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A New Tautological Relation in $\overline{\mathcal{M}}_{3,1}$ via the Invariance Constraint

*Dedicated to Matteo S. Arcara**

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Abstract. A new tautological relation of $\overline{\mathcal{M}}_{3,1}$ in codimension 3 is derived and proved, using an invariance constraint from our previous work.

1 Introduction

This work is a continuation of [1]. We apply the same technique to $R^3(\overline{\mathcal{M}}_{3,1})$ to find a tautological relation. A general scheme and practical steps, as well as notations used in this paper, can be found in [1] and [10].

1.1 Tautological Rings

The study of the tautological rings is one of the central problems in moduli of curves. The reader is referred to Vakil's survey article [12] and references therein for the various definitions, examples and motivation. In this section we merely try to fix notations.

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack of stable curves of genus g with n marked points. The stacks $\overline{\mathcal{M}}_{g,n}$ are proper, irreducible, smooth Deligne–Mumford stacks. The Chow rings $A^*(\overline{\mathcal{M}}_{g,n})$ over $\mathbb{Q}$ are isomorphic to the Chow rings of their coarse moduli spaces. The tautological rings $R^*(\overline{\mathcal{M}}_{g,n})$ are subrings of $A^*(\overline{\mathcal{M}}_{g,n})$, or subrings of $H^{2*}(\overline{\mathcal{M}}_{g,n})$ via cycle maps. Note that the tautological rings are defined by generators and relations. Since the generators are explicitly given, *the study of tautological rings is the study of relations of tautological classes.*

1.2 Notations

The moduli of curves can be stratified by topological type. Each boundary stratum can be conveniently presented by the (dual) graph of its generic curve defined in the following way. To each stable curve C with marked points, one can associate a dual graph Γ. Vertices of Γ correspond to irreducible components. They are labeled by their geometric genus. Assign an edge joining two vertices each time the two

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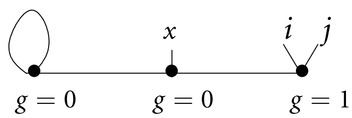
components intersect. To each marked point one draws an half-edge incident to the vertex, with the same label as the point. Now, the stratum corresponding to Γ is the closure of the subset of all stable curves in $\overline{\mathcal{M}}_{g,n}$ which have the same topological type as C. For each dual graph Γ, one can decorate the graph by assigning a monomial, or more generally a polynomial, in the ψ classes to each half-edge and one in the κ classes to each vertex. The tautological classes in $R^k(\overline{\mathcal{M}}_{g,n})$ can be represented by $\mathbb{Q}$-linear combinations of *decorated graphs. Since there are no κ or λ-classes involved in this paper, they will be left out of our discussion below.*

For typesetting reasons, it is more convenient to denote a decorated graph by another notation, inspired by Gromov–Witten theory, called *gwi*. Given a decorated graph Γ:

- To the vertices of Γ of genera $g_1, g_2, \dots,$ we assign a product of "brackets" $\langle \cdot \rangle_{g_1} \langle \cdot \rangle_{g_2} \dots$. To simplify the notation, $\langle \cdot \rangle := \langle \cdot \rangle_0$.
- Assign to each half-edge a symbol ∂^*. The external half-edges use super-indices $\partial^{x_1}, \partial^{x_2}, \dots,$ corresponding to their labeling. For each pair of half-edges determined by an edge, a single super-index will by used, denoted by $\mu_1, \mu_2, \dots$. Otherwise, all half-edges should use different super-indices.
- A decoration to a half-edge a by ψ^k is denoted by ∂^a_k.
- For each vertex $\langle \cdot \rangle_g$ with m half-edges and n external half-edges, an insertion is placed in the vertex $\langle \partial^{x_1}_{k_1} \partial^{x_2}_{k_2} \cdots \partial^{\mu_1}_{h_1} \partial^{\mu_2}_{h_2} \cdots \rangle_g$.

For example, the following graph



has gwi $\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \partial^{\mu_3} \partial^{\mu_3} \rangle \langle \partial^{\mu_2} \rangle_1$, which appears in (40).

The main tool employed in this paper is the following theorem on the Invariance Constraint.

Theorem **([11] Theorem 5)** *There exist linear operators $\mathfrak{r}_l$*

$$\mathfrak{r}_l : R^k(\overline{\mathcal{M}}_{g,n}) \to R^{k-l+1}(\overline{\mathcal{M}}^{\bullet}_{g-1,n+2}), \quad l = 1, 2, \dots,$$

where the symbol $^\bullet$ denotes the moduli of possibly disconnected curves. In particular, if $E = 0$ is a tautological relation in $R^k(\overline{\mathcal{M}}_{g,n})$, then $\mathfrak{r}_l(E) = 0$.

The definition of $\mathfrak{r}_l$ will be given in the next section.

1.3 Motivation

Our motivation is quite simple. In earlier works [1, 7], we applied the Invariance Constraint in genus one and two. The choice of codimension 3 in $\overline{\mathcal{M}}_{3,1}$ as a next step is almost obvious. First of all, the Invariance Constraint works inductively. Given what we know about genus one and two, it is only reasonable to proceed to either $\overline{\mathcal{M}}_{2,n}$ for $n \geq 4$ or $\overline{\mathcal{M}}_{3,1}$. Secondly, one also knows from the Graber–Vakil theorem in

[8] that ψ^3 on $\overline{\mathcal{M}}_{3,1}$ is rationally equivalent to a sum of boundary strata containing at least one rational component. Thirdly, Getzler and Looijenga [7] have shown that there is only one relation in codimension 3 in $\overline{\mathcal{M}}_{3,1}$. That makes it a reasonable place to start.

1.4 Main Result

The main result of this paper is the following.

Main Theorem *The new tautological relation for codimension 3 strata in* $\overline{\mathcal{M}}_{3,1}$ *is*

$$\begin{aligned}
\langle\partial_3^x\rangle_3 =& \tfrac{5}{72}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_3}\rangle_2 + \tfrac{1}{252}\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\rangle\langle\partial_1^x\partial^{\mu_1}\rangle_2\\
&+ \tfrac{5}{72}\langle\partial^x\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial_1^{\mu_2}\rangle_2 + \tfrac{5}{42}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial_1^{\mu_1}\partial^{\mu_2}\rangle_2\\
&+ \tfrac{41}{21}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\rangle_1\langle\partial_1^{\mu_2}\rangle_2 + \tfrac{11}{40320}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_3}\partial^{\mu_3}\rangle_1\\
&+ \tfrac{1}{13440}\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\rangle\langle\partial^x\partial^{\mu_1}\partial^{\mu_3}\partial^{\mu_3}\rangle_1 + \tfrac{1}{8064}\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_1}\rangle_1\\
&+ \tfrac{191}{120960}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\rangle_1 + \tfrac{1}{5040}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle_1\\
&+ \tfrac{1}{4032}\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle_1 + \tfrac{17}{2880}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\rangle_1\\
&+ \tfrac{1}{840}\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\rangle\langle\partial^x\partial^{\mu_3}\rangle_1\langle\partial^{\mu_1}\partial^{\mu_3}\rangle_1 + \tfrac{1}{336}\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\rangle\langle\partial^x\partial^{\mu_1}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_3}\rangle_1\\
&+ \tfrac{1}{126}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_3}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_2}\rangle_1 + \tfrac{23}{5040}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_3}\rangle_1\\
&+ \tfrac{17}{5040}\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_1}\rangle_1\langle\partial^{\mu_2}\rangle_1 + \tfrac{113}{2520}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\rangle_1\langle\partial^{\mu_2}\rangle_1\\
&+ \tfrac{1}{210}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle_1 + 0\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle_1\\
&+ \tfrac{1}{84}\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_1}\rangle_1\langle\partial^{\mu_2}\partial^{\mu_3}\rangle_1 + \tfrac{211}{1260}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1\\
&+ \tfrac{1}{1260}\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1 + \tfrac{1}{630}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_3}\rangle_1\\
&+ \tfrac{11}{140}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_2}\partial^{\mu_3}\rangle_1 + \tfrac{4}{35}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_1}\rangle_1\langle\partial^{\mu_3}\rangle_1\\
&+ \tfrac{2}{105}\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_1}\rangle_1\langle\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1\\
&+ \tfrac{89}{210}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\rangle_1\langle\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1 + \tfrac{1}{53760}\langle\partial^x\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle.
\end{aligned}$$

Remarks. (i) While this paper was in preparation, a preprint by T. Kimura and X. Liu [9] appeared on the arXiv. There are two major differences between our results and theirs. First, their choice of basis of codimension 3 strata in $\overline{\mathcal{M}}_{3,1}$ is different. They use $\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\rangle\langle\partial^x\partial_1^{\mu_1}\rangle_2$ instead of $\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\rangle\langle\partial_1^x\partial^{\mu_1}\rangle_2$ We have checked that their relation is equivalent to ours. Second, their approach is "traditional": knowing that there must be a relation from Graber–Vakil, they can then proceed to find the coefficients based on the evaluation of the Gromov–Witten invariants of $\mathbb{P}^1$.

