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HIGHER-ORDER EVOLUTION INEQUALITIES INVOLVING CONVECTION AND HARDY-LERAY POTENTIAL TERMS IN A BOUNDED DOMAIN

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Abstract We consider a class of nonlinear higher-order evolution inequalities posed in $(0,\infty)\times B_1\backslash\{0\}$, subject to inhomogeneous Dirichlet-type boundary conditions, where B_1 is the unit ball in $\mathbb{R}^N$. The considered class involves differential operators of the form

$$\mathcal{L}_{\mu_1,\mu_2}=-\Delta+\frac{\mu_1}{|x|^2}x\cdot\nabla+\frac{\mu_2}{|x|^2},\qquad x\in\mathbb{R}^N\backslash\{0\},$$

where $\mu_1\in\mathbb{R}$ and $\mu_2\geq-\left(\frac{\mu_1-N+2}{2}\right)^2$. Optimal criteria for the nonexistence of weak solutions are established. Our study yields naturally optimal nonexistence results for the corresponding class of elliptic inequalities. Notice that no restriction on the sign of solutions is imposed.

Keywords: evolution inequalities; bounded domain; degenerate operator; nonexistence; potential terms

2020 Mathematics subject classification: Primary 35R45; 35B44; 35B33

1. Introduction

For a natural number $N\geq 2$ and $\mu_1,\mu_2\in\mathbb{R}$, we consider differential operators of the form

$$\mathcal{L}_{\mu_1,\mu_2}=-\Delta+\frac{\mu_1}{|x|^2}x\cdot\nabla+\frac{\mu_2}{|x|^2},\qquad x\in\mathbb{R}^N\backslash\{0\},$$

where $\cdot$ denotes the inner product in $\mathbb{R}^N$. The considered operators arise from the critical Caffarelli–Kohn–Nirenberg inequality, see [5, 9, 35] for more details. Notice that for $\mu_i\neq 0$, $i=1,2$, $\mathcal{L}_{\mu_1,\mu_2}$ is degenerate at the origin both for the gradient term and the critical Hardy term.

In this paper, we are concerned with the study of existence and non-existence of weak solutions to evolution inequalities of the form

$$\begin{cases} \partial_t^k u + \mathcal{L}_{\mu_1,\mu_2}(|u|^{q-1}u) \geq |x|^{-a}|u|^p & \text{in } (0,\infty)\times B_1\backslash\{0\}, \\ |u|^{q-1}u(t,x) \geq f(x) & \text{for } (t,x)\in(0,\infty)\times\partial B_1, \end{cases} \tag{1.1}$$

where $k \geq 1$ is a natural number, $\partial_t^k := \frac{\partial^k}{\partial t^k}$, $\mu_1 \in \mathbb{R}$, $\mu_2 \geq -\left(\frac{\mu_1-N+2}{2}\right)^2$, $p > q \geq 1$, $a \in \mathbb{R}$, $B_1 = \{x \in \mathbb{R}^N : |x| \leq 1\}$ and $f \in L^1(\partial B_1)$ is a non-trivial function.

In the special case $\mu_2 = 0$, the operator $\mathcal{L}_{\mu_1,0} = -\Delta + \frac{\mu_1}{|x|^2}x\cdot\nabla$ is a type of degenerate elliptic operator, which together with its divergence form plays an important role in the harmonic analysis, see for example [29]. Some studies related to regularities and qualitative properties for elliptic equations with more general degenerate operators in divergence form can be found in [13, 31]. The study of existence and non-existence of solutions to evolution equations and inequalities involving operators of the form $\mathcal{L}_{\mu,1^0}$ has been considered in infinite domains of $\mathbb{R}^N$, see for example [18, 19, 30, 37, 38] and the references therein. For instance, in [37], the authors investigated parabolic equations of the form

$$\begin{cases} |x|^{\lambda_1}\partial_t u + \mathcal{L}_{\mu_1,0}u^m = |x|^{\lambda_2}u^p, \quad u \geq 0 & \text{in } (0,\infty)\times\mathbb{R}^N\backslash\overline{\Omega}, \\ u(t,\cdot) = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}^N\backslash\overline{\Omega}, \end{cases} \tag{1.2}$$

where Ω is a regular bounded domain in $\mathbb{R}^N$ containing the origin, $m \geq 1$, $p > m$, $-2 < \lambda_1 \leq \lambda_2$ and $\mu_1 < N-2$. It was shown that Equation (1.2) admits as Fujita critical exponent the real number

$$p_c = m + \frac{\lambda_2 + 2}{\lambda_1 + N - \mu_1}.$$

More precisely, it was proven that

(i) if $0 \leq u_0 \in C_0\left(\mathbb{R}^N\backslash\Omega\right)$, $u_0|_{\partial\Omega} = 0$, $u_0 \not\equiv 0$ and $m < p < p_c$, then any solution to Equation (1.2) blows up in a finite time;
(ii) if $p = p_c$, then any nontrivial solution to Equation (1.2) blows up in a finite time;
(iii) if $p > p_c$, then Equation (1.2) admits non-trivial global solutions for some small initial value u_0.

Observe that in the special case $\lambda_1 = \lambda_2 = \mu_1 = 0$, $m=1$ and $N \geq 3$, one has $p_c = 1 + \frac{2}{N}$, which is the Fujita critical exponent for the semilinear heat equation

$$\partial_t u - \Delta u = u^p \quad \text{in } (0,\infty)\times\mathbb{R}^N.$$

For more references related to the study of evolution equations and inequalities in exterior domains, see for example [20, 21, 23, 32, 36].

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When $\mu_1 = 0$, $\mathcal{L}_{0,\mu_2} = -\Delta + \frac{\mu_2}{|x|^2}$ is the Hardy–Leray operator. Elliptic equations involving such operators have been investigated extensively in the last decades, for instance, the analysis of isolated singular solutions [6–8, 11, 17], existence and non-existence of solutions [14, 15, 24, 25, 34] and qualitative properties of solutions [10, 22, 26, 28]. The study of existence and non-existence of solutions to evolution equations and inequalities involving Hardy–Leray potential in infinite domains has been considered in several papers. In [12], Hamidi and Laptev investigated the nonexistence of weak solutions to higher-order evolution inequalities of the form

$$\begin{cases} \partial_t^k u + \mathcal{L}_{0,\mu_2} u \geq |u|^p \quad \text{in} \ \ (0,\infty)\times\mathbb{R}^N, \\ \partial_t^{k-1} u(0,\cdot) \geq 0 \quad \text{in} \ \ \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $N \geq 3$, $\mu_2 \geq -\left(\frac{N-2}{2}\right)^2$ and $p > 1$. Namely, it was shown that, if either

$$\mu_2 \geq 0, \qquad 1 < p \leq 1 + \frac{2}{\frac{2}{k} + s^*};$$

or

$$-\left(\frac{N-2}{2}\right)^2 \leq \mu_2 < 0, \qquad 1 < p \leq 1 + \frac{2}{\frac{2}{k} - s_*},$$

where

$$s^* = \frac{N-2}{2} + \sqrt{\mu_2 + \left(\frac{N-2}{2}\right)^2}, \qquad s_* = s^* + 2 - N,$$

then Equation (1.3) admits no non-trivial weak solution. In [20], the authors considered hyperbolic inequalities of the form

$$\begin{cases} \partial_t^2 u + \mathcal{L}_{0,\mu_2} u \geq |u|^p & \text{in} \ \ (0,\infty)\times\mathbb{R}^N\backslash B_1, \\ \alpha \dfrac{\partial u}{\partial \nu}(t,x) + \beta u(t,x) \geq f(x) & \text{for} \ \ (t,x) \in (0,\infty)\times\partial B_1, \end{cases} \tag{1.4}$$

where $N \geq 2$, $\mu_2 \geq -((N-2)/2)^2$, $\alpha, \beta \geq 0$, $(\alpha,\beta) \neq (0,0)$ and ν is the outward unit normal vector on ∂B_1, relative to $\Omega = \mathbb{R}^N\backslash B_1$. It was shown that Equation (1.4) admits a Fujita critical exponent

$$p_c(\mu_2, N) = \begin{cases} \infty & \text{if} \ \ N - 2 + 2\mu_N = 0, \\ 1 + \dfrac{4}{N-2+2\mu_N} & \text{if} \ \ N - 2 + 2\mu_N > 0, \end{cases}$$

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where

$$\mu_N = \sqrt{\mu_2 + \left(\frac{N-2}{2}\right)^2}.$$

More precisely, it was proven that

(i) if $1 < p < p_c(\mu_2, N)$ and $\int_{\partial B_1} f(x)\,\mathrm{d}\sigma > 0$, then Equation (1.4) admits no global weak solution;
(ii) if $p > p_c(\mu_2, N)$, then Equation (1.4) admits global solutions for some $f > 0$.

