VECTORS AND INVARIANTS.

Einleitung in die Theorie der Invarianten linearer Transformationen auf Grund der Vectorenrechnung,

Von E. Study. Vol. 1. Braunschweig (1923), pp. 1-268 (paper cover; 7/-).

This work is the seventy-first volume of the Braunschweig series of treatises on Natural Philosophy and Technology, edited by Professor Dr Eilhard Wiedemann.

The number of authors who, in these days of unavoidable specializing, have a ripe and effective knowledge of widely varied fields of mathematics and physics is sadly few. But perhaps it is well that most writers shrink from embarking on too broad a programme, and, rather, confine themselves to the narrow lines along which their knowledge is expert. When, however, a really competent man, endowed with the requisite gifts and powers of lucid exposition, undertakes this wider task, we may expect something noteworthy and enlivening. The present volume is a case in point. It is a modest little volume in size, yet it covers the ground extremely thoroughly.

The aim of the book is to draw more closely together two branches of mathematics, the theory of invariants on the one hand and the theory of vectors on the other. It may be assumed that, at any rate before this book appeared, these two subjects were not identical: and that for most students of mathematics, pure or applied, the content of these subjects was roughly known and certainly considered entirely distinct. The author who can make good his claim to relate two such distinct fields of knowledge and to shew their inherent likeness, does a notable service in the interests of mathematics itself. It is no mere labour-saving device, that of learning one thing instead of two: it suggests new possibilities in both fields. Is the pleasure, for instance, of learning that the theorems of Pascal and Brianchon, relative to hexagons inscribed and circumscribed to a conic, are essentially the same, entirely due to a delight in economizing thought? The pleasure surely lies in grasping the underlying principle of duality (or reciprocation). An unwary author can still be found to write an elementary geometry book with these two theorems kept entirely distinct; and no doubt books on vectors will continue to be written which will foster the same disintegrating effect among theorems on vectors, by neglect of the binding underlying principles which are there for the asking. Probably there has been in the past too much of this private and independent development of theories of vectors, for special physical or geometrical purposes, and not enough care taken to enquire into the heart of the subject and its general relations with algebra (in its widest sense). Consequently, its progress has been unnecessarily hampered.

These matters are considered in the present book. No earlier knowledge either of vectors or of invariants is presupposed; so the book is accessible to anyone who can read German and whose mathematics includes the rudiments of the theory of determinants. This is the starting point. The author then proceeds to develop an analytical theory of vectors, incorporating those points of view which have been variously taken by pioneers in the nineteenth century, in each case the point of view being carefully introduced with historical notes and relevant references. In particular this applies to the work of Grassmann, Sophus Lie, Aronhold, Clebsch, and Frobenius. But why not Hamilton? To the genius of Hamilton we owe the elegant theory of quaternions, nowhere more appreciated than in Edinburgh. Yet may not the very brilliance of its geometrical aspect handicap it in its ultimate survival, at a time when vectors in a field of four (or n) dimensions are freely discussed not merely by geometers but by physicists? Grassmann, a contemporary of Hamilton, hit upon very much the same idea for a calculus of vectors. (Whether independently or not, there appears to be doubt. At all events the present reviewer does not know; and in any case we are deeply indebted to both.) But Grassmann found a notation which, being more algebraic and less geometrical, has proved capable of great extension, and, with such modifications as experience has brought, is likely to be a valuable method in the future.

All of us, whether we are interested in mathematical analysis or geometry or physics, learn more or less distinctly to recognise certain properties or functions of the material with which we operate. Such properties or functions assume importance because they meet us at every turn and persist most unexpectedly through the intricacies of a mathematical operation. To give them a specific name they are ***invariants of the group of transformations*** with which we are concerned. They mark off what is of permanent interest in the field of enquiry from what is ephemeral. A simple example is

$$ll' + mm' + nn'$$

which gives the cosine of the angle between two straight lines in terms of their direction cosines (l, m, n) and (l', m', n'). This expression remains constant however much we shift the origin and orientation of the frame of reference, provided the axes are kept mutually orthogonal. The analytical aspect of this kinematical procedure is characterized by calling this function an *invariant for the group of orthogonal transformations.* Similar examples could be quoted from almost all departments of pure and applied mathematics, not the least famous being a certain integral invariant, due to the genius of Hamilton again, which expresses one of the few physical findings of last century which have survived the revolution denoted by the name Relativity.