Our approach is quite different. There are no computer-aided calculations of Gromov–Witten invariants. Only linear algebra is involved in the calculation.

(ii) This technique has been applied to prove some Faber type results in tautological rings [2]. A corollary of [2] is that there is no relation between ψ_1 and κ_1 in

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$R^1(\overline{\mathcal{M}}_{3,1})$. It is easy to see, however, that there is a relation between the monomials in κ- and ψ-classes. On the other hand, the reader may amuse him or herself with the following result.

Proposition 1 *There is no (new) relation in $R^2(\overline{\mathcal{M}}_{3,1})$ involving ψ_1^2 and the classes in $R^2(\overline{\mathcal{M}}_{3,1}) \setminus R^2(\mathcal{M}_{3,1})$.*

This can be shown, for example, by the same technique used in this paper. Let $\{\Gamma_k\}$ be a basis of these strata. When one sets a hypothetical equation $\sum_k c_k \Gamma_k = 0$ and imposes the invariance constraint, the only solution is $c_k = 0$ for all k.

2 Invariance Constraints

Here some ingredients from [10] and [11] will be briefly recalled.

2.1 The Operators $\mathfrak{r}_l$

The invariance constraint is imposed by applying the linear operators $\mathfrak{r}_l$. These linear operators are defined as operations on the decorated graphs. The output graphs have two more markings, which are denoted by i, j. In terms of gwis,

$$
\begin{aligned}
\mathfrak{r}_l(\langle \partial^{\mu}_{k'} \cdots \rangle_{g'} \cdots \langle \partial^{\mu}_{k''} \cdots \rangle_{g''}) &= \tfrac{1}{2}(\langle \partial^{i}_{k'+l} \cdots \rangle_{g'} \cdots \langle \partial^{j}_{k''} \cdots \rangle_{g''} \\
&+ \langle \partial^{j}_{k'} \cdots \rangle_{g'} \langle \partial^{i}_{k''+l} \cdots \rangle_{g''}) \\
&+ \tfrac{1}{2}(-1)^{l-1}(\langle \partial^{j}_{k'+l} \cdots \rangle_{g'} \cdots \langle \partial^{i}_{k''} \cdots \rangle_{g''} + \langle \partial^{i}_{k'} \cdots \rangle_{g'} \langle \partial^{j}_{k''+l} \cdots \rangle_{g''}) + \cdots \\
&+ \tfrac{1}{2} \sum_{m=0}^{l-1} (-1)^{m+1} \langle \partial^{i}_{l-1-m} \partial^{j}_{m} \partial^{\mu}_{k'} \cdots \rangle_{g'-1} \cdots \langle \partial^{\mu}_{k''} \cdots \rangle_{g''} + \cdots \\
&+ \tfrac{1}{2} \sum_{m=0}^{l-1} (-1)^{m+1} \langle \partial^{\mu}_{k'} \cdots \rangle_{g'} \cdots \langle \partial^{i}_{l-1-m} \partial^{j}_{m} \partial^{\mu}_{k''} \cdots \rangle_{g''-1} \\
&+ \tfrac{1}{2} \Big(\sum_{m=0}^{l-1} (-1)^{m+1} \sum_{g=0}^{g'} \partial^{\mu}_{k'} \cdots (\langle \partial^{i}_{l-1-m} \rangle_{g} \langle \partial^{j}_{m} \rangle_{g'-g}) \Big) \langle \partial^{\mu}_{k''} \cdots \rangle_{g''} + \cdots \\
&+ \tfrac{1}{2} \langle \partial^{\mu}_{k'} \cdots \rangle_{g'} \Big(\sum_{m=0}^{l-1} (-1)^{m+1} \sum_{g=0}^{g''} \partial^{\mu}_{k''} \cdots (\langle \partial^{i}_{l-1-m} \rangle_{g} \langle \partial^{j}_{m} \rangle_{g''-g}) \Big),
\end{aligned}
$$

where the notation $\partial^{\mu}_{k} \cdots (\langle \partial^{i}_{l-1-m} \rangle_{g_1} \langle \partial^{j}_{m} \rangle_{g_2})$ means that the half-edge insertions $\partial^{\mu}_{k} \cdots$ act on the product of vertices $\langle \partial^{i}_{l-1-m} \rangle_{g_1} \langle \partial^{j}_{m} \rangle_{g_2}$ by the Leibniz rule. Note that $\langle \cdots \rangle_{-1} := 0$.

One can also denote gwi's as decorated graphs as explained in Section 1.2. In terms of decorated graphs, $\mathfrak{r}_l$ can be defined as follows.

- *Cutting edges.* Cut one edge and create two new half-edges. Label two new half-edges with $i, j \notin \{1, 2, \ldots, n\}$ in two different ways. Produce a formal sum of 4 graphs by decorating extra ψ^l to i-labeled new half-edges with coefficient $1/2$ and by decorating extra ψ^l to j-labeled new half-edges with coefficient $(-1)^{l-1}/2$. (By "extra" decoration we mean that ψ^l is multiplied by whatever decorations which

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are already there.) Produce more graphs by repeating the above proceedure on the other edges of the original graph. Retain only the stable graphs. Take the formal sum of these final graphs.

- *Genus reduction.* For each vertex, produce l graphs. Reduce the genus of this given vertex by one and add two new half-edges. Label two new half-edges with i, j and decorate them by ψ^{l-1-m}, ψ^m (respectively) where $0 \leq m \leq l-1$. Do this to all vertices, and retain only the stable graphs. Take the formal sum of these graphs with coefficient $\frac{1}{2}(-1)^{m+1}$.
- *Splitting vertices.* Split one vertex into two. Add one new half-edge to each of the two new vertices. Label them with i, j and decorate them by ψ^{l-1-m}, ψ^m (respectively) where $0 \leq m \leq l-1$. Produce new graphs by splitting the genus g between the two new vertices (g', g'' such that $g' + g'' = g$), and distributing to the two new vertices the (old) half-edges which belong to the original chosen vertex, in all possible ways. Do this to all vertices, and retain only the stable graphs. Take the formal sum of these graphs with coefficient $\frac{1}{2}(-1)^{m+1}$.

Remarks. (i) In terms of graphical operations, the first two lines of the equation above which defines $\mathfrak{r}_l$ in terms of gwis stand for "cutting edges"; the middle two for "genus reduction"; the last two for "splitting vertices". These are explained in [10].

(ii) In this paper, only the $l = 1$ case will be used. In fact, it has been shown recently, by Faber, Shadrin, and Zvonkine, and independently Pandharipande (and the second author) that the $l = 1$ case implies general l. See Section 3 of [4].

2.2 The Algorithm of Finding Tautological Relations

Our method of finding this relation is fairly simple.

(a) By Graber and Vakil's ($*$) Theorem [8] or Getzler and Looijenga's Hodge number calculations [7], there is a new relation in $R^3(\overline{\mathcal{M}}_{3,1})$.

(b) Apply the invariance constraint [10, Theorem 5] to obtain the coefficients of the relation.

Applying (b) gives a *necessary* condition. Combined with (a), this generates and proves the new relation.

In the case of $R^3(\overline{\mathcal{M}}_{3,1})$, we first identify thirty "potentially independent" decorated graphs with decorations coming from ψ-classes only. They are the 30 graphs listed in the Main Theorem. More precisely, out of the tautological classes of codimension 3 in $\overline{\mathcal{M}}_{3,1}$ generated by ψ-class and boundary classes, many of them can be written in terms of the others using WDVV, TRR, Mumford–Getzler, Getzler [5, 6] and Belorousski–Pandharipande [3] relations. After applying those relations, we can write all of the strata in terms of the thirty strata appearing in the Main Theorem.