In the case of bounded domains, some results related to parabolic equations have been obtained. For instance, Abdellaoui et al. [1] considered parabolic equations of the form

$$\begin{cases} \partial_t u + \mathcal{L}_{0,\mu_2} u = u^p + f, \quad u \geq 0 & \text{in } (0,\infty)\times\Omega, \\ u(t,\cdot) = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded regular domain containing the origin, $p > 1$, $\mu_2 < 0$, and $u_0, f \geq 0$ belong to a suitable class of functions. Namely, the existence of a critical exponent $p_+(\mu_2)$ was shown such that for $p \geq p_+(\mu_2)$, there is no distributional solution to Equation (1.5), while for $p < p_+(\mu_2)$ and under some additional conditions on the data, Equation (1.5) admits solutions. Notice that in [1], the positivity of u is essential in the proof of the obtained results. Moreover, in this reference, the authors used the comparison principle for the heat equation, which cannot be applied for our problem (1.1) in the case $k \geq 2$. For other contributions related to the study of parabolic equations with Hardy–Leray potential in bounded domains, see for example [2–4, 16, 33] and the references therein. To the best of our knowledge, the study of sign-changing solutions to evolution equations or inequalities involving Hardy–Leray potential in bounded domains has not been previously considered in the literature.

Very recently, the authors [9] studied some basic properties of the operator $\mathcal{L}_{\mu_1,\mu_2}$. Namely, they analyzed the fundamental solutions in a weighted distributional identity and derived a Liouville-type result for positive solutions to the elliptic inequality

$$\begin{cases} \mathcal{L}_{\mu_1,\mu_2} u \geq V u^p & \text{in } \Omega\backslash\{0\}, \\ u \geq 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $p > 1$, Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 3$) containing the origin, $V > 0$, $V \in C^{\beta}_{\mathrm{loc}}(\mathbb{R}^N\backslash\{0\})$, $0 < \beta < 1$ and

$$\liminf_{|x|\to 0^+} V(x)|x|^{-\rho} > 0$$

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for some $\rho > -2$. It was proven that if

$$\mu_1 < N-2, \qquad -\left(\frac{2-N+\mu_1}{2}\right)^2 \le \mu_2 < 0,$$

then for all $p \ge p^*(\mu_1, \mu_2, \rho)$, Equation (1.3) admits no positive solution, where

$$p^*(\mu_1,\mu_2,\rho) = 1 + \frac{2+\rho}{\tau_+(\mu_1,\mu_2)} \tag{1.7}$$

and

$$\tau_+(\mu_1,\mu_2) = \frac{N-2-\mu_1}{2} - \sqrt{\mu_2 + \left(\frac{2-N+\mu_1}{2}\right)^2}.$$

The used approach in [9] is based on the classification of isolated singular solutions to the related Poisson problem.

Motivated by the above-mentioned contributions, problem (1.1) is investigated in this paper. Notice that no restriction on the sign of solutions is imposed. Moreover, our obtained results yield naturally existence and non-existence results for the corresponding stationary problem.

It is interesting to observe that in the special case $q = 1$, making use of the change of variable

$$u(t,x) = v(t,x)|x|^{\frac{\mu_1}{2}}, \quad t > 0,\ x \in B_1\backslash\{0\},$$

problem (1.1) reduces to

$$\begin{cases} \partial_t^k v + \mathcal{L}_{0,\mu} v \ge |x|^{\frac{\mu_1(p-1)-2a}{2}} |v|^p & \text{in } (0,\infty)\times B_1\backslash\{0\}, \\ v(t,x) \ge f(x) & \text{for } (t,x)\in(0,\infty)\times\partial B_1, \end{cases}$$

where

$$\mu = \frac{1}{4}\mu_1^2 - \frac{N-2}{2}\mu_1 + \mu_2.$$

Before stating our obtained results, we need to define weak solutions to the considered problem. Let

$$Q = (0,\infty)\times B_1\backslash\{0\} \quad \text{and} \quad \Gamma = (0,\infty)\times\partial B_1.$$

Notice that $\Gamma \subset \partial Q$. We introduce the functional space Φ defined as follows.

Definition 1.1. *We say that $\varphi = \varphi(t,x)$ belongs to Φ if the following conditions are satisfied:*

(i) $\varphi \in C^{k,2}_{t,x}(Q), \quad \varphi \geq 0$;
(ii) $\mathrm{supp}(\varphi) \subset\subset Q$;
(iii) $\varphi|_\Gamma = 0$, $\left.\dfrac{\partial \varphi}{\partial \nu}\right|_\Gamma \leq 0$, *where* ν *denotes the outward unit normal vector on* ∂B_1.

Using standard integration by parts, we define weak solutions to Equation (1.1) as follows.

Definition 1.2. Weak solutions *We say that* $u \in L^p_{\mathrm{loc}}(Q)$ *is a weak solution to Equation (1.1), if for all* $\varphi \in \Phi$, *there holds*

$$\int_Q |x|^{-a}|u|^p \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_\Gamma \frac{\partial \varphi}{\partial \nu} f \,\mathrm{d}\sigma\,\mathrm{d}t \leq (-1)^k \int_Q u \partial_t^k \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_Q |u|^{q-1} u \,\mathcal{L}^*_{\mu_1,\mu_2} \varphi \,\mathrm{d}x\,\mathrm{d}t, \tag{1.8}$$

where $\mathcal{L}^*_{\mu_1,\mu_2}$ *is the adjoint operator of* $\mathcal{L}_{\mu_1,\mu_2}$, *given by*

$$\mathcal{L}^*_{\mu_1,\mu_2}\varphi = -\Delta\varphi - \mu_1 \,\mathrm{div}\left(\frac{\varphi x}{|x|^2}\right) + \frac{\mu_2}{|x|^2}\varphi. \tag{1.9}$$

For $\mu_1 \in \mathbb{R}$ and $\mu_2 \geq -\left(\frac{\mu_1 - N + 2}{2}\right)^2$, we introduce the parameter α given by

$$\alpha = \frac{2 - N - \mu_1}{2} + \sqrt{\mu_2 + \left(\frac{\mu_1 - N + 2}{2}\right)^2}. \tag{1.10}$$

For $f \in L^1(\partial B_1)$, let

$$I_f = \int_{\partial B_1} f(x) \,\mathrm{d}\sigma.$$

We denote by $L^{1,+}(\partial B_1)$ the functional space defined by

$$L^{1,+}(\partial B_1) = \left\{ f \in L^1(\partial B_1) : I_f > 0 \right\}.$$

Our main result is stated in the following theorem.

Theorem 1.3. *Let* $k \geq 1$, $N \geq 2$, $p > q \geq 1$, $\mu_1 \in \mathbb{R}$ *and* $\mu_2 \geq -\left(\frac{\mu_1 - N + 2}{2}\right)^2$.

(I) *If* $f \in L^{1,+}(\partial B_1)$ *and*

$$(\mu_1 + \alpha)p < (a - 2 + \mu_1 + \alpha)q, \tag{1.11}$$

then Equation (1.1) admits no weak solution.

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(II) If

$$(\mu_1 + \alpha)p > (a - 2 + \mu_1 + \alpha)q, \tag{1.12}$$

then Equation (1.1) admits positive solutions (stationary solutions) for some $f \in L^{1,+}(\partial B_1)$.