For any field of mathematical enquiry it is clear that three useful questions can at once be asked:—(1) Does an invariant exist?; (2) if it occurs can it be recognised?; (3) Can all possible invariants of this field be found?; or can enough be found to render the search for others superfluous? The relevance of these is best seen by taking an illustration from the theory of differential equations. For field take such an equation; for invariant substitute solution, and these questions are at once seen to go to the heart of the matter.

The answers to these questions is the concern of this book, especially in two fields which interest physicists, the fields of rigid displacements and of the restricted theory of relativity. The existence of what Grassmann calls inner and outer products ($ll' + mm' + nn'$ is an inner product) gives an answer to question (1). The invention of a good notation, originally due to Aronhold (although both Grassmann and Cayley deserve some of the credit) answers (2). The extension to these other fields, of results first given by Gordan and Hilbert for the projective field, answers (3). This extension has only recently been made by Weitzenböck and by Study himself.

It is curious that until a good notation was found this question (3) was never answered. Once found, however, it completely revolutionized the invariant theory of the projective group, for which, as the author remarks, we have had a highly developed theory for over fifty years "and for more than twenty-five years at least the fundamentals of an invariant theory of the other groups" already mentioned. "But not a glimmer of light seems to have fallen from these investigations upon the highly loved 'Vector Analysis' of to-day. . . . So far as I know the question is never once raised in such writings as to what are all possible algebraic, and in particular, rational, invariants of these groups." In illustration of this the Weyl hypothesis of general relativity affords an example. By great physical and analytical insight Weyl produced certain functions with the desired Hamiltonian properties to suit not merely the original Einstein conditions of transformation, but also a change in the guage of distance itself. Weyl thereby answered question (1) affirmatively. Whereupon Weitzenböck undertook to examine the further problems involved in (2) and (3). Not only did he shew* these functions to be structurally of a pattern perfectly well known to the last generation of algebraic-geometers, but he proved that Weyl had in fact exhausted his possible constructions. This is surely a remarkable result which intensifies the interest in the physical aspect of Weyl's achievement.

Not only has the subject of vectors suffered from discordant notations, but it has also suffered from a lavish confusion of unnecessary names for one and the same thing. Why, for instance, should a ternary bilinear form be called—tensor of second rank, affinor, dyadic, tensortriple, complete dyad, asymmetric tensor, diatensor, vectorial homography, and in special cases, deviator, antitensor, axiator, idemfaktor, versor, perversor, etc? The author does a real service in protesting (p. 8) against this ill-considered confusion.

The book is provocative of thought and is keenly critical in the best interests of mathematical economy. The criticism attacks what appear to be blemishes in details of the work of great pioneers in the past, as well as the preliminary obstacles to a wider appreciation of what the physicists are doing to-day because they have fettered themselves with " an unlucky notation " (p. 101). This may be too severe, for until Vol. 2 appears one is not in a position to judge; it may be there is no convenient alternative, in the higher branches of tensor analysis, to the suffix notation " with its emphasis on the unessential," to which objection is made.

The book may be misunderstood by those who are unwilling to grant the right for a mathematician to define the fundamental elements of geometry, points, lines, planes, and, here, vectors, by means of ordered sets of numbers. This is not, indeed, the only point of view, but it is a fruitful point of view, more especially as it stands midway between the classical notation of pure geometry and that of analytical methods With it " the clatter of the co-ordinate mill " is soon silenced, and the processes are developed not with co-ordinates, but with an explicit use of the elements themselves, in a manner reminiscent of a proposition of Euclid.

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* *Cf.* Wiener Berichte, 129 (1920); 130 (1921).