Let Γ_k denote the tautological class in the k-th term of the Main Theorem. A general combination of these 30 decorated graphs is written as $E = \sum_{k=1}^{30} c_k \Gamma_k$ where c_k is the corresponding unknown coefficient to be found. Suppose that $E = 0$ is a tautological equation. The Invariance Theorem implies that $\mathfrak{r}_1(E) = 0$, where $\mathfrak{r}_1 \colon R^3(\overline{\mathcal{M}}_{3,1}) \to R^3(\overline{\mathcal{M}}^{\bullet}_{2,3})$. By analyzing the properties of the image in $R^3(\overline{\mathcal{M}}^{\bullet}_{2,3})$, which is known from earlier work in genus one and two [1, 3, 5, 6], we obtain a

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system of homogeneous linear equations on the c_k's, which will be equations (1)–(49). This system is potentially very over-determined, as there are more equations than variables. However, the Invariance Constraint theorem predicts that this system of linear equations uniquely determines the c_k's if there is a nontrivial tautological relation. (We always have the trivial solution with $c_k = 0$ for all k.)

3 Proof of the Main Theorem

We will spare the reader the explicit formula for $\mathfrak{r}_1(E)$, which would occupy several pages. It is possible to reconstruct it from the equations below. For example, let Γ_i be the decorated graph corresponding to the i-th term in the theorem. Then $\mathfrak{r}_1(\Gamma_i)$ would contain the sum of the basis elements below for which a c_i appears in their equation, with the corresponding coefficient. To illustrate how we went about calculating $\mathfrak{r}_1(E)$, though, we include two sample calculations in Appendix A, where we go through all of the steps for calculating and simplifying $\mathfrak{r}_1(\Gamma_3)$ and $\mathfrak{r}_1(\Gamma_{10})$.

Note that the (new) half-edges i and j in the output graphs are always assumed to be symmetrized. Some of the output graphs will be disconnected. They are easier to deal with as they involve fewer relations (*e.g.* WDVV). So we shall start with the disconnected terms.

We now choose a basis of $R^3(\overline{\mathcal{M}}^{\bullet}_{2,3})$, and we set the coefficient in $\mathfrak{r}_1(E)$ of each element in the basis equal to zero, obtaining an equation that the c_k's have to satisfy. In the following list, for each equation, we write its number, the corresponding basis vector in $R^3(\overline{\mathcal{M}}^{\bullet}_{2,3})$, and its coefficient in $\mathfrak{r}_1(E)$, which is set equal to zero. If a basis vector is disconnected, we use a vertical bar to separate its connected components.

(1)	$\langle\partial^j_1\rangle_2 \vert \langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^i\partial^{\mu_1}\partial^{\mu_2}\rangle$	$c_2 + c_4 = 0.$
(2)	$\langle\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle \vert \langle\partial^x\partial^j\partial^{\mu_2}\rangle\langle\partial^{\mu_2}_1\rangle_2$	$3c_3 - c_4 + \frac{1}{24}c_6 = 0.$
(3)	$\langle\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle \vert \langle\partial^{\mu_2}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^j\partial^{\mu_3}\rangle_1$	$-\frac{1}{80}c_3 - c_8 + \frac{1}{24}c_{15} = 0.$
(4)	$\langle\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle_1 \vert \langle\partial^x\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle$	$c_7 - c_8 - c_{11} = 0.$
(5)	$\langle\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle \vert \langle\partial^x\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle_1$	$\frac{1}{30}c_3 - c_{11} + \frac{1}{24}c_{25} = 0.$
(6)	$\langle\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle \vert \langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_2}\partial^{\mu_3}\rangle_1$	$\frac{1}{30}c_3 + 2c_8 - c_{12} + \frac{1}{24}c_{24} = 0.$
(7)	$\langle\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle \vert \langle\partial^x\partial^j\partial^{\mu_2}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle_1$	$-\frac{1}{30}c_3 - c_7 + c_8 + \frac{1}{24}c_{16} = 0.$
(8)	$\langle\partial^x\partial^i\rangle_1 \vert \langle\partial^j\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^{\mu_2}\rangle_1$	$c_{14} - c_{15} + c_{18} - c_{24} = 0.$
(9)	$\langle\partial^i\partial^{\mu_1}\rangle_1\langle\partial^{\mu_1}\rangle_1 \vert \langle\partial^x\partial^j\partial^{\mu_2}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle$	$c_{15} - c_{17} + c_{25} = 0.$
(10)	$\langle\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle \vert \langle\partial^x\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1$	$\frac{4}{5}c_3 - c_{16} + \frac{1}{24}c_{27} = 0.$
(11)	$\langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^{\mu_1}\rangle_1 \vert \langle\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_2}\rangle_1$	$-c_3 - \frac{1}{240}c_6 + c_{14} - c_{15} = 0.$
(12)	$\langle\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle \vert \langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1$	$\frac{4}{5}c_3 + c_{14} + c_{15} - c_{18} + \frac{1}{12}c_{28} = 0.$
(13)	$\langle\partial^j\partial^{\mu_1}\partial^{\mu_1}\rangle \vert \langle\partial^x\partial^i\partial^{\mu_2}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_3}\rangle_1$	$-\frac{4}{5}c_3 + c_{15} - c_{17} + \frac{1}{24}c_{27} = 0.$
(14)	$\langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^{\mu_1}\rangle_1 \vert \langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\rangle_1$	$\frac{1}{10}c_6 + 2c_{16} + c_{22} - c_{23} = 0.$
(15)	$\langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^{\mu_1}\rangle_1 \vert \langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1$	$\frac{7}{10}c_6 + c_{27} + c_{28} - 3c_{29} = 0.$
(16)	$\langle\partial^x\partial^i\rangle_1 \vert \langle\partial^j\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle =$	
	$\langle\partial^x\partial^i\rangle_1 \vert \langle\partial^j\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_3}\partial^{\mu_3}\rangle$	$-c_8 + c_9 + \frac{1}{24}c_{14} - c_{21} = 0.$

(17)	$\langle\partial^x\partial^i\rangle_1\vert\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_1}\partial^{\mu_2}\rangle$	$2c_9 - c_{12} = 0.$
(18)	$\langle\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle\vert\langle\partial^x\partial^{\mu_2}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_3}\rangle_1$	$\frac{1}{48}c_3 - c_7 + \frac{1}{24}c_{17} = 0.$
(19)	$\langle\partial^x\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle\vert\langle\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_2}\rangle_1$	$-\frac{1}{24}c_3 - \frac{1}{240}c_4 - c_8 + \frac{1}{24}c_{14} = 0.$
(20)	$\langle\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle\vert\langle\partial^j\partial^{\mu_2}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_3}\rangle_1$	$\frac{1}{48}c_3 + c_8 - 2c_9 + \frac{1}{24}c_{14} + \frac{1}{24}c_{18} = 0.$
(21)	$\langle\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle\vert\langle\partial^x\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\rangle_1$	$\frac{7}{30}c_3 + 2c_8 - c_{13} + \frac{1}{24}c_{23} = 0.$
(22)	$\langle\partial^x\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle\vert\langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\rangle_1$	$\frac{1}{10}c_4 + 2c_7 - c_{13} + \frac{1}{24}c_{22} = 0.$
(23)	$\langle\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle\vert\langle\partial^x\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1$	$\frac{13}{10}c_3 + c_{15} - c_{19} + \frac{1}{8}c_{29} = 0.$
(24)	$\langle\partial^x\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle\vert\langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1$	$\frac{7}{10}c_4 + c_{17} - c_{19} + \frac{1}{24}c_{28} = 0.$
(25)	$\langle\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle\vert\langle\partial^x\partial^j\partial^{\mu_2}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_3}\rangle_1$	$\frac{23}{240}c_3 + c_8 - 2c_{10} + \frac{1}{24}c_{15} + \frac{1}{12}c_{19} = 0.$
(26)	$\langle\partial^i\partial^x\partial^{\mu_1}\partial^{\mu_1}\rangle\vert\langle\partial^j\partial^{\mu_2}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_3}\rangle_1$	$\frac{13}{240}c_4 + c_7 - 2c_{10} + \frac{1}{24}c_{17} + \frac{1}{24}c_{18} = 0.$
(27)	$\langle\partial^i\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\rangle\vert\langle\partial^x\partial^j\partial^{\mu_3}\rangle\langle\partial^{\mu_3}\rangle_1$	$\frac{1}{960}c_6 + c_9 - c_{10} + \frac{1}{24}c_{16} = 0.$
(28)	$\langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^{\mu_1}\rangle_1\vert\langle\partial^j\partial^{\mu_2}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_3}\rangle_1$	$\frac{13}{240}c_6 + c_{16} + c_{18} - 2c_{19} + \frac{1}{24}c_{27} = 0.$
(29)	$\langle\partial^i\partial^{\mu_1}\partial^{\mu_1}\rangle\vert\langle\partial^x\partial^j\partial^{\mu_2}\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle$	$\frac{1}{576}c_3 + \frac{1}{24}c_8 + \frac{1}{24}c_{10} - 3c_{30} = 0.$
(30)	$\langle\partial^i\partial^x\partial^{\mu_1}\partial^{\mu_1}\rangle\vert\langle\partial^j\partial^{\mu_2}\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle$	$\frac{1}{960}c_4 + \frac{1}{24}c_7 + \frac{1}{24}c_9 - 3c_{30} = 0.$

