\cos 2z\cos 2nz$ in the form $\cos(2n-2)z+\cos(2n+2)z$, and equate coefficients of the various cosines to zero; we thus get the set of relations*

$$a+8q^2A_1(q)=0$$
$$(4-a)A_1(q)=16+8q_2A^2(q)$$
$$(4n^2-a)A_n(q)=8\{A_{n-1}(q)+A_{n+1}(q)\} \qquad n>1.$$

If we define $A_0(q)$ to be 2, this system of equations may be written (or substituting for a)

$$\{n^2+2q^2A_1(q)\}A_n(q)=2\{A_{n-1}(q)+q^2A_{n+1}(q)\} \qquad (n\geq 1).$$

Now assume

$$A_n(q)=\sum_{r=0}^{\infty}(-)^r B_{n,r}\,q^{2r} \qquad (n\geq 1)$$

where the numbers $B_{n,r}$ are functions of n and r only.

* The first of these does not seem to have been noticed previously; it would not be obvious from Mathieu's method, and Whittaker's method does not introduce a at all

For the values given above for $B_{n,0}$, $B_{n,1}$, $B_{n,2}$, it seems likely that $B_{n,r}$ is positive.*

Substituting in the recurrence formula for A_n and picking out the coefficient of q^{2r}, we get

$$\tfrac{1}{2}n^2 B_{n,r} - B_{n-1,r} = \{B_{n,r-1}B_{1,0} + B_{n,r-2}B_{1,1} + \;..+ B_{n,0}B_{1,r-1}\} - B_{n+1,r-1} \qquad \text{(I.)}$$

This holds when $r = 0, 1, 2, \ldots$, $n = 1, 2, \ldots$, with the convention that the terms with negative suffices vanish, and that $B_{0,0} = 2$, $B_{0,1} = B_{0,2} = \ldots = 0$.

Writing $r = 0$, $r = 1$ in turn, we get

$$\tfrac{1}{2}n^2 B_{n,0} = B_{n-1,0}\,;\; \tfrac{1}{2}n^2 B_{n,1} - B_{n-1,1} = B_{n,0}B_{1,0} - B_{n+1,0}\,.$$

From these we can re-establish in turn the known results

$$B_{n,0} = \frac{2^{n+1}}{(n\,!)^2},\quad B_{n,1} = \frac{2^{n+3}n(3n+4)}{(n+1\,!)^2}.$$

We see at once that, if $s = 0$ or 1, then

$$0 < B_{n+1,s} < 2B_{n,s} \qquad (n = 1, 2, \ldots).$$

We proceed to prove this inequality for every $B_{n,r}$.

Suppose it true when $s = 0, 1, \ldots r-1$ for every n.

Then since $B_{n,r-1}B_{1,0} - B_{n+1,r-1} = 4B_{n,r-1} - B_{n+1,r-1} > 0$, we have from (I.)

$$\tfrac{1}{2}n^2 B_{n,r} - B_{n-1,r} > \sum_{p=1}^{r-1} B_{n,r-p-1}B_{1,p} > 0 \qquad (n = 1, 2, \ldots).$$

The first equation of this system is $\tfrac{1}{2} \cdot 1^2 B_{1,r} > 0$.

Therefore $\qquad B_{n,r} > 0 \qquad (n = 1, 2, \ldots).$

Also, writing

$$(n-1\,!)^2\{B_{n,r-1}B_{1,0} + B_{n,r-2}B_{1,1} + \ldots + B_{n,0}B_{1,r-1}\} = 2^{n-1}U_{n,r-1}$$

$$(n-1\,!)^2 B_{n+1,r-1} = 2^{n-1}V_{n,r-1}$$

we have $$\frac{(n\,!)^2}{2^n}B_{n,r} - \frac{(n-1\,!)^2}{2^{n-1}}B_{n-1,r} = U_{n,r-1} - V_{n,r-1}\,,$$

and so, summing, $$B_{n,r} = \frac{2^n}{(n\,!)^2}\left\{\sum_{m=1}^{n}(U_{m,r-1} - V_{m,r-1})\right\}.$$

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* This is the reason for introducing the factor $(-)^r$.

Consequently, since $U_{m,\,r-1} - V_{m,\,r-1} > 0 \quad (m = 1, 2, \ldots n-1)$,

$$ {}_{n,\,r} - B_{n-1,\,r} = \frac{2^{n+1}}{(n\,!)^2}\left\{1 - \frac{1}{(n+1)^2}\right\}\sum_{m=1}^{n}(U_{m,\,r-1} - V_{m,\,r-1}) - \frac{2^{n+1}}{(n+1\,!)^2}(U_{n+1,\,r-1} - V_{n+1,\,r-1}), $$

$$ > \frac{2^{n+1}}{n\,!^2}\left\{1 - \frac{1}{(n+1)^2}\right\}\{U_{n,\,r-1} - V_{n,\,r-1}\} - \frac{2^{n+1}}{(n+1\,!)^2}(U_{n+1,\,r-1} - V_{n+1,\,r-1}) $$

$$ = \sum_{p=1}^{r-1} B_{1,p}\left\{\frac{4(n+2)}{n(n+1)^2}B_{n,\,r-p-1} - \frac{2}{(n+1)^2}B_{n+1,\,r-p-1}\right\} $$

$$ + \frac{4(n+2)}{n(n+1)^2}\{4B_{n,\,r-1} - B_{n+1,\,r-1}\} - \frac{2}{(n+1)^2}\{4B_{n+1,\,r-1} - B_{n+2,\,r-1}\}. $$

