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Some extensions of a dual of the Hahn-Banach Theorem, with applications to separation and Helly type theorems

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In previous papers we have proved that if G is a w^*-closed subspace of the conjugate space B^* of a normed linear space B, then every $b \in B$ can be extended within B, from G to the whole B^*, with an arbitrarily small increase of the norm. Here we give some extensions of this result to the case when B^* is replaced by a normed linear space E and B by any linear subspace V of E^*, and some applications to separation and Helly type theorems.

1.

In [7], [8], [9] we have proved and given some applications of the following theorem, which is dual, in a certain sense, to the classical theorem of Hahn-Banach on the norm-preserving extension of continuous linear functionals.

THEOREM DHB. *Let B be a (real or complex) normed linear space and G a $\sigma(B^*, B)$-closed linear subspace of the conjugate space B^*. Then for every $b \in B$ and $\varepsilon > 0$ there exists an element $b_\varepsilon \in B$ such that*

$$x(b_\varepsilon) = x(b) \quad (x \in G) , \tag{1}$$

$$\|b_\varepsilon\| \leq \sup_{\substack{x\in G \\ \|x\|\leq 1}} |x(b)| + \varepsilon . \tag{2}$$

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Some results related to Theorem DHB have been obtained by Günzler [6]. Recently Alfsen and Effros have rediscovered Theorem DHB ([1], p. 116), with our proof given in [9] and they have given a sufficient condition on G under which it is possible to delete the $\varepsilon > 0$ in (2) above ([1], Corollary 5.5), as well as some related extension theorems (see, for example, [1], Theorem 5.4). Apparently Alfsen and Effros [1] have been unaware of the papers [7], [8], [9], and [6].

In §2 of the present paper we shall give an extension of Theorem DHB to the more general case when B^* is replaced by an arbitrary normed linear space E and B is replaced by an arbitrary linear subspace V of E^* (Theorem 1), and some applications to separation and Helly type results (Theorems 2-4). In §3 we shall give some related results on the factor $\frac{1}{r'r}$ and the $\varepsilon > 0$ occurring in Theorem 1 (Propositions 1-4), which may have their own interest even in the particular case when $E = B^*$ and $V = \pi(B)$, where $\pi : B \to B^{**}$ is the canonical embedding (Corollaries 1 and 2).

Let us recall the terminology and notations, which we shall use in the sequel.

Let E be a normed linear space and V a linear subspace of E^*. We shall denote by $\tilde{V}$ the $\sigma(E^*, E)$-closure of V in E^*. Following Dixmier [4], the *characteristic* $r'_{\tilde{V}}(V)$ *of* V *with respect to* $\tilde{V}$ is the greatest number r' such that the unit ball $S_V = \{f \in V \mid \|f\| \leq 1\}$ of V is $\sigma(E^*, E)$-dense in the r'-ball $r'S_{\tilde{V}} = \{h \in \tilde{V} \mid \|h\| \leq r'\}$ of $\tilde{V}$. It is known ([4], Theorem 7') that

$$r'_{\tilde{V}}(V) = \inf_{\substack{x \in E \\ x \notin V_\perp}} \frac{\sup_{f \in S_V} |f(x)|}{\sup_{h \in S_{\tilde{V}}} |h(x)|}, \tag{3}$$

where $V_\perp = \{x \in E \mid f(x) = 0 \ (f \in V)\}$. Clearly, $0 \leq r'_{\tilde{V}}(V) \leq 1$. The *characteristic* $r(V)$ of V is [4] the number

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$$(4) \qquad r(V) = \begin{cases} r'_{E^*}(V) = \inf\limits_{\substack{x \in E \\ x \neq 0}} \sup\limits_{f \in S_V} \left| f\left(\frac{x}{\|x\|}\right) \right| & \text{if } \tilde{V} = E^* , \\ 0 & \text{if } \tilde{V} \neq E^* . \end{cases}$$

The *canonical mapping* u *of* E *into* V^* is the continuous linear mapping defined by

$$(5) \qquad u(x)(f) = f(x) \quad (x \in E,\ f \in V) .$$

Clearly, u is one-to-one if and only if V is *total* on E (that is, $V_\perp = \{0\}$, or, what is equivalent, V is $\sigma(E^*, E)$-dense in E^*). Also, by (5), u is an isomorphism of E into V^* if and only if $r(V) > 0$, and in this case

$$(6) \qquad \|u^{-1}\| = \frac{1}{r(V)} .$$

Finally, dually to the notation $V_\perp$, for a subspace G of E we shall use the notation $G^\perp = \{f \in E^* \mid f(x) = 0 \ (x \in G)\}$.

2.

We have the following extension of Theorem DHB (the proof is rather short, since it uses Theorem DHB itself).

THEOREM 1. *Let* E *be a normed linear space,* G *a linear subspace of* E *, and* V *a linear subspace of* E^* *. Then for every* $f \in V$ *and* $\varepsilon > 0$ *there exists an element* $f_\varepsilon \in V$ *such that*

$$(7) \qquad f_\varepsilon(x) = f(x) \quad (x \in G) ,$$

$$(8) \qquad \|f_\varepsilon\| \le \frac{1}{r'_{\widetilde{u(G)}}\big(u(G)\big) r(V)} \|f|_G\| + \varepsilon ,$$

where u *is the canonical mapping of* E *into* V^* .

Proof. If either $r'_{\widetilde{u(G)}}\big(u(G)\big) = 0$ or $r(V) = 0$, then condition (8) is void, so we may take $f_\varepsilon = f$.