The proof of part (I) of Theorem 1.3 relies on nonlinear capacity estimates specifically adapted to the operator $\mathcal{L}_{\mu_1,\mu_2}$, the boundedness of the domain and the considered boundary condition. Namely, we use the nonlinear capacity method (see, e.g., [27] for more details about this approach) with a judicious choice of a family of test functions belonging to Φ and involving a function $H \geq 0$, solution to

$$-\Delta H - \mu_1 \operatorname{div}\left(\frac{Hx}{|x|^2}\right) + \frac{\mu_2}{|x|^2} H = 0 \text{ in } B_1 \backslash \{0\}, \quad H = 0 \text{ on } \partial B_1.$$

The existence result given by part (II) of Theorem 1.3 is established by the construction of explicit solutions.

Remark 1.4. Theorem 1.3 leaves open the issue of existence and non-existence in the critical case:

$$(\mu_1 + \alpha)p = (a - 2 + \mu_1 + \alpha)q.$$

Remark 1.5. Consider the case $\mu_2 > -\left(\frac{\mu_1 - N + 2}{2}\right)^2$.

(i) Let $\mu_1 \geq N - 2$. In this case, one has $\mu_1 + \alpha > 0$. Hence, Equation (1.11) reduces to

$$a > 2, \qquad 1 \leq q < p < q\left(1 + \frac{a-2}{\mu_1 + \alpha}\right). \tag{1.13}$$

(ii) Let $\mu_1 < N - 2$.
- If $\mu_2 = 0$, then $\mu_1 + \alpha = 0$ and Equation (1.11) reduces to $a > 2$.
- If $\mu_2 > 0$, then $\mu_1 + \alpha > 0$ and Equation (1.11) reduces to Equation (1.13).
- If $-\left(\frac{\mu_1 - N + 2}{2}\right)^2 < \mu_2 < 0$, then $\mu_1 + \alpha < 0$ and Equation (1.11) reduces to

$$a \geq 2; \quad \text{or} \quad a < 2, \quad p > q\left(1 + \frac{a-2}{\mu_1 + \alpha}\right). \tag{1.14}$$

Remark 1.6. Consider now the case $\mu_2 = -\left(\frac{\mu_1 - N + 2}{2}\right)^2$.

(i) Let $\mu_1 = N - 2$. In this case, one has $\mu_1 + \alpha = 0$. Hence, Equation (1.11) reduces to $a > 2$.

(ii) Let $\mu_1 > N-2$. In this case, we have $\mu_1+\alpha > 0$ and Equation (1.11) reduces to Equation (1.13).
(iii) Let $\mu_1 < N-2$. In this case, we get $\mu_1+\alpha < 0$ and Equation (1.11) reduces to Equation (1.14).

Clearly, Theorem 1.3 yields naturally existence and non-existence results for the corresponding stationary problem

$$\begin{cases} \mathcal{L}_{\mu_1,\mu_2}\left(|u|^{q-1}u\right)(x) \geq |x|^{-a}|u(x)|^p & \text{in } B_1\backslash\{0\}, \\ |u|^{q-1}u \geq f & \text{on } \partial B_1. \end{cases} \tag{1.15}$$

Corollary 1.7. *Let $N \geq 2$, $p > q \geq 1$, $\mu_1 \in \mathbb{R}$ and $\mu_2 \geq -\left(\frac{\mu_1-N+2}{2}\right)^2$.*

(I) If $f \in L^{1,+}(\partial B_1)$ and Equation (1.11) holds, then Equation (1.15) admits no weak solution.
(II) If Equation (1.12) holds, then Equation (1.15) admits positive solutions for some $f \in L^{1,+}(\partial B_1)$.

Remark 1.8. Notice that in the special case

$$a < 2, \quad q = 1, \quad \mu_1 < N-2, \quad -\left(\frac{\mu_1-N+2}{2}\right)^2 \leq \mu_2 < 0,$$

condition (1.11) reduces to (see Remark 1.5) that obtained in [9] (for the non-existence of positive solutions to Equation (1.6))

$$p > 1 + \frac{a-2}{\mu_1+\alpha} = p^*(\mu_1,\mu_2,\rho),$$

where $\rho = -a$ and $p^*(\mu_1,\mu_2,\rho)$ is given by Equation (1.7).

The rest of the paper is organized as follows. In § 2, we establish some preliminary results that will be useful in the proof of Theorem 1.3. Namely, we first establish an a priori estimate for problem (1.1). Next, we introduce a certain class of test functions belonging to Φ and specifically adapted to our problem and prove some useful estimates involving such functions. Finally, the proof of Theorem 1.3 is given in § 3.

Throughout this paper, the symbol C denotes always a generic positive constant, which is independent of the scaling parameters T, R and the solution u. Its value could be changed from one line to another.

2. Preliminary estimates

Let $k \geq 1$, $N \geq 2$, $a \in \mathbb{R}$, $p > q \geq 1$, $\mu_1 \in \mathbb{R}$ and $\mu_2 \geq -\left(\frac{\mu_1 - N+2}{2}\right)^2$. For $\varphi \in \Phi$, let

$$J_1(\varphi) = \int_{\operatorname{supp}(\varphi)} \varphi^{\frac{-1}{p-1}} \left|\partial_t^k \varphi\right|^{\frac{p}{p-1}} |x|^{\frac{a}{p-1}} \,\mathrm{d}x\,\mathrm{d}t, \tag{2.1}$$

$$J_2(\varphi) = \int_{\operatorname{supp}(\varphi)} \varphi^{\frac{-q}{p-q}} \left|\mathcal{L}^*_{\mu_1,\mu_2}\varphi\right|^{\frac{p}{p-q}} |x|^{\frac{aq}{p-q}} \,\mathrm{d}x\,\mathrm{d}t, \tag{2.2}$$

where $\mathcal{L}^*_{\mu_1,\mu_2}$ is the differential operator defined by Equation (1.9).

2.1. (A priori estimate).

We have the following a priori estimate.

Lemma 2.1. A priori estimate *Let $u \in L^p_{\mathrm{loc}}(Q)$ be a weak solution to Equation (1.1). Then, there holds*

$$-\int_\Gamma f(x)\,\frac{\partial \varphi}{\partial \nu}\,\mathrm{d}\sigma\,\mathrm{d}t \leq C \sum_{i=1}^{2} J_i(\varphi) \tag{2.3}$$

for all $\varphi \in \Phi$, provided that $J_i(\varphi) < \infty$, $i = 1, 2$.

Proof. Let $u \in L^p_{\mathrm{loc}}(Q)$ be a weak solution to Equation (1.1). By Equation (1.8), there holds

$$\int_Q |x|^{-a}|u|^p\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_\Gamma f(x)\,\frac{\partial \varphi}{\partial \nu}\,\mathrm{d}\sigma\,\mathrm{d}t \leq \int_Q |u| \left|\partial_t^k \varphi\right| \,\mathrm{d}x\,\mathrm{d}t + \int_Q |u|^q \left|\mathcal{L}^*_{\mu_1,\mu_2}\varphi\right| \,\mathrm{d}x\,\mathrm{d}t \tag{2.4}$$

for all $\varphi \in \Phi$. On the other hand, by means of Young's inequality, we obtain

$$\begin{aligned}\int_Q |u| \left|\partial_t^k \varphi\right| \,\mathrm{d}x\,\mathrm{d}t &= \int_{\operatorname{supp}(\varphi)} \left(|x|^{\frac{-a}{p}}|u|\varphi^{\frac{1}{p}}\right)\left(\varphi^{\frac{-1}{p}}\left|\partial_t^k\varphi\right||x|^{\frac{a}{p}}\right)\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{1}{2}\int_Q |x|^{-a}|u|^p\varphi\,\mathrm{d}x\,\mathrm{d}t + CJ_1(\varphi).\end{aligned} \tag{2.5}$$

Similarly, we get

$$\begin{aligned}\int_Q |u|^q \left|\mathcal{L}^*_{\mu_1,\mu_2}\varphi\right| \,\mathrm{d}x\,\mathrm{d}t &= \int_Q \left(|x|^{\frac{-aq}{p}}|u|^q\varphi^{\frac{q}{p}}\right)\left(|x|^{\frac{aq}{p}}\varphi^{\frac{-q}{p}}\left|\mathcal{L}^*_{\mu_1,\mu_2}\varphi\right|\right)\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{1}{2}\int_Q |x|^{-a}|u|^p\varphi\,\mathrm{d}x\,\mathrm{d}t + CJ_2(\varphi).\end{aligned} \tag{2.6}$$