There are several terms of the form $*\langle\partial^i\rangle_1$. If we remove the $\langle\partial^i\rangle_1$, they become terms in $\overline{\mathcal{M}}_{2,2}$ of codimension 3, and there is a relation between them which we can find by using Getzler's relation [6] with ψ^2 on x and ψ on j (see Appendix B). The relation is

$$\begin{aligned}
0 = &-\tfrac{3}{40}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\rangle\langle\partial^j\partial^{\mu_1}\partial^{\mu_3}\rangle\langle\partial^{\mu_3}\rangle_1 + \tfrac{3}{40}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_3}\rangle_1\\
&-\tfrac{7}{120}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^j\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_3}\rangle_1 + \tfrac{7}{120}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\rangle_1\\
&+\tfrac{1}{120}\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_3}\rangle_1 - \tfrac{1}{120}\langle\partial^x\partial^j\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^{\mu_2}\rangle_1\\
&-\tfrac{1}{120}\langle\partial^x\partial^j\partial^{\mu_1}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\rangle_1 + \tfrac{1}{120}\langle\partial^j\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_2}\rangle_1\\
&+\text{other terms with all vertices of genus 0.}
\end{aligned}$$

We are going to solve this relation for the term

$$\langle\partial^i\rangle_1\vert\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_3}\rangle_1$$

and find an equation for all of the other terms. Among those, seven of them are of the form $*\langle\partial^\mu\rangle_1\langle\partial^i\rangle_1$ and are related by WDVV. They can be written in terms of the following four independent vectors.

(31) $\langle\partial^i\rangle_1\vert\langle\partial^j\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^x\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_3}\rangle_1$

$$-\tfrac{3}{20}c_1 - c_2 - 2c_3 - 2c_5 - \tfrac{1}{24}c_6 - c_{16} + 2c_{19} - c_{24} + \tfrac{1}{24}c_{27} = 0.$$

(32) $\langle\partial^i\rangle_1\vert\langle\partial^x\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_3}\rangle_1$ $\quad -c_2 - c_4 = 0.$

(33) $\langle\partial^i\rangle_1\vert\langle\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^j\partial^{\mu_2}\rangle\langle\partial^{\mu_3}\rangle_1$ $\quad \frac{11}{240}c_1 + c_2 + 2c_3 + c_5 - c_{18} + c_{24} = 0.$

(34) $\langle\partial^i\rangle_1\vert\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_3}\rangle_1$ $\quad -\frac{1}{10}c_1 - 2c_3 - c_5 + 4c_{19} - c_{23} - c_{24} = 0.$

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There are four additional independent vectors

(35) $\langle\partial^i\rangle_1|\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^j\partial^{\mu_1}\partial^{\mu_3}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\rangle_1$ $\frac{1}{10}c_1 + c_5 + c_{22} - c_{23} - 2c_{25} + 2c_{26} = 0.$

(36) $\langle\partial^i\rangle_1|\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\rangle_1\langle\partial^{\mu_3}\rangle_1$ $-\frac{7}{10}c_1 - c_6 - c_{28} + 3c_{29} = 0.$

(37) $\langle\partial^i\rangle_1|\langle\partial^j\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_2}\rangle_1$ $\frac{1}{20}c_1 + c_2 + c_3 + c_5 - c_{14} + c_{15} + c_{18} - c_{22} = 0.$

(38) $\langle\partial^i\rangle_1|\langle\partial^x\partial^j\partial^{\mu_1}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\rangle_1$ $-\frac{11}{240}c_1 - c_2 - 2c_3 - c_5 - c_{14} + c_{15} = 0.$

The remaining connected strata are all of codimension 3 in $\overline{\mathcal{M}}_{2,3}$. We found relations between them in two different ways (see Appendix C). First, by taking Getzler's relation in $\overline{\mathcal{M}}_{1,4}$ [5], adding another marked point, and then identifying either two of the first four marked points or the fifth marked point with one of the others. Secondly, by taking the Belorousski–Pandharipande relation [3], adding ψ at the marked point x, and then simplifying. The relations we obtain are the following:

$$\begin{aligned}
0 = {} & \langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle_1 + \langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle_1\\
& + \langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^i\partial^j\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle_1 - \langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^j\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle_1\\
& - \langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle_1 - \langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle_1\\
& - \langle\partial^i\partial^j\partial^{\mu_1}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_2}\partial^{\mu_3}\rangle_1 + \text{other terms},\\
0 = {} & \langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^{\mu_2}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_1}\partial^{\mu_3}\rangle_1 - \langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^i\partial^j\partial^{\mu_3}\rangle_1\\
& - \langle\partial^i\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^j\partial^{\mu_3}\rangle_1 + \text{other terms},\\
0 = {} & \langle\partial^i\partial^j\partial^{\mu_1}\rangle\langle\partial^{\mu_2}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_1}\partial^{\mu_3}\rangle_1 - \langle\partial^i\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^x\partial^{\mu_3}\rangle_1\\
& - \langle\partial^i\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^x\partial^{\mu_3}\rangle_1 + \text{other terms},\\
0 = {} & -\langle\partial^i\partial^j\partial^{\mu_1}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1 - \langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1\\
& - \langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1 - \langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^i\partial^j\partial^{\mu_1}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_3}\rangle_1\\
& + \langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^i\partial^j\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_2}\rangle_1 + \langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1\\
& + \langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1 + \text{other terms}.
\end{aligned}$$

After solving the four relations above for the following four terms

$$\begin{gathered}
\langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^j\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle_1, \langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^{\mu_2}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_1}\partial^{\mu_3}\rangle_1,\\
\langle\partial^i\partial^j\partial^{\mu_1}\rangle\langle\partial^{\mu_2}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_1}\partial^{\mu_3}\rangle_1, \langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^i\partial^j\partial^{\mu_1}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_3}\rangle_1,
\end{gathered}$$

we obtain the following equations for the other terms.

(39) $\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^i\partial^j\partial^{\mu_2}\rangle\langle\partial_1^{\mu_1}\rangle_2$ $-\frac{1}{2}c_1 + 4c_5 - \frac{1}{2}c_6 = 0.$

(40) $\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^i\partial^j\partial^{\mu_2}\rangle_1$ $-\frac{1}{40}c_1 - \frac{1}{2}c_2 - \frac{1}{2}c_3 - \frac{1}{2}c_5 + 2c_8 - \frac{1}{2}c_{15} = 0.$

(41) $\langle\partial^i\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^j\partial^x\partial^{\mu_2}\rangle_1$ $\frac{1}{240}c_1 - c_3 - \frac{1}{60}c_5 + 4c_8 + 2c_{12} - c_{15} - 3c_{21} = 0.$

(42) $\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^i\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_1}\partial^{\mu_3}\rangle_1 \quad -\frac{1}{10}c_1-\frac{29}{30}c_5-4c_{11}+c_{16}+3c_{20}-3c_{21}=0.$

(43) $\langle\partial^i\partial^j\partial^{\mu_1}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_2}\partial^{\mu_3}\rangle_1 \quad \frac{1}{30}c_5-2c_{11}+\frac{1}{2}c_{16}+6c_{21}-\frac{1}{2}c_{24}=0.$

(44) $\langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle_1 \quad \frac{1}{15}c_5+8c_{11}-c_{16}-3c_{20}+3c_{21}=0.$

(45) $\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^i\partial^j\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle_1 \quad -\frac{1}{30}c_5+4c_{11}-\frac{1}{2}c_{16}-\frac{1}{2}c_{25}=0.$

(46) $\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^i\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^j\partial^{\mu_3}\rangle_1\langle\partial^{\mu_1}\rangle_1$

$$-\frac{7}{5}c_1+\frac{7}{5}c_5-c_6+2c_{23}-2c_{24}-6c_{25}+2c_{26}+c_{27}=0.$$

(47) $\langle\partial^i\partial^j\partial^{\mu_1}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^x\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1 \quad \frac{4}{5}c_5+2c_{22}+2c_{24}-2c_{25}+\frac{1}{2}c_{27}-c_{28}=0.$

(48) $\langle\partial^x\partial^i\partial^{\mu_1}\rangle\langle\partial^j\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1 \quad \frac{3}{5}c_5-2c_{23}+2c_{24}+6c_{25}+2c_{26}-c_{27}=0.$

(49) $\langle\partial^x\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^i\partial^j\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_2}\rangle_1 \quad -\frac{4}{5}c_5+2c_{16}+2c_{25}-c_{27}=0.$

Solving the equations (1)–(49) gives the coefficients of the terms in the relation in the Main Theorem.