But $\quad 4(n+2)B_{n,\,r-p-1} > 2B_{n+1,\,r-p-1}, \quad B_{n+2,\,r-1} > 0,$

and so

$$ 2B_{n,\,r} - B_{n+1,\,r} > \frac{4(n+2)}{n(n+1)^2}\{4B_{n,\,r-1} - B_{n+1,\,r-1}\} - \frac{8}{(n+1)^2}B_{n+1,\,r-1} $$

$$ > B_{n+1,\,r-1}\left\{\frac{4(n+2)\,.\,2}{n(n+1)^2} - \frac{8}{(n+1)^2}\right\}. $$

$$ > 0. $$

Hence the induction holds from $r-1$ to r; and as $2B_{n,\,r} > B_{n+1,\,r}$ (for every n), when $r = 0$, it is always true.

We have therefore proved that $0 < B_{n+1,r} < 2B_{n,\,r} \quad (n = 1, 2, \ldots)$.

Now choose numbers $C_{n,\,s}$ such that

$$ B_{n,\,s} \leq C_{n,\,s} \qquad (s = 0, 1, \ldots r-1) $$

and define $C_{n,\,r}$ by the equation

$$ \frac{(n\,!)^2}{2^n}C_{n,\,r} = \sum_{m=1}^{n}\frac{(m-1\,!)^2}{2^{n-1}}\{C_{m,\,r-1}C_{1,\,0} + C_{m,\,r-2}C_{1,\,1} + \ldots + C_{m,\,0}C_{1,r-1}\} $$

Then since, from (I.),

$$ \frac{(n\,!)^2}{2^n}B_{n,r} = \sum_{m=1}^{n}\frac{(m-1\,!)^2}{2^{m-1}}[B_{m,r-1}B_{1,0} + B_{m,r-2}B_{1,1} + \ldots + B_{m,\,0}B_{1,\,r-1} - B_{m+1,\,r-1}], $$

and since $B_{m+1,\,r-1} > 0$, we have

$$ B_{n,\,r} < C_{n,\,r}. $$

Therefore, by induction, if $B_{n,\,0} \leq C_{n,\,0} \qquad$ (every n),

we have $\qquad B_{n,\,r} \leq C_{n,\,r} \qquad$ (every n and r).

Now, if $\sum_{r=0}^{\infty}(-)^r C_{n,r}\, q^{2r} = D_n(q)$, the functions $D_n(q)$ are determined by the relations

$$\{1+2q^2 D_1(q)\}\, D_1(q) = 4$$

$$\{n^2+2q^2 D_1(q)\}\, D_n(q) = 2D_{n-1}(q) \qquad (n>1).$$

Therefore $D_1(q) = -\dfrac{1+(1+32q^2)^{\frac{1}{2}}}{4q^2}$

$$D_n(q) = \frac{2^{n+1}}{\prod_{n=1}^{n}\{m^2+2q^2 D_1(q)\}}.$$

Therefore $D_1(q)$ can be expanded into an absolutely convergent series of powers of q if $|32q^2|<1$; and $D_n(q)$ can be expanded in powers of $2q^2 D_1(q)$, and then rearranged in powers of q if *

$$2\,|q|^2\,\Delta_1(\,|q|\,) < 1$$

where $\Delta_n(\,|q|\,)$ is the same function of $|q|$ as D_n is of q, but having the coefficients of all powers of $|q|$ positive; and so

$$\Delta_1(\,|q|\,) = \frac{1-(1-32\,|q|^2)^{\frac{1}{2}}}{4\,|q|^2},$$

which gives $2|q|^2\Delta_1(\,|q|\,) < 1$ whenever $1-(1-32\,|q|^2)^{\frac{1}{2}} < 2$ (which is the case when $32\,|q|^2 < 1$).

Hence, since the coefficients in $A_n(q)$ are numerically less than those in $D_n(q)$, and *a fortiori* less than those in $\Delta_n(\,|q|\,)$, where

$$\Delta_n(\,|q|\,) = \frac{2^{n+1}}{\prod_{m=1}^{n}\{m^2-2\,|q|^2\,\Delta_1(\,|q|\,)\}},$$

$A_n(q)$ can be expanded into an absolutely convergent series of powers of q when $32\,|q|^2 < 1$, and obviously

$$|\,A_n(q)\,| < \Delta_n(\,|q|\,).$$

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* See Bromwich, *Infinite Series*, p. 67. It is obvious that $C_{n,0} = B_{n,0}$.

Now $1+\frac{1}{2}\sum_{n=1}^{\infty} q^n \Delta_n |q| \{e^{2niz}+e^{-2niz}\}$ converges absolutely for all values of z since $q\,\Delta_{n+1}(q)e^{\pm 2iz} \div \Delta_n(q)$ tends to zero as n tends to ∞; and *a fortiori* $1+\sum_{n=1}^{\infty} q^n A_n(q)\cos 2nz$ converges absolutely for all values of z.

The absolute convergence of the series for $ce_0(z)$ has thus been established when $32\,|q|^2 < 1$.

Also, since $1+\sum_{n=1}^{\infty} q^n \Delta_n(|q|)\cos 2nz$ converges absolutely, the series for $ce_0(z)$ may be re-arranged in powers of q.

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