Thus, Equation (2.3) follows from Equations (2.4), (2.5) and (2.6). □

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2.2. Test functions

Let us introduce the function H defined in $B_1\backslash\{0\}$ by

$$H(x) = \begin{cases} |x|^{2-N-\mu_1-\alpha}\left(1-|x|^{2\alpha-2+N+\mu_1}\right) & \text{if } \mu_2 > -\left(\frac{\mu_1-N+2}{2}\right)^2, \\ -|x|^{\alpha}\ln|x| & \text{if } \mu_2 = -\left(\frac{\mu_1-N+2}{2}\right)^2, \end{cases} \tag{2.7}$$

where the parameter α is given by Equation (1.10). It can be easily seen that $H \geq 0$ in $B_1\backslash\{0\}$. Moreover, elementary calculations show that

$$\begin{cases} \mathcal{L}^*_{\mu_1,\mu_2}H(x) = 0 & \text{in } B_1\backslash\{0\}, \\ H(x) = 0 & \text{on } \partial B_1. \end{cases} \tag{2.8}$$

Let $\eta, \xi \in C^\infty([0,\infty))$ be two cutoff functions satisfying, respectively,

$$\eta \geq 0, \quad \operatorname{supp}(\eta) \subset\subset (0,1) \tag{2.9}$$

and

$$0 \leq \xi \leq 1, \quad \xi(s) = 0 \quad \text{if } 0 \leq s \leq \frac{1}{2}, \quad \xi(s) = 1 \quad \text{if } s \geq 1. \tag{2.10}$$

For sufficiently large T, R and ℓ, let

$$\eta_T(t) = \eta\left(\frac{t}{T}\right)^\ell, \quad t \geq 0 \tag{2.11}$$

and

$$\xi_R(x) = H(x)\xi(R|x|)^\ell, \quad x \in B_1\backslash\{0\}, \tag{2.12}$$

that is,

$$\xi_R(x) = \begin{cases} 0 & \text{if } 0 < |x| \leq (2R)^{-1}, \\ H(x)\xi(R|x|)^\ell & \text{if } (2R)^{-1} \leq |x| \leq R^{-1}, \\ H(x) & \text{if } R^{-1} \leq |x| \leq 1. \end{cases} \tag{2.13}$$

We introduce test functions of the form

$$\varphi(t,x) = \eta_T(t)\xi_R(x), \quad (t,x) \in Q. \tag{2.14}$$

Lemma 2.2. *For sufficiently large T, R and ℓ, the function φ defined by Equation (2.14) belongs to Φ.*

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Proof. By Equations (2.7), (2.9), (2.10), (2.11) and (2.12), it can be easily seen that for sufficiently large T, R and ℓ, the function φ defined by Equation (2.14) satisfies properties (i) and (ii) of Definition 1.1. Moreover, $H|_{\partial B_1} = 0$ implies that $\varphi|_\Gamma = 0$. So, we have just to show that

$$\frac{\partial \varphi}{\partial \nu}(t,x) \leq 0, \quad (t,x) \in \Gamma. \tag{2.15}$$

In view of Equations (2.13) and (2.14) , we obtain

$$\frac{\partial \varphi}{\partial \nu}(t,x) = \eta_T(t)\,\frac{\partial \xi_R}{\partial \nu}(x) = \eta_T(t)\,\frac{\partial H}{\partial \nu}(x), \quad (t,x) \in \Gamma. \tag{2.16}$$

On the other hand, by Equation (2.7), for all $x \in B_1\backslash\{0\}$, if $\mu_2 > -\left(\frac{\mu_1-N+2}{2}\right)^2$, we have

$$\nabla H(x) = \left((2-N-\mu_1-\alpha)|x|^{1-N-\mu_1-\alpha} - \alpha|x|^{\alpha-1}\right)\frac{x}{|x|};$$

if $\mu_2 = -\left(\frac{\mu_1-N+2}{2}\right)^2$, we have

$$\nabla H(x) = \left(-\alpha|x|^{\alpha-1}\,\ln|x| - |x|^{\alpha-1}\right)\frac{x}{|x|}.$$

Hence, by Equation (1.10), we get

$$\frac{\partial H}{\partial \nu}(x) = \begin{cases} -\left(2\alpha-2+N+\mu_1\right) < 0 & \text{if } \ \mu_2 > -\left(\frac{\mu_1-N+2}{2}\right)^2, \\ -1 & \text{if } \ \mu_2 = -\left(\frac{\mu_1-N+2}{2}\right)^2. \end{cases} \tag{2.17}$$

Using Equations (2.9), (2.11), (2.16) and (2.17), for all $(t,x) \in \Gamma$, we obtain

$$\frac{\partial \varphi}{\partial \nu}(t,x) = \begin{cases} -\left(2\alpha-2+N+\mu_1\right)\eta_T(t) \leq 0 & \text{if } \ \mu_2 > -\left(\frac{\mu_1-N+2}{2}\right)^2, \\ -\eta_T(t) \leq 0 & \text{if } \ \mu_2 = -\left(\frac{\mu_1-N+2}{2}\right)^2, \end{cases} \tag{2.18}$$

which proves Equation (2.15). □

2.3. Estimates of $J_i(\varphi)$

The aim of this subsection is to estimate the terms $J_1(\varphi)$ and $J_2(\varphi)$ defined, respectively, by Equations (2.1) and (2.2), where φ is the function defined by Equation (2.14). Such estimates will play a crucial role in the proof of our non-existence results.

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Lemma 2.3. *For sufficiently large T and ℓ, there holds*

$$\int_{\mathrm{supp}(\eta_T)} \eta_T(t)^{\frac{-1}{p-1}} \left|\eta_T^{(k)}(t)\right|^{\frac{p}{p-1}} \,\mathrm{d}t \le C T^{1-\frac{kp}{p-1}}, \tag{2.19}$$

where $\eta_T^{(k)} = \dfrac{\mathrm{d}^k \eta}{\mathrm{d}t^k}$.

Proof. In view of Equations (2.9) and (2.11), we obtain

$$\int_{\mathrm{supp}(\eta_T)} \eta_T(t)^{\frac{-1}{p-1}} \left|\eta_T^{(k)}(t)\right|^{\frac{p}{p-1}} \,\mathrm{d}t = \int_0^T \eta\left(\frac{t}{T}\right)^{\frac{-\ell}{p-1}} \left|\left[\eta\left(\frac{t}{T}\right)^{\ell}\right]^{(k)}\right|^{\frac{p}{p-1}} \,\mathrm{d}t$$

and

$$\left|\left[\eta\left(\frac{t}{T}\right)^{\ell}\right]^{(k)}\right| \le C T^{-k} \eta\left(\frac{t}{T}\right)^{\ell-k}, \quad 0 < t < T.$$

Hence, there holds

$$\begin{aligned} \int_{\mathrm{supp}(\eta_T)} \eta_T(t)^{\frac{-1}{p-1}} \left|\eta_T^{(k)}(t)\right|^{\frac{p}{p-1}} \,\mathrm{d}t &\le C T^{\frac{-kp}{p-1}} \int_0^T \eta\left(\frac{t}{T}\right)^{\ell-\frac{kp}{p-1}} \,\mathrm{d}t \\ &= C T^{1-\frac{kp}{p-1}} \int_0^1 \eta(s)^{\ell-\frac{k}{p-1}} \,\mathrm{d}s, \end{aligned}$$

which proves Equation (2.19). □

Lemma 2.4.

(i) *Let $\mu_2 > -\left(\frac{\mu_1-N+2}{2}\right)^2$. For sufficiently large R, there holds*

$$\int_{\mathrm{supp}(\xi_R)} \xi_R(x)|x|^{\frac{a}{p-1}} \,\mathrm{d}x \le C\left(\ln R + R^{\alpha+\mu_1-2-\frac{a}{p-1}}\right). \tag{2.20}$$

(ii) *Let $\mu_2 = -\left(\frac{\mu_1-N+2}{2}\right)^2$. For sufficiently large R, there holds*

$$\int_{\mathrm{supp}(\xi_R)} \xi_R(x)|x|^{\frac{a}{p-1}} \,\mathrm{d}x \le C \ln R\left(\ln R + R^{-\left(\frac{a}{p-1}+\alpha+N\right)}\right). \tag{2.21}$$

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Proof.