A Sample Calculations of $\mathfrak{r}_1(\Gamma_k)$

We calculate $\mathfrak{r}_1(\Gamma_3)$ and $\mathfrak{r}_1(\Gamma_{10})$ here to illustrate the process.

A.1 Calculation of $\mathfrak{r}_1(\Gamma_3)$.

Recall that $\Gamma_3=\langle\partial^x\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial_1^{\mu_2}\rangle_2$.

The first step in calculating $\mathfrak{r}_1(\Gamma_3)$ is to cut the edges. There are two edges here: μ_1 and μ_2. If we cut μ_1, we obtain $2\langle\partial^x\partial_1^i\partial^j\partial^{\mu_2}\rangle\langle\partial_1^{\mu_2}\rangle_2$, and if we cut μ_2, we obtain $\langle\partial^x\partial^{\mu_1}\partial^{\mu_1}\partial_1^i\rangle\langle\partial_1^j\rangle_2+\langle\partial^x\partial^{\mu_1}\partial^{\mu_1}\partial^j\rangle\langle\partial_2^i\rangle_2$. The next step is genus reduction, which can only be applied to the genus 2 term to obtain $-\frac{1}{2}\langle\partial^x\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^i\partial^j\partial_1^{\mu_2}\rangle_1$. The third and last step is splitting the vertices. There are two vertices to split. If we split the genus 0 vertex into two vertices, we obtain $-\frac{1}{2}\partial^x\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}(\langle\partial^i\rangle\langle\partial^j\rangle)\langle\partial_1^{\mu_2}\rangle_2$, where the notation $\partial^x\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}(\langle\partial^i\rangle\langle\partial^j\rangle)$ here stands for a Leibniz rule (graphically, it corresponds to attaching the four half-edges in all possible ways, some to the first genus 0 vertex, and the rest to the other one). We shall simplify it later. If we split the genus 2 vertex, it can only be split into two genus 1 vertices (or else, if we use a genus 2 vertex and a genus 0 vertex, the genus 0 vertex would not have enough half-edges to be stable) to obtain $-\frac{1}{2}\langle\partial^x\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\partial_1^{\mu_2}(\langle\partial^i\rangle_1\langle\partial^j\rangle_1)$.

Putting these together we obtain

$$\begin{aligned}\mathfrak{r}_1(\Gamma_3)=\;&2\langle\partial^x\partial_1^i\partial^j\partial^{\mu_2}\rangle\langle\partial_1^{\mu_2}\rangle_2+\langle\partial^x\partial^{\mu_1}\partial^{\mu_1}\partial_1^i\rangle\langle\partial_1^j\rangle_2+\langle\partial^x\partial^{\mu_1}\partial^{\mu_1}\partial^j\rangle\langle\partial_2^i\rangle_2\\&-\frac{1}{2}\langle\partial^x\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^i\partial^j\partial_1^{\mu_2}\rangle_1-\frac{1}{2}\partial^x\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}(\langle\partial^i\rangle\langle\partial^j\rangle)\langle\partial_1^{\mu_2}\rangle_2\\&-\frac{1}{2}\langle\partial^x\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\partial_1^{\mu_2}(\langle\partial^i\rangle_1\langle\partial^j\rangle_1).\end{aligned}$$

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Next, we expand the last two terms to obtain

$$\begin{aligned}
\mathfrak{r}_1(\Gamma_3) = {} & 2\langle \partial^x \partial_1^i \partial^j \partial^{\mu_2} \rangle \langle \partial_1^{\mu_2} \rangle_2 + \langle \partial^x \partial^{\mu_1} \partial^{\mu_1} \partial_1^i \rangle \langle \partial_1^j \rangle_2 + \langle \partial^x \partial^{\mu_1} \partial^{\mu_1} \partial^j \rangle \langle \partial_2^i \rangle_2 \\
& - \tfrac{1}{2} \langle \partial^x \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^j \partial_1^{\mu_2} \rangle_1 - 2\langle \partial^i \partial^x \partial^{\mu_1} \rangle \langle \partial^j \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial_1^{\mu_2} \rangle_2 \\
& - \langle \partial^i \partial^x \partial^{\mu_2} \rangle \langle \partial^j \partial^{\mu_1} \partial^{\mu_1} \rangle \langle \partial_1^{\mu_2} \rangle_2 - \langle \partial^x \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial_1^{\mu_2} \rangle_1 \langle \partial^j \rangle_1.
\end{aligned}$$

The terms of genus 0 or 1 with a ∂_1^* in them can be simplified using Tautological Recursive Relations, and the term $\langle \partial_2^i \rangle_2$ can be simplified using Mumford's Relation in $\overline{\mathcal{M}}_{2,1}$.

For the reader's convenience, let us recall those relations here.

- TRR in $g = 0$: $\langle \partial_{k+1}^a \partial^b \partial^c \rangle = \langle \partial_k^a \partial^\mu \rangle \langle \partial^\mu \partial^b \partial^c \rangle$.
- TRR in $g = 1$: $\langle \partial_{k+1}^a \rangle_1 = \langle \partial_k^a \partial^\mu \rangle \langle \partial^\mu \rangle_1 + \frac{1}{24} \langle \partial_k^a \partial^\mu \partial^\mu \rangle$.
- Mumford's Relation in $g = 2$:

$$\begin{aligned}
\langle \partial_{k+2}^x \rangle_2 = {} & \langle \partial_{k+1}^x \partial^{\mu_1} \rangle \langle \partial^{\mu_1} \rangle_2 + \langle \partial_k^x \partial^{\mu_1} \rangle \langle \partial_1^{\mu_1} \rangle_2 - \langle \partial_k^x \partial^{\mu_1} \rangle \langle \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_2} \rangle_2 \\
& + \tfrac{7}{10} \langle \partial_k^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \rangle_1 \langle \partial^{\mu_2} \rangle_1 + \tfrac{1}{10} \langle \partial_k^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \partial^{\mu_2} \rangle_1 \\
& - \tfrac{1}{240} \langle \partial_k^x \partial^{\mu_1} \rangle_1 \langle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_2} \rangle + \tfrac{13}{240} \langle \partial_k^x \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_2} \rangle_1 \\
& + \tfrac{1}{960} \langle \partial_k^x \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_2} \rangle.
\end{aligned}$$