(i) In view of Equations (2.7), (2.10) and (2.12), for sufficiently large R, we obtain

$$\begin{aligned}\int_{\mathrm{supp}(\xi_R)} \xi_R(x)|x|^{\frac{a}{p-1}}\,\mathrm{d}x &= \int_{\frac{1}{2R}<|x|<1} |x|^{\frac{a}{p-1}} H(x)\xi(R|x|)^{\ell}\,\mathrm{d}x \\ &\leq \int_{\frac{1}{2R}<|x|<1} |x|^{\frac{a}{p-1}} H(x)\,\mathrm{d}x \\ &\leq \int_{\frac{1}{2R}<|x|<1} |x|^{2-N-\mu_1-\alpha+\frac{a}{p-1}}\,\mathrm{d}x \\ &= C\int_{r=\frac{1}{2R}}^{1} r^{1-\mu_1-\alpha+\frac{a}{p-1}}\,\mathrm{d}r \\ &\leq \begin{cases} C\ \ln R & \text{if } (2-\mu_1-\alpha)(p-1)+a=0, \\ C & \text{if } (2-\mu_1-\alpha)(p-1)+a>0, \\ CR^{\mu_1+\alpha-2-\frac{a}{p-1}} & \text{if } (2-\mu_1-\alpha)(p-1)+a<0, \end{cases}\end{aligned}$$

which proves Equation (2.20).

(ii) Similarly, using Equations (2.7), (2.10) and (2.12), for sufficiently large R, we obtain

$$\begin{aligned}\int_{\mathrm{supp}(\xi_R)} \xi_R(x)|x|^{\frac{a}{p-1}}\,\mathrm{d}x &= \int_{\frac{1}{2R}<|x|<1} |x|^{\frac{a}{p-1}} H(x)\xi(R|x|)^{\ell}\,\mathrm{d}x \\ &\leq \int_{\frac{1}{2R}<|x|<1} |x|^{\frac{a}{p-1}} H(x)\,\mathrm{d}x \\ &= \int_{\frac{1}{2R}<|x|<1} |x|^{\frac{a}{p-1}+\alpha}\ \ln\left(\frac{1}{|x|}\right)\,\mathrm{d}x \\ &\leq C\ \ln R\int_{r=\frac{1}{2R}}^{1} r^{\frac{a}{p-1}+\alpha+N-1}\,\mathrm{d}r \\ &\leq \ln R\begin{cases} C\ \ln R & \text{if } (\alpha+N)(p-1)+a=0, \\ C & \text{if } (\alpha+N)(p-1)+a>0, \\ CR^{-\left(\frac{a}{p-1}+\alpha+N\right)} & \text{if } (\alpha+N)(p-1)+a<0, \end{cases}\end{aligned}$$

which proves Equation (2.21). □

Lemma 2.5.

(i) Let $\mu_2 > -\left(\frac{\mu_1-N+2}{2}\right)^2$. For sufficiently large T, R and ℓ, there holds

$$J_1(\varphi) \leq CT^{1-\frac{kp}{p-1}}\left(\ln R + R^{\alpha+\mu_1-2-\frac{a}{p-1}}\right), \tag{2.22}$$

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where φ is the function defined by Equation (2.14).

(ii) Let $\mu_2 = -\left(\frac{\mu_1-N+2}{2}\right)^2$. For sufficiently large T, R and ℓ, there holds

$$J_1(\varphi) \leq CT^{1-\frac{kp}{p-1}} \ln R \left(\ln R + R^{-\left(\frac{a}{p-1}+\alpha+N\right)} \right). \tag{2.23}$$

Proof. By Equations (2.1) and (2.14), we obtain

$$J_1(\varphi) = \left(\int_{\mathrm{supp}(\eta_T)} \eta_T(t)^{\frac{-1}{p-1}} \left| \eta_T^{(k)}(t) \right|^{\frac{p}{p-1}} \mathrm{d}t \right) \left(\int_{\mathrm{supp}(\xi_R)} \xi_R(x) |x|^{\frac{a}{p-1}} \mathrm{d}x \right). \tag{2.24}$$

Using Lemma 2.3, Lemma 2.4 and Equation (2.24), we obtain Equations (2.22) and (2.23). □

Lemma 2.6.

(i) Let $\mu_2 > -\left(\frac{\mu_1-N+2}{2}\right)^2$. For sufficiently large R and ℓ, there holds

$$\int_{\mathrm{supp}(\xi_R)} \xi_R^{\frac{-q}{p-q}} \left| \mathcal{L}^*_{\mu_1,\mu_2} \xi_R \right|^{\frac{p}{p-q}} |x|^{\frac{aq}{p-q}} \mathrm{d}x \leq CR^{\frac{(\mu_1+\alpha)p-(a-2+\mu_1+\alpha)q}{p-q}}. \tag{2.25}$$

(ii) Let $\mu_2 = -\left(\frac{\mu_1-N+2}{2}\right)^2$. For sufficiently large R and ℓ, there holds

$$\int_{\mathrm{supp}(\xi_R)} \xi_R^{\frac{-q}{p-q}} \left| \mathcal{L}^*_{\mu_1,\mu_2} \xi_R \right|^{\frac{p}{p-q}} |x|^{\frac{aq}{p-q}} \mathrm{d}x \leq CR^{\frac{(2-\alpha-N)p+(\alpha-a+N)q}{p-q}} \ln R. \tag{2.26}$$

Proof.

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(i) By Equations (1.9) and (2.12), for $x \in B_1\backslash\{0\}$, we obtain

$$\begin{aligned}
\mathcal{L}^*_{\mu_1,\mu_2}\xi_R(x) &= \mathcal{L}^*_{\mu_1,\mu_2}(H(x)\xi(R|x|)^\ell) \\
&= -\Delta(H(x)\xi(R|x|)^\ell) - \mu_1 \operatorname{div}\left(\frac{H(x)\xi(R|x|)^\ell x}{|x|^2}\right) \\
&\quad + \frac{\mu_2}{|x|^2}H(x)\xi(R|x|)^\ell \\
&= -\xi(R|x|)^\ell \Delta H(x) - H(x)\Delta(\xi(R|x|)^\ell) \\
&\quad - 2\nabla H(x)\cdot\nabla(\xi(R|x|)^\ell) - \mu_1\left(\xi(R|x|)^\ell \operatorname{div}\left(\frac{H(x)x}{|x|^2}\right)\right. \\
&\quad \left. + \frac{H(x)x}{|x|^2}\cdot\nabla(\xi(R|x|)^\ell)\right) + \frac{\mu_2}{|x|^2}H(x)\xi(R|x|)^\ell \\
&= \xi(R|x|)^\ell\left(-\Delta H(x) - \mu_1 \operatorname{div}\left(\frac{H(x)x}{|x|^2}\right) + \frac{\mu_2}{|x|^2}H(x)\right) \\
&\quad - H(x)\Delta\left(\xi(R|x|)^\ell\right) - 2\nabla H(x)\cdot\nabla(\xi(R|x|)^\ell) \\
&\quad - \mu_1\frac{H(x)x}{|x|^2}\cdot\nabla\left(\xi(R|x|)^\ell\right) \\
&= \xi(R|x|)^\ell\mathcal{L}^*_{\mu_1,\mu_2}H(x) - H(x)\Delta\left(\xi(R|x|)^\ell\right) \\
&\quad - 2\nabla H(x)\cdot\nabla\left(\xi(R|x|)^\ell\right) - \mu_1\frac{H(x)x}{|x|^2}\cdot\nabla\left(\xi(R|x|)^\ell\right).
\end{aligned}$$

In view of Equation (2.8), we get

$$\mathcal{L}^*_{\mu_1,\mu_2}\xi_R(x) = -H(x)\Delta(\xi(R|x|)^\ell) - 2\nabla H(x)\cdot\nabla\left(\xi(R|x|)^\ell\right) - \mu_1\frac{H(x)x}{|x|^2}\cdot\nabla\left(\xi(R|x|)^\ell\right),$$

which implies by Equation (2.10) and Cauchy–Schwarz inequality that

$$\int_{\operatorname{supp}(\xi_R)} \xi_R^{\frac{-q}{p-q}}\left|\mathcal{L}^*_{\mu_1,\mu_2}\xi_R\right|^{\frac{p}{p-q}}|x|^{\frac{aq}{p-q}}\,\mathrm{d}x = \int_{\frac{1}{2R}<|x|<\frac{1}{R}} \xi_R^{\frac{-q}{p-q}}\left|\mathcal{L}^*_{\mu_1,\mu_2}\xi_R\right|^{\frac{p}{p-q}}|x|^{\frac{aq}{p-q}}\,\mathrm{d}x \tag{2.27}$$

and

$$\left|\mathcal{L}^*_{\mu_1,\mu_2}\xi_R\right| \le H(x)\left|\Delta\left(\xi(R|x|)^\ell\right)\right| + C\left|\nabla\left(\xi(R|x|)^\ell\right)\right|\left(|\nabla H(x)| + \frac{H(x)}{|x|}\right). \tag{2.28}$$

On the other hand, using Equations (2.7) and (2.10), for $\frac{1}{2R} < |x| < \frac{1}{R}$, we obtain

$$\left|\Delta\left(\xi(R|x|)^\ell\right)\right| \le CR^2\xi(R|x|)^{\ell-2}, \qquad \left|\nabla\left(\xi(R|x|)^\ell\right)\right| \le CR\xi(R|x|)^{\ell-1}. \tag{2.29}$$

In view of Equations (2.7) and (2.29), for $\frac{1}{2R} < |x| < \frac{1}{R}$, we get

$$H(x)\left|\Delta\left(\xi(R|x|)^\ell\right)\right| \le CR^2|x|^{2-N-\mu_1-\alpha}\left(1-|x|^{2\alpha-2+N+\mu_1}\right)\xi(R|x|)^{\ell-2}$$

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$$[1mm] \leq CR^{N+\mu_1+\alpha}\xi(R|x|)^{\ell-2} \tag{2.30}$$

and

$$\begin{aligned}
&\left|\nabla\left(\xi(R|x|)^{\ell}\right)\right|\left(|\nabla H(x)|+\frac{H(x)}{|x|}\right)\\
&\leq CR\left(|x|^{1-N-\mu_1-\alpha}+|x|^{\alpha-1}+|x|^{1-N-\mu_1-\alpha}\left(1-|x|^{2\alpha-2+N+\mu_1}\right)\right)\xi(R|x|)^{\ell-1}\\
&\leq CR\left(|x|^{1-N-\mu_1-\alpha}+|x|^{\alpha-1}\right)\xi(R|x|)^{\ell-1}\\
&\leq CR|x|^{1-N-\mu_1-\alpha}\left(1+|x|^{2\alpha+N+\mu_1-2}\right)\xi(R|x|)^{\ell-1}\\
&\leq CR^{N+\mu_1+\alpha}\left(1+R^{-2\alpha-N-\mu_1+2}\right)\xi(R|x|)^{\ell-1}.
\end{aligned}$$

Notice that $-2\alpha-N-\mu_1+2<0$, which implies that for sufficiently large R,

$$\left|\nabla\left(\xi(R|x|)^{\ell}\right)\right|\left(|\nabla H(x)|+\frac{H(x)}{|x|}\right)\leq CR^{N+\mu_1+\alpha}\xi(R|x|)^{\ell-1}. \tag{2.31}$$

Thus, in view of Equations (2.10), (2.28), (2.30) and (2.31), we obtain

$$\left|\mathcal{L}^*_{\mu_1,\mu_2}\xi_R\right|^{\frac{p}{p-q}}\leq CR^{\frac{(N+\mu_1+\alpha)p}{p-q}}\xi(R|x|)^{\frac{(\ell-2)p}{p-q}},\quad \frac{1}{2R}<|x|<\frac{1}{R}. \tag{2.32}$$

Moreover, we have (for $\frac{1}{2R}<|x|<\frac{1}{R}$),

$$\begin{aligned}
\xi_R(x)^{\frac{-q}{p-q}}|x|^{\frac{aq}{p-q}} &= |x|^{\frac{aq}{p-q}}H(x)^{\frac{-q}{p-q}}\xi(R|x|)^{\frac{-\ell q}{p-q}}\\
&= |x|^{\frac{aq}{p-q}}|x|^{\frac{(-2+N+\mu_1+\alpha)q}{p-q}}\left(1-|x|^{2\alpha-2+N+\mu_1}\right)^{\frac{-q}{p-q}}\xi(R|x|)^{\frac{-\ell q}{p-q}}\\
&\leq C|x|^{\frac{aq+q(-2+N+\mu_1+\alpha)}{p-q}}\xi(R|x|)^{\frac{-\ell q}{p-q}}\\
&\leq CR^{\frac{q(-a+2-N-\mu_1-\alpha)}{p-q}}\xi(R|x|)^{\frac{-\ell q}{p-q}}.
\end{aligned} \tag{2.33}$$

Combining Equation (2.32) with Equation (2.33), we get

$$\begin{aligned}
&\xi_R^{\frac{-q}{p-q}}\left|\mathcal{L}^*_{\mu_1,\mu_2}\xi_R\right|^{\frac{p}{p-q}}|x|^{\frac{aq}{p-q}}\leq CR^{\frac{(N+\mu_1+\alpha)p+q(-a+2-N-\mu_1-\alpha)}{p-q}}\xi(R|x|)^{\ell-\frac{2p}{p-q}},\\
&\frac{1}{2R}<|x|<\frac{1}{R}.
\end{aligned} \tag{2.34}$$

Thus, using Equations (2.10), (2.27) and (2.34), we obtain

$$\begin{aligned}
&\int_{\mathrm{supp}(\xi_R)} \xi_R^{\frac{-q}{p-q}} \left|\mathcal{L}^*_{\mu_1,\mu_2}\xi_R\right|^{\frac{p}{p-q}} |x|^{\frac{aq}{p-q}}\,\mathrm{d}x \\
&\leq CR^{\frac{(N+\mu_1+\alpha)p+q(-a+2-N-\mu_1-\alpha)}{p-q}} \int_{\frac{1}{2R}<|x|<\frac{1}{R}} \xi(R|x|)^{\ell-\frac{2p}{p-q}}\,\mathrm{d}x \\
&\leq CR^{\frac{(N+\mu_1+\alpha)p+q(-a+2-N-\mu_1-\alpha)}{p-q}} R^{-N},
\end{aligned}$$

which proves Equation (2.25).

(ii) In view of Equations (2.7) and (2.29), for $\frac{1}{2R} < |x| < \frac{1}{R}$, we get

$$\begin{aligned}
H(x)\left|\Delta\left(\xi(R|x|)^{\ell}\right)\right| &\leq CR^2|x|^{\alpha}\ln\left(\frac{1}{|x|}\right)\xi(R|x|)^{\ell-2} \\
&\leq CR^{2-\alpha}\ln R\,\xi(R|x|)^{\ell-2}
\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}
\left|\nabla\left(\xi(R|x|)^{\ell}\right)\right|\left(|\nabla H(x)|+\frac{H(x)}{|x|}\right) &\leq CR\left(|x|^{\alpha-1}-|x|^{\alpha-1}\ln|x|\right)\xi(R|x|)^{\ell-1} \\
&\leq CR^{2-\alpha}(1+\ln R)\,\xi(R|x|)^{\ell-1} \\
&\leq CR^{2-\alpha}\ln R\xi(R|x|)^{\ell-1}.
\end{aligned} \tag{2.36}$$

Thus, in view of Equations (2.10), (2.28), (2.35) and (2.36), we obtain

$$\left|\mathcal{L}^*_{\mu_1,\mu_2}\xi_R\right|^{\frac{p}{p-q}} \leq CR^{\frac{(2-\alpha)p}{p-q}}(\ln R)^{\frac{p}{p-q}}\xi(R|x|)^{\frac{(\ell-2)p}{p-q}}, \quad \frac{1}{2R}<|x|<\frac{1}{R}. \tag{2.37}$$