Using these relations, our $\mathfrak{r}_1(\Gamma_3)$ becomes

$$\begin{aligned}
\mathfrak{r}_1(\Gamma_3) = {} & 2\langle \partial^i \partial^{\mu_2} \partial^{\mu_3} \rangle \langle \partial^{\mu_3} \partial^j \partial^x \rangle \langle \partial_1^{\mu_2} \rangle_2 + \langle \partial^i \partial^{\mu_1} \partial^{\mu_3} \rangle \langle \partial^{\mu_3} \partial^x \partial^{\mu_1} \rangle \langle \partial_1^j \rangle_2 \\
& + \tfrac{7}{10} \langle \partial^j \partial^x \partial^{\mu_1} \partial^{\mu_1} \rangle \langle \partial^i \partial^{\mu_3} \partial^{\mu_4} \rangle \langle \partial^{\mu_3} \rangle_1 \langle \partial^{\mu_4} \rangle_1 + \tfrac{1}{10} \langle \partial^j \partial^x \partial^{\mu_1} \partial^{\mu_1} \rangle \langle \partial^i \partial^{\mu_3} \partial^{\mu_4} \rangle \langle \partial^{\mu_3} \partial^{\mu_4} \rangle_1 \\
& - \tfrac{1}{240} \langle \partial^j \partial^x \partial^{\mu_1} \partial^{\mu_1} \rangle \langle \partial^{\mu_3} \partial^{\mu_4} \partial^{\mu_4} \rangle \langle \partial^i \partial^{\mu_3} \rangle_1 + \tfrac{13}{240} \langle \partial^j \partial^x \partial^{\mu_1} \partial^{\mu_1} \rangle \langle \partial^i \partial^{\mu_3} \partial^{\mu_3} \partial^{\mu_4} \rangle \langle \partial^{\mu_4} \rangle_1 \\
& + \tfrac{1}{960} \langle \partial^j \partial^x \partial^{\mu_1} \partial^{\mu_1} \rangle \langle \partial^i \partial^{\mu_3} \partial^{\mu_3} \partial^{\mu_4} \partial^{\mu_4} \rangle - \tfrac{1}{2} \langle \partial^x \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^j \partial^{\mu_2} \partial^{\mu_3} \rangle \langle \partial^{\mu_3} \rangle_1 \\
& - \langle \partial^x \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_2} \partial^{\mu_3} \rangle \langle \partial^j \partial^{\mu_3} \rangle_1 - \tfrac{1}{48} \langle \partial^x \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^j \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle \\
& - 2\langle \partial^i \partial^x \partial^{\mu_1} \rangle \langle \partial^j \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial_1^{\mu_2} \rangle_2 - \langle \partial^i \partial^x \partial^{\mu_2} \rangle \langle \partial^j \partial^{\mu_1} \partial^{\mu_1} \rangle \langle \partial_1^{\mu_2} \rangle_2 \\
& - \langle \partial^x \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_2} \partial^{\mu_3} \rangle \langle \partial^{\mu_3} \rangle_1 \langle \partial^j \rangle_1 - \tfrac{1}{24} \langle \partial^x \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle \langle \partial^j \rangle_1.
\end{aligned}$$

A.2 Calculation of $\mathfrak{r}_1(\Gamma_{10})$.

Recall that $\Gamma_{10} = \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle_1$.

The first step in calculating $\mathfrak{r}_1(\Gamma_3)$ is to cut the edges. There are three edges here, μ_1, μ_2, μ_3, and we obtain

$$\langle \partial^x \partial^j \partial^{\mu_2} \rangle \langle \partial_1^i \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle_1 + \langle \partial^x \partial^{\mu_1} \partial^j \rangle \langle \partial^{\mu_1} \partial_1^i \partial^{\mu_3} \partial^{\mu_3} \rangle_1 + 2\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \partial^{\mu_2} \partial_1^i \partial^j \rangle_1.$$

when we cut them (note that $\langle \partial^x \partial^i_1 \partial^{\mu_2} \rangle$ and $\langle \partial^x \partial^{\mu_1} \partial^i_1 \rangle$ are 0 by a simple dimension count). The next step is genus reduction, which can only be applied to the genus 1 term to obtain $-\frac{1}{2}\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^j \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle$. Finally, we need to split the vertices. The genus 0 vertex cannot be split. Indeed, the only way to do so would be to split it into two genus 0 vertices. But since we only have three half-edges to attach to them, one of them would not be stable. If we split the genus 1 vertex, we obtain $-\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} (\langle \partial^i \rangle \langle \partial^j \rangle_1)$.

Putting these together we obtain

$$\begin{aligned} \mathrm{r}_1(\Gamma_{10}) = {} & \langle \partial^x \partial^j \partial^{\mu_2} \rangle \langle \partial^i_1 \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle_1 + \langle \partial^x \partial^{\mu_1} \partial^j \rangle \langle \partial^{\mu_1} \partial^i_1 \partial^{\mu_3} \partial^{\mu_3} \rangle_1 \\ & + 2\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \partial^{\mu_2} \partial^i_1 \partial^j \rangle_1 - \tfrac{1}{2}\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^j \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle \\ & - \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} (\langle \partial^i \rangle \langle \partial^j \rangle_1). \end{aligned}$$

We now leave it as an exercise to the reader to check that, after expanding the last term and simplifying everything using TRRs, one obtains

$$\begin{aligned} \mathrm{r}_1(\Gamma_{10}) = {} & 2\langle \partial^x \partial^j \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \partial^{\mu_4} \rangle \langle \partial^{\mu_4} \rangle_1 + 4\langle \partial^x \partial^j \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_4} \rangle \langle \partial^{\mu_3} \partial^{\mu_4} \rangle_1 \\ & + 2\langle \partial^x \partial^j \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_3} \partial^{\mu_3} \partial^{\mu_4} \rangle \langle \partial^{\mu_2} \partial^{\mu_4} \rangle_1 + 2\langle \partial^x \partial^j \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_2} \partial^{\mu_4} \rangle \langle \partial^{\mu_3} \partial^{\mu_3} \partial^{\mu_4} \rangle_1 \\ & + 4\langle \partial^x \partial^j \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_3} \partial^{\mu_4} \rangle \langle \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_4} \rangle_1 + \tfrac{1}{12}\langle \partial^x \partial^j \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \partial^{\mu_4} \partial^{\mu_4} \rangle \\ & + 2\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_1} \partial^{\mu_2} \partial^j \partial^{\mu_4} \rangle \langle \partial^{\mu_4} \rangle_1 + 2\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_4} \rangle \langle \partial^j \partial^{\mu_4} \rangle_1 \\ & + 4\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_1} \partial^j \partial^{\mu_4} \rangle \langle \partial^{\mu_2} \partial^{\mu_4} \rangle_1 + 4\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_1} \partial^{\mu_4} \rangle \langle \partial^{\mu_2} \partial^j \partial^{\mu_4} \rangle_1 \\ & + 2\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^j \partial^{\mu_4} \rangle \langle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_4} \rangle_1 + \tfrac{1}{12}\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_1} \partial^{\mu_2} \partial^j \partial^{\mu_4} \partial^{\mu_4} \rangle \\ & - \tfrac{1}{2}\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^j \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle - \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^j \partial^{\mu_3} \partial^{\mu_3} \rangle_1 \\ & - 4\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_1} \partial^{\mu_3} \rangle \langle \partial^j \partial^{\mu_2} \partial^{\mu_3} \rangle_1 - \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_3} \partial^{\mu_3} \rangle \langle \partial^j \partial^{\mu_1} \partial^{\mu_2} \rangle_1 \\ & - 2\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \rangle \langle \partial^j \partial^{\mu_3} \rangle_1 - 2\langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_1} \partial^{\mu_3} \partial^{\mu_3} \rangle \langle \partial^j \partial^{\mu_2} \rangle_1 \\ & - \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^i \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle \langle \partial^j \rangle_1. \end{aligned}$$

B A Relation in $\overline{\mathcal{M}}_{2,2}$ in Codimension 3

We use Getzler's and Mumford's relations to derive a new relation for $\overline{\mathcal{M}}_{2,2}$ in codimension 3. Start with Getzler's relation with $k = 1$ and $l = 0$:

$$\begin{aligned} \langle \partial^{x_1}_2 \partial^{x_2}_1 \rangle_2 = {} & \tfrac{13}{10}\langle \partial^{x_1}_1 \partial^{x_2} \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \rangle_1 \langle \partial^{\mu_2} \rangle_1 + \tfrac{4}{5}\langle \partial^{x_1}_1 \partial^{\mu_1} \rangle_1 \langle \partial^{x_2} \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_2} \rangle_1 \\ & + \tfrac{23}{240}\langle \partial^{x_1}_1 \partial^{x_2} \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_2} \rangle_1 + \tfrac{1}{48}\langle \partial^{x_1}_1 \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{x_2} \partial^{\mu_2} \rangle_1 \\ & + \tfrac{1}{48}\langle \partial^{x_1}_1 \partial^{\mu_2} \rangle_1 \langle \partial^{x_2} \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \rangle - \tfrac{1}{80}\langle \partial^{x_1}_1 \partial^{x_2} \partial^{\mu_2} \rangle_1 \langle \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \rangle \\ & + \tfrac{7}{30}\langle \partial^{x_1}_1 \partial^{x_2} \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \partial^{\mu_2} \rangle_1 + \tfrac{1}{30}\langle \partial^{x_1}_1 \partial^{\mu_1} \partial^{\mu_2} \rangle_1 \langle \partial^{x_2} \partial^{\mu_1} \partial^{\mu_2} \rangle \\ & + \tfrac{1}{576}\langle \partial^{x_1}_1 \partial^{x_2} \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_2} \rangle. \end{aligned}$$

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We can then use Mumford's relation to rewrite the left-hand side, and TRRs to simplify the right-hand side. If we set the two sides equal to each other and simplify, we obtain the equation

$$\begin{aligned}0 = &-\tfrac{3}{40}\langle\partial^{x_1}\partial^{\mu_1}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle\partial^{x_2}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_2}\rangle_1 + \tfrac{3}{40}\langle\partial^{x_1}\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_3}\partial^{x_2}\partial^{\mu_1}\rangle\langle\partial^{\mu_2}\rangle_1\\ &-\tfrac{7}{120}\langle\partial^{x_1}\partial^{\mu_1}\partial^{\mu_3}\rangle\langle\partial^{\mu_3}\partial^{x_2}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_2}\rangle_1 + \tfrac{7}{120}\langle\partial^{x_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_3}\partial^{x_2}\partial^{\mu_1}\partial^{\mu_1}\rangle\langle\partial^{\mu_2}\rangle_1\\ &+\tfrac{1}{120}\langle\partial^{x_1}\partial^{\mu_1}\partial^{\mu_3}\rangle\langle\partial^{\mu_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{x_2}\partial^{\mu_2}\rangle_1 - \tfrac{1}{120}\langle\partial^{x_1}\partial^{x_2}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_3}\rangle_1\langle\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\\ &-\tfrac{1}{120}\langle\partial^{x_1}\partial^{x_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle + \tfrac{1}{120}\langle\partial^{x_2}\partial^{\mu_1}\partial^{\mu_3}\rangle\langle\partial^{x_1}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\rangle\\ &+\text{other terms with all vertices of genus 0.}\end{aligned}$$

C Relations in $\overline{\mathcal{M}}_{2,3}$ in Codimension 3

First of all, we use Getzler's relation in $\overline{\mathcal{M}}_{1,4}$ to find relations in $\overline{\mathcal{M}}_{2,3}$ in codimension 3. Since we are only interested in terms with a total of three markings at genus 1 vertices, we start with Getzler's relation written as follows:

$$3\langle\partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle\partial^{x_3}\partial^{x_4}\partial^{\mu_2}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\rangle_1 - 4\langle\partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle\partial^{x_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{x_4}\partial^{\mu_2}\rangle_1 + \text{other terms} = 0.$$

Since in Getzler's relation the four marked points were symmetrized, we need to desymmetrize it first. After we add a fifth marked point, we obtain[1]

$$\begin{aligned}0 = &\langle\partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle\partial^{x_3}\partial^{x_4}\partial^{\mu_2}\rangle\langle\partial^{x_5}\partial^{\mu_1}\partial^{\mu_2}\rangle_1 + \langle\partial^{x_1}\partial^{x_3}\partial^{\mu_1}\rangle\langle\partial^{x_2}\partial^{x_4}\partial^{\mu_2}\rangle\langle\partial^{x_5}\partial^{\mu_1}\partial^{\mu_2}\rangle_1\\ &+\langle\partial^{x_1}\partial^{x_4}\partial^{\mu_1}\rangle\langle\partial^{x_2}\partial^{x_3}\partial^{\mu_2}\rangle\langle\partial^{x_5}\partial^{\mu_1}\partial^{\mu_2}\rangle_1 - \langle\partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle\partial^{x_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{x_4}\partial^{x_5}\partial^{\mu_2}\rangle_1\\ &-\langle\partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle\partial^{x_4}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{x_3}\partial^{x_5}\partial^{\mu_2}\rangle_1 - \langle\partial^{x_1}\partial^{x_3}\partial^{\mu_1}\rangle\langle\partial^{x_4}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{x_2}\partial^{x_5}\partial^{\mu_2}\rangle_1\\ &-\langle\partial^{x_2}\partial^{x_3}\partial^{\mu_1}\rangle\langle\partial^{x_4}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{x_1}\partial^{x_5}\partial^{\mu_2}\rangle_1 + \text{other terms.}\end{aligned}$$

If we glue the fourth and fifth marked points into an edge, we obtain that

$$\begin{aligned}0 = &\langle\partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle\partial^{x_3}\partial^{\mu_3}\partial^{\mu_2}\rangle\langle\partial^{\mu_3}\partial^{\mu_1}\partial^{\mu_2}\rangle_1 + \langle\partial^{x_1}\partial^{x_3}\partial^{\mu_1}\rangle\langle\partial^{x_2}\partial^{\mu_3}\partial^{\mu_2}\rangle\langle\partial^{\mu_3}\partial^{\mu_1}\partial^{\mu_2}\rangle_1\\ &+\langle\partial^{x_1}\partial^{\mu_3}\partial^{\mu_1}\rangle\langle\partial^{x_2}\partial^{x_3}\partial^{\mu_2}\rangle\langle\partial^{\mu_3}\partial^{\mu_1}\partial^{\mu_2}\rangle_1 - \langle\partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle\partial^{x_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{\mu_3}\partial^{\mu_3}\partial^{\mu_2}\rangle_1\\ &-\langle\partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle\partial^{\mu_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{x_3}\partial^{\mu_3}\partial^{\mu_2}\rangle_1 - \langle\partial^{x_1}\partial^{x_3}\partial^{\mu_1}\rangle\langle\partial^{\mu_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{x_2}\partial^{\mu_3}\partial^{\mu_2}\rangle_1\\ &-\langle\partial^{x_2}\partial^{x_3}\partial^{\mu_1}\rangle\langle\partial^{\mu_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{x_1}\partial^{\mu_3}\partial^{\mu_2}\rangle_1 + \text{other terms.}\end{aligned}$$

[1] Throughout this appendix, we label the original edges μ_1 and μ_2, and use μ_3 for any new edge appearing when we simplify. Therefore, the relations found here need to be relabeled before they look like the ones in Section 3.

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If we glue the third and fifth marked points (and rename the fourth marked point x_3), we obtain that

$$0 = \langle \partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle \partial^{\mu_3}\partial^{\mu_3}\partial^{\mu_2}\rangle\langle \partial^{x_3}\partial^{\mu_1}\partial^{\mu_2}\rangle_1 - \langle \partial^{x_1}\partial^{\mu_3}\partial^{\mu_1}\rangle\langle \partial^{\mu_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{x_2}\partial^{x_3}\partial^{\mu_2}\rangle_1 \\ - \langle \partial^{x_2}\partial^{\mu_3}\partial^{\mu_1}\rangle\langle \partial^{\mu_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{x_1}\partial^{x_3}\partial^{\mu_2}\rangle_1 + \text{other terms.}$$

Now we use Belorousski–Pandharipande's, Getzler's and Mumford's relations to derive other relations for $\overline{\mathcal{M}}_{2,3}$ in codimension 3. Start with the Belorousski–Pandharipande relation and add one descendent to the first marked point (note that to do this, we had to first desymmetrize the relation):