Moreover, we have (for $\frac{1}{2R} < |x| < \frac{1}{R}$)

$$\begin{aligned}
\xi_R(x)^{\frac{-q}{p-q}}|x|^{\frac{aq}{p-q}} &= |x|^{\frac{aq}{p-q}}H(x)^{\frac{-q}{p-q}}\xi(R|x|)^{\frac{-\ell q}{p-q}} \\
&= |x|^{\frac{aq}{p-q}}|x|^{\frac{-\alpha q}{p-q}}\left(\ln\left(\frac{1}{|x|}\right)\right)^{\frac{-q}{p-q}}\xi(R|x|)^{\frac{-\ell q}{p-q}} \\
&\leq CR^{\frac{q(\alpha-a)}{p-q}}(\ln R)^{\frac{-q}{p-q}}\xi(R|x|)^{\frac{-\ell q}{p-q}}.
\end{aligned} \tag{2.38}$$

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Combining Equation (2.37) with Equation (2.38), we get

$$\xi_R^{\frac{-q}{p-q}} \left|\mathcal{L}^*_{\mu_1,\mu_2}\xi_R\right|^{\frac{p}{p-q}} |x|^{\frac{aq}{p-q}} \leq CR^{\frac{(2-\alpha)p+q(\alpha-a)}{p-q}} \ln R\xi(R|x|)^{\ell-\frac{2p}{p-q}},$$
$$\frac{1}{2R} < |x| < \frac{1}{R}. \tag{2.39}$$

Thus, using Equations (2.10), (2.27) and (2.39), we obtain

$$\begin{aligned}
&\int_{\mathrm{supp}(\xi_R)} \xi_R^{\frac{-q}{p-q}} \left|\mathcal{L}^*_{\mu_1,\mu_2}\xi_R\right|^{\frac{p}{p-q}} |x|^{\frac{aq}{p-q}}\,\mathrm{d}x\\
&\leq CR^{\frac{(2-\alpha)p+q(\alpha-a)}{p-q}} \ln R \int_{\frac{1}{2R}<|x|<\frac{1}{R}} \xi(R|x|)^{\ell-\frac{2p}{p-q}}\,\mathrm{d}x\\
&\leq CR^{\frac{(2-\alpha)p+q(\alpha-a)}{p-q}} R^{-N} \ln R,
\end{aligned}$$

which proves Equation (2.26).

□

Lemma 2.7.

(i) Let $\mu_2 > -\left(\frac{\mu_1-N+2}{2}\right)^2$. For sufficiently large T, R and ℓ, there holds

$$J_2(\varphi) \leq CTR^{\frac{(\mu_1+\alpha)p-(a-2+\mu_1+\alpha)q}{p-q}}, \tag{2.40}$$

where φ is the function defined by Equation (2.14).

(ii) Let $\mu_2 = -\left(\frac{\mu_1-N+2}{2}\right)^2$. For sufficiently large T, R and ℓ, there holds

$$J_2(\varphi) \leq CTR^{\frac{(2-\alpha-N)p+(\alpha-a+N)q}{p-q}} \ln R. \tag{2.41}$$

Proof. By Equation (2.2) and (2.14), we have

$$J_2(\varphi) = \left(\int_{\mathrm{supp}(\eta_T)} \eta_T(t)\,\mathrm{d}t\right)\left(\int_{\mathrm{supp}(\xi_R)} \xi_R^{\frac{-q}{p-q}} \left|\mathcal{L}^*_{\mu_1,\mu_2}\xi_R\right|^{\frac{p}{p-q}} |x|^{\frac{aq}{p-q}}\,\mathrm{d}x\right). \tag{2.42}$$

On the other hand, using Equations (2.9) and (2.11), we obtain

$$\int_{\mathrm{supp}(\eta_T)} \eta_T(t)\,\mathrm{d}t = \int_0^T \eta\left(\frac{t}{T}\right)^\ell \mathrm{d}t = T\int_0^1 \eta(s)^\ell\,\mathrm{d}s. \tag{2.43}$$

Hence, using Lemma 2.6 and Equations (2.42) and (2.43), we obtain Equation (2.40) and (2.41). □

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3. Proof of Theorem 1.3

3.1. Proof of part (I)

We use the contradiction argument by supposing that $u \in L^p_{\mathrm{loc}}(Q)$ is a weak solution to Equation (1.1). By Lemma 2.1, Equation (2.3) holds for all $\varphi \in \Phi$ (with $J_i(\varphi) < \infty$, $i = 1, 2$). Hence, by Lemma 2.2, we deduce that for sufficiently large T, R and ℓ,

$$-\int_\Gamma f(x)\,\frac{\partial \varphi}{\partial \nu}\,\mathrm{d}\sigma\,\mathrm{d}t \le C\sum_{i=1}^{2} J_i(\varphi), \tag{3.1}$$

where φ is the function defined by Equation (2.14). We first consider the following:

- The case: $\mu_2 > -\left(\frac{\mu_1 - N + 2}{2}\right)^2$. In view of Equations (2.18) and (2.43), we obtain

$$\begin{aligned}-\int_\Gamma f(x)\,\frac{\partial \varphi}{\partial \nu}\,\mathrm{d}\sigma\,\mathrm{d}t &= (2\alpha - 2 + N + \mu_1)\int_{\mathrm{supp}(\eta_T)}\int_{\partial B_1} f(x)\eta_T(t)\,\mathrm{d}\sigma\,\mathrm{d}t\\ &= (2\alpha - 2 + N + \mu_1)\left(\int_0^1 \eta(s)^\ell\,\mathrm{d}s\right) T\int_{\partial B_1} f(x)\,\mathrm{d}\sigma.\end{aligned}$$

Notice that

$$2\alpha - 2 + N + \mu_1 > 0.$$

Hence, there holds

$$-\int_\Gamma f(x)\,\frac{\partial \varphi}{\partial \nu}\,\mathrm{d}\sigma\,\mathrm{d}t = CTI_f. \tag{3.2}$$

Using Equations (2.22), (2.40), (3.1) and (3.2), we obtain

$$TI_f \le C\left(T^{1-\frac{kp}{p-1}}\left(\ln R + R^{\alpha+\mu_1-2-\frac{a}{p-1}}\right) + TR^{\frac{(\mu_1+\alpha)p-(a-2+\mu_1+\alpha)q}{p-q}}\right),$$

that is,

$$I_f \le C\left(T^{\frac{-kp}{p-1}}\left(\ln R + R^{\alpha+\mu_1-2-\frac{a}{p-1}}\right) + R^{\frac{(\mu_1+\alpha)p-(a-2+\mu_1+\alpha)q}{p-q}}\right). \tag{3.3}$$

Next, taking $T = R^\theta$, where

$$\theta > \max\left\{\frac{(\alpha+\mu_1-2-\frac{a}{p-1})(p-1)}{kp}, 0\right\}, \tag{3.4}$$

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Equation (3.3) reduces to

$$I_f \le C\left(R^{\frac{-kp\theta}{p-1}} \ln R + R^{\lambda_1} + R^{\lambda_2}\right), \tag{3.5}$$

where

$$\lambda_1 = \alpha + \mu_1 - 2 - \frac{a}{p-1} - \frac{kp\theta}{p-1} \tag{3.6}$$

and

$$\lambda_2 = \frac{(\mu_1+\alpha)p - (a-2+\mu_1+\alpha)q}{p-q}. \tag{3.7}$$

Observe that by the choice (3.4) of the parameter θ, one has $\lambda_1 < 0$. Moreover, by Equation (1.11), there holds $\lambda_2 < 0$. Thus, passing to the limit as $R \to \infty$ in Equation (3.5), we get $I_f \le 0$, which contradicts the positivity of I_f. Next, we consider the following:

- The case: $\mu_2 = -\left(\frac{\mu_1 - N + 2}{2}\right)^2$. In view of Equations (2.18) and (2.43), we obtain

$$\begin{aligned} -\int_\Gamma f(x)\,\frac{\partial\varphi}{\partial\nu}\,\mathrm{d}\sigma\,\mathrm{d}t &= \int_{\mathrm{supp}(\eta_T)}\int_{\partial B_1} f(x)\eta_T(t)\,\mathrm{d}\sigma\,\mathrm{d}t \\ &= \left(\int_0^1 \eta(s)^\ell\,\mathrm{d}s\right) T \int_{\partial B_1} f(x)\,\mathrm{d}\sigma, \end{aligned}$$

which yields Equation (3.2). Using Equations (2.23), (2.41), (3.1) and (3.2), we obtain

$$TI_f \le C\left(T^{1-\frac{kp}{p-1}} \ln R \left(\ln R + R^{-\left(\frac{a}{p-1}+\alpha+N\right)}\right) + TR^{\frac{(2-\alpha-N)p+(\alpha-a+N)q}{p-q}} \ln R\right).$$

Notice that

$$2-\alpha-N = \mu_1+\alpha, \qquad \alpha - a + N = -(a-2+\mu_1+\alpha).$$

Hence, the above estimate is equivalent to

$$I_f \le C\left(T^{-\frac{kp}{p-1}} \ln R \left(\ln R + R^{\alpha+\mu_1-2-\frac{a}{p-1}}\right) + R^{\frac{(\mu_1+\alpha)p-(a-2+\mu_1+\alpha)q}{p-q}} \ln R\right). \tag{3.8}$$

Taking $T = R^\theta$, where the parameter θ satisfies Equation (3.4), Equation (3.8) reduces to

$$I_f \le C\left(R^{\frac{-kp\theta}{p-1}} (\ln R)^2 + R^{\lambda_1} \ln R + R^{\lambda_2} \ln R\right), \tag{3.9}$$

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where λ_1 and λ_2 are given, respectively, by Equations (3.6) and (3.7). As in the previous case, due to Equations (3.4) and (1.11), one has $\lambda_i < 0$, $i = 1, 2$. Thus, passing to the limit as $R \to \infty$ in Equation (3.9), we get a contradiction with $I_f > 0$.

Consequently, Equation (1.1) admits no weak solution. This completes the proof of part (I) of Theorem 1.3. □

3.2. Proof of part (II)

We first consider the following:

- Case 1: $\mu_2 > -\left(\frac{\mu_1-N+2}{2}\right)^2$. Let δ and ϵ be two real numbers satisfying, respectively,

$$\delta_1 < \delta < \min\left\{\delta_2, \frac{2-a}{p-q}\right\} \tag{3.10}$$

and

$$0 < \epsilon < \left[P_{q,\mu_1,\mu_2,N}(\delta)\right]^{\frac{1}{p-q}}, \tag{3.11}$$

where

$$P_{q,\mu_1,\mu_2,N}(\delta) = -q^2\delta^2 + q(N-\mu_1-2)\delta + \mu_2$$

and δ_i, $i = 1, 2$, are the roots of $P_{q,\mu_1,\mu_2,N}(\delta)$, given by

$$\delta_1 = -\frac{\mu_1+\alpha}{q} < \delta_2 = \frac{N-2+\alpha}{q}.$$

Notice that by Equation (1.12), one has

$$\delta_1 < \frac{2-a}{p-q}.$$

Hence, the set of δ satisfying Equation (3.10) is non-empty. Moreover, for $\delta_1 < \delta < \delta_2$, one has $P_{q,\mu_1,\mu_2,N}(\delta) > 0$. Hence, $\left[P_{q,\mu_1,\mu_2,N}(\delta)\right]^{\frac{1}{p-q}}$ is well-defined, and the set of ϵ satisfying Equation (3.11) is non-empty. Let us consider functions of the form

$$u_{\delta,\epsilon}(x) = \epsilon|x|^{-\delta}, \quad x \in B_1\backslash\{0\}. \tag{3.12}$$

Elementary calculations show that

$$\mathcal{L}_{\mu_1,\mu_2} u^q_{\delta,\epsilon}(x) = \epsilon^q P_{q,\mu_1,\mu_2,N}(\delta)|x|^{-q\delta-2}, \quad x \in B_1\backslash\{0\}. \tag{3.13}$$

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In view of Equations (3.10), (3.11), (3.12) and (3.13), for all $x \in B_1\backslash\{0\}$, we obtain

$$\begin{aligned}\mathcal{L}_{\mu_1,\mu_2} u^q_{\delta,\epsilon}(x) &\geq \epsilon^q \epsilon^{p-q}|x|^{-q\delta-2}\\ &\geq \epsilon^p |x|^{-\delta p-a}\\ &= |x|^{-a}u^p_{\delta,\epsilon}(x).\end{aligned}$$

Hence, for any δ and ϵ satisfying, respectively, Equations (3.10) and (3.11), functions of the form (3.12) are stationary positive solutions to Equation (1.1) with $f \equiv \epsilon^q$. Next, we consider the following:

- Case 2: $\mu_2 = -\left(\frac{\mu_1-N+2}{2}\right)^2$. For

$$0 < \delta < \frac{1}{q} \tag{3.14}$$

and

$$0 < \epsilon < \left[\delta q(1-\delta q)\right]^{\frac{1}{p-q}}, \tag{3.15}$$

let

$$u_{\delta,\epsilon}(x) = \begin{cases} 0 & \text{if } \ 0 < |x| \leq \mathrm{e}^{-1}, \\ \epsilon |x|^{\frac{\mu_1+\alpha}{q}} \left[\ln(e|x|)\right]^{\delta} & \text{if } \ e^{-1} < |x| \leq 1. \end{cases} \tag{3.16}$$

Elementary calculations show that

$$\mathcal{L}_{\mu_1,\mu_2} u^q_{\delta,\epsilon}(x) = \begin{cases} 0 = |x|^{-a}u^p_{\delta,\epsilon}(x) & \text{if } \ 0 < |x| \leq \mathrm{e}^{-1}, \\ \epsilon^q \delta q(1-\delta q)|x|^{\mu_1+\alpha-2}\left[\ln(e|x|)\right]^{\delta q-2} & \text{if } \ \mathrm{e}^{-1} < |x| \leq 1. \end{cases} \tag{3.17}$$

Using Equations (3.15) and (3.16), for $\mathrm{e}^{-1} < |x| \leq 1$, we obtain

$$\begin{aligned}&\epsilon^q \delta q(1-\delta q)|x|^{\mu_1+\alpha-2}\left[\ln(e|x|)\right]^{\delta q-2}\\ &\geq \epsilon^q\epsilon^{p-q}|x|^{\mu_1+\alpha-2}\left[\ln(e|x|)\right]^{\delta q-2}\\ &= |x|^{-a}u^p_{\delta,\epsilon}(x)|x|^{a+\mu_1+\alpha-2-\frac{(\mu_1+\alpha)p}{q}}\left[\ln(e|x|)\right]^{\delta(q-p)-2}.\end{aligned} \tag{3.18}$$

Notice that, in view of Equation (1.12), one has

$$a+\mu_1+\alpha-2-\frac{(\mu_1+\alpha)p}{q} < 0. \tag{3.19}$$

Moreover, since $\delta > 0$ and $p > q$, there holds

$$\delta(q-p)-2 < 0. \tag{3.20}$$

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Hence, it follows from Equations (3.19) and (3.20) that

$$|x|^{a+\mu_1+\alpha-2-\frac{(\mu_1+\alpha)p}{q}}\left[\ln(e|x|)\right]^{\delta(q-p)-2}\geq 1,\quad e^{-1}<|x|\leq 1. \tag{3.21}$$

Then, Equations (3.18) and (3.21) yield

$$\epsilon^q\delta q(1-\delta q)|x|^{\mu_1+\alpha-2}\left[\ln(e|x|)\right]^{\delta q-2}\geq |x|^{-a}u^p_{\delta,\epsilon}(x),\quad e^{-1}<|x|\leq 1. \tag{3.22}$$

Thus, in view of Equations (3.17) and (3.22), we obtain

$$\mathcal{L}_{\mu_1,\mu_2}u^q_{\delta,\epsilon}(x)\geq |x|^{-a}u^p_{\delta,\epsilon}(x),\quad x\in B_1\backslash\{0\}.$$

Consequently, for any δ and ϵ satisfying, respectively, Equations (3.14) and (3.15), functions of the form (3.16) are stationary positive solutions to Equation (1.1) with $f\equiv\epsilon^q$. This completes the proof of part (II) of Theorem 1.3. □

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