$$\begin{aligned}
0 = {} & 12\langle \partial_1^{x_1}\partial^{x_2}\partial^{x_3}\partial^{\mu_1}\rangle\langle \partial_1^{\mu_1}\rangle_2 + 6\langle \partial_2^{x_1}\partial^{\mu_1}\rangle_2\langle \partial^{\mu_1}\partial^{x_2}\partial^{x_3}\rangle \\
& - 6\langle \partial_1^{x_1}\partial_1^{\mu_1}\rangle_2\langle \partial^{\mu_1}\partial^{x_2}\partial^{x_3}\rangle + \tfrac{6}{5}\langle \partial_1^{x_1}\partial^{x_2}\partial^{x_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_1}\rangle_1\langle \partial^{\mu_2}\rangle_1 \\
& - \tfrac{12}{5}\langle \partial_1^{x_1}\partial^{x_2}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_1}\rangle_1\langle \partial^{\mu_2}\partial^{x_3}\rangle_1 - \tfrac{12}{5}\langle \partial_1^{x_1}\partial^{x_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_1}\rangle_1\langle \partial^{\mu_2}\partial^{x_2}\rangle_1 \\
& - \tfrac{12}{5}\langle \partial^{x_2}\partial^{x_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_1}\rangle_1\langle \partial^{\mu_2}\partial_1^{x_1}\rangle_1 + \tfrac{24}{5}\langle \partial^{x_2}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_1}\rangle_1\langle \partial^{\mu_2}\partial_1^{x_1}\partial^{x_3}\rangle_1 \\
& + \tfrac{24}{5}\langle \partial^{x_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_1}\rangle_1\langle \partial^{\mu_2}\partial_1^{x_1}\partial^{x_2}\rangle_1 - \tfrac{36}{5}\langle \partial_1^{x_1}\partial^{\mu_1}\rangle_1\langle \partial^{\mu_1}\partial^{x_2}\partial^{\mu_2}\rangle\langle \partial^{\mu_2}\partial^{x_3}\rangle_1 \\
& - \tfrac{36}{5}\langle \partial_1^{x_1}\partial^{\mu_1}\rangle_1\langle \partial^{\mu_1}\partial^{x_3}\partial^{\mu_2}\rangle\langle \partial^{\mu_2}\partial^{x_2}\rangle_1 - \tfrac{36}{5}\langle \partial_1^{x_1}\partial^{x_2}\partial^{x_3}\partial^{\mu_1}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\rangle_1\langle \partial^{\mu_2}\rangle_1 \\
& + \tfrac{18}{5}\langle \partial_1^{x_1}\partial^{\mu_1}\rangle_1\langle \partial^{\mu_1}\partial^{\mu_2}\rangle_1\langle \partial^{\mu_2}\partial^{x_2}\partial^{x_3}\rangle - \tfrac{12}{5}\langle \partial^{\mu_1}\rangle_1\langle \partial^{\mu_1}\partial_1^{x_1}\partial^{\mu_2}\rangle_1\langle \partial^{\mu_2}\partial^{x_2}\partial^{x_3}\rangle \\
& + \tfrac{1}{20}\langle \partial_1^{x_1}\partial^{x_2}\partial^{x_3}\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_2}\rangle_1 - \tfrac{3}{20}\langle \partial_1^{x_1}\partial^{x_2}\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_2}\partial^{x_3}\rangle_1 \\
& - \tfrac{3}{20}\langle \partial_1^{x_1}\partial^{x_3}\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_2}\partial^{x_2}\rangle_1 - \tfrac{3}{20}\langle \partial^{x_2}\partial^{x_3}\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_2}\partial_1^{x_1}\rangle_1 \\
& + \tfrac{3}{20}\langle \partial_1^{x_1}\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_2}\partial^{x_2}\partial^{x_3}\rangle_1 + \tfrac{3}{20}\langle \partial^{x_2}\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_2}\partial_1^{x_1}\partial^{x_3}\rangle_1 \\
& + \tfrac{3}{20}\langle \partial^{x_3}\partial^{\mu_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_2}\partial_1^{x_1}\partial^{x_2}\rangle_1 - \tfrac{1}{20}\langle \partial_1^{x_1}\partial^{x_2}\partial^{x_3}\partial^{\mu_1}\rangle_1\langle \partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\rangle \\
& + \tfrac{3}{5}\langle \partial_1^{x_1}\partial^{x_2}\partial^{x_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\rangle_1 - \tfrac{3}{5}\langle \partial_1^{x_1}\partial^{x_2}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial^{x_3}\rangle_1 \\
& - \tfrac{3}{5}\langle \partial_1^{x_1}\partial^{x_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial^{x_2}\rangle_1 - \tfrac{3}{5}\langle \partial^{x_2}\partial^{x_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial_1^{x_1}\rangle_1 \\
& + \tfrac{1}{5}\langle \partial^{x_2}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial_1^{x_1}\partial^{x_3}\rangle_1 + \tfrac{1}{5}\langle \partial^{x_3}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial_1^{x_1}\partial^{x_2}\rangle_1 \\
& - \tfrac{3}{10}\langle \partial_1^{x_1}\partial^{x_2}\partial^{x_3}\partial^{\mu_1}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\rangle_1 - \tfrac{1}{10}\langle \partial^{x_2}\partial^{x_3}\partial^{\mu_1}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_2}\partial_1^{x_1}\rangle_1.
\end{aligned}$$

Then use Getzler's and Mumford's relations and TRRs to simplify and obtain

$$\begin{aligned}
0 = {} & \tfrac{1}{10}\langle \partial^{x_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle \partial^{x_2}\partial^{x_3}\partial^{\mu_3}\rangle_1 - \tfrac{1}{5}\langle \partial^{x_2}\partial^{x_3}\partial^{\mu_1}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle \partial^{x_1}\partial^{\mu_2}\partial^{\mu_3}\rangle_1 \\
& - \tfrac{1}{5}\langle \partial^{x_1}\partial^{x_3}\partial^{\mu_1}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle \partial^{x_2}\partial^{\mu_2}\partial^{\mu_3}\rangle_1 - \tfrac{1}{5}\langle \partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle \partial^{x_3}\partial^{\mu_2}\partial^{\mu_3}\rangle_1 \\
& + \tfrac{1}{20}\langle \partial^{x_2}\partial^{x_3}\partial^{\mu_1}\rangle\langle \partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle \partial^{x_1}\partial^{\mu_1}\partial^{\mu_2}\rangle_1 - \tfrac{1}{20}\langle \partial^{x_1}\partial^{x_3}\partial^{\mu_1}\rangle\langle \partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle \partial^{x_2}\partial^{\mu_1}\partial^{\mu_2}\rangle_1 \\
& - \tfrac{1}{20}\langle \partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle \partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle\langle \partial^{x_3}\partial^{\mu_1}\partial^{\mu_2}\rangle_1 - \tfrac{1}{5}\langle \partial^{x_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{x_2}\partial^{x_3}\partial^{\mu_1}\rangle\langle \partial^{\mu_2}\partial^{\mu_3}\partial^{\mu_3}\rangle_1 \\
& + \tfrac{1}{5}\langle \partial^{x_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle \partial^{x_2}\partial^{x_3}\partial^{\mu_3}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle_1 + \tfrac{1}{5}\langle \partial^{x_1}\partial^{x_3}\partial^{\mu_1}\rangle\langle \partial^{x_2}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle_1 \\
& + \tfrac{1}{5}\langle \partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle \partial^{x_3}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle_1 - \tfrac{24}{5}\langle \partial^{x_2}\partial^{x_3}\partial^{\mu_1}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle \partial^{x_1}\partial^{\mu_2}\rangle_1\langle \partial^{\mu_3}\rangle_1 \\
& - \tfrac{24}{5}\langle \partial^{x_1}\partial^{x_3}\partial^{\mu_1}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle \partial^{x_2}\partial^{\mu_2}\rangle_1\langle \partial^{\mu_3}\rangle_1 - \tfrac{24}{5}\langle \partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle \partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle \partial^{x_3}\partial^{\mu_2}\rangle_1\langle \partial^{\mu_3}\rangle_1
\end{aligned}$$

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$$
\begin{aligned}
&-\tfrac{24}{5}\langle\partial^{x_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{x_2}\partial^{x_3}\partial^{\mu_1}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_3}\rangle_1+\tfrac{24}{5}\langle\partial^{x_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{x_2}\partial^{x_3}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_2}\rangle_1\\
&+\tfrac{24}{5}\langle\partial^{x_1}\partial^{x_3}\partial^{\mu_1}\rangle\langle\partial^{x_2}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1+\tfrac{24}{5}\langle\partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle\partial^{x_3}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1\\
&+\text{other terms.}
\end{aligned}
$$

Using the two relations we found above from Getzler's relation in $\overline{\mathcal{M}}_{1,4}$, this simplifies into

$$
\begin{aligned}
0=&-\langle\partial^{x_2}\partial^{x_3}\partial^{\mu_1}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{x_1}\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1-\langle\partial^{x_1}\partial^{x_3}\partial^{\mu_1}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{x_2}\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1\\
&-\langle\partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{x_3}\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1-\langle\partial^{x_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{x_2}\partial^{x_3}\partial^{\mu_1}\rangle\langle\partial^{\mu_2}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_3}\rangle_1\\
&+\langle\partial^{x_1}\partial^{\mu_1}\partial^{\mu_2}\rangle\langle\partial^{x_2}\partial^{x_3}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_3}\rangle_1\langle\partial^{\mu_2}\rangle_1+\langle\partial^{x_1}\partial^{x_3}\partial^{\mu_1}\rangle\langle\partial^{x_2}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1\\
&+\langle\partial^{x_1}\partial^{x_2}\partial^{\mu_1}\rangle\langle\partial^{x_3}\partial^{\mu_2}\partial^{\mu_3}\rangle\langle\partial^{\mu_1}\partial^{\mu_2}\rangle_1\langle\partial^{\mu_3}\rangle_1+\text{other terms.}
\end{aligned}
$$

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