Assume now that both $r'_{\widetilde{u(G)}}\big(u(G)\big) = r' > 0$ and $r(V) = r > 0$. Then, by Theorem DHB applied to V and $\widetilde{u(G)} \subset V^*$, for every $f \in V$ and $\varepsilon > 0$

there exists an $f_\varepsilon \in V$ such that

$$f_\varepsilon(x) = u(x)(f_\varepsilon) = u(x)(f) = f(x) \quad (x \in G) ,$$

$$\|f_\varepsilon\| \leq \sup_{\substack{\psi \in \widetilde{u(G)} \\ \|\psi\| \leq 1}} |\psi(f)| + \varepsilon \leq \frac{1}{r'} \sup_{\substack{u(x) \in u(G) \\ \|u(x)\| \leq 1}} |u(x)(f)| + \varepsilon \leq$$

$$\leq \frac{1}{r'} \sup_{\substack{x \in G \\ \|x\| \leq \|u^{-1}\|}} |f(x)| + \varepsilon = \frac{1}{r'r} \|f|_G\| + \varepsilon ,$$

which completes the proof of Theorem 1.

REMARK 1. (a) Theorem DHB can be obtained as a consequence of Theorem 1, as follows. Let B be a normed linear space, $E = B^*$, $V = \pi(B) \subset B^{**} = E^*$, where $\pi : B \to B^{**}$ is the canonical embedding, and let G be a $\sigma(B^*, B)$-closed subspace of $B^* = E$. Then $r(V) = 1$ and the canonical mapping $u : E \to V^*$ is an isometry of E *onto* V (actually, $u = (\pi^{-1})^*$). Also, u is an isomorphism for $\sigma(B^*, B)$, $\sigma(V^*, V)$, whence, since G is $\sigma(B^*, B)$-closed, $\widetilde{u(G)} = u(G)$ and therefore $r'_{\widetilde{u(G)}}(u(G)) = 1$. Consequently, by Theorem 1, we obtain Theorem DHB.

(b) If $E = B^*$, $V = \pi(B)$, and $G \subset B^*$ is not necessarily $\sigma(B^*, B)$-closed, we still have $r(V) = 1$ and, since u is an isometry and a w^*-isomorphism of $E = B^*$ onto V^*, we have $r'_{\widetilde{u(G)}}(u(G)) = r'_{\tilde{G}}(G)$. Hence, in this case, from Theorem 1 we obtain the following generalization of Theorem DHB, due to Günzler ([6], Theorem 3). *If B is a normed linear space and G is a linear subspace of B^*, then for every $b \in B$ and $\varepsilon > 0$ there exists an element $b_\varepsilon \in B$ satisfying* (1) *and*

$$\|b_\varepsilon\| \leq \frac{1}{r'_{\tilde{G}}(G)} \sup_{\substack{x \in G \\ \|x\| \leq 1}} |x(b)| + \varepsilon .$$

REMARK 2. The method of Remark 1 also shows how to construct examples of norm-closed $G \subset E$ with $r'_{\widetilde{u(G)}}(u(G)) = 0$. Indeed, take again $E = B^*$, $V = \pi(B) \subset E^*$ and then take any norm-closed $\sigma(B^*, B)$-dense subspace G of $B^* = E$, with $r(G) = 0$ (for examples of such $G \subset c_0^*$ or $(l^1)^*$ or $(l^\infty)^*$ see, for instance, [4]). Then, since $u = (\pi^{-1})^*$ is an

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isomorphism for $\sigma(B^*, B)$, $\sigma(V^*, V)$, and an isometry of E onto V^*, we have $\widetilde{u(G)} = V^*$ and $r'_{\widetilde{u(G)}}(u(G)) = 0$.

REMARK 3. In other words, Theorem 1 says that

$$(9) \quad \operatorname{dist}(f, G^{\perp} \cap V) = \inf_{h \in G^{\perp} \cap V} \|f-h\| = \inf_{\substack{h' \in V \\ h'|_G = f|_G}} \|h'\| \leq$$

$$\leq \frac{1}{r'_{\widetilde{u(G)}}(u(G))\, r(V)} \|f|_G\| \quad (f \in V)$$

Hence, since $\|f|_G\| \leq \operatorname{dist}(f, G^{\perp} \cap V) = \|f+(G^{\perp} \cap V)\|_{V/G^{\perp} \cap V}$ for all $f \in V$, it follows that *if* $r(V) > 0$ *and* $r'_{\widetilde{u(G)}}(u(G)) > 0$, *then* $V|_G$ *is isomorphic to* $V/G^{\perp} \cap V$, *by the mapping* $f|_G \to f + (G^{\perp} \cap V)$. Consequently, *in this case the subspace* $(G^{\perp} \cap V)^{\perp} \equiv (V/G^{\perp} \cap V)^*$ *of* V^* *is isomorphic to* $(V|_G)^*$, *by the mapping* $\Phi \to \psi$, *where* $\psi(f|_G) = \Phi(f)$ $(f \in V)$.

Let us give now some applications of Theorem 1 to separation and Helly type theorems.

We recall the following well known results (see, for example, [5], p. 422, Corollary 12, and [3], Chapter I, §4), which we shall use in the sequel.

LEMMA 1. *Let* E *be a normed linear space and* V *a total linear subspace of* E^*. *Then*

(a) *a linear subspace* G *of* E *is* $\sigma(E, V)$-*closed if and only if for each* $x \notin G$ *there exists* $f \in G^{\perp} \cap V$ *with* $f(x) = 1$;

(b) *every finite-dimensional subspace* G *of* E *is* $\sigma(E, V)$-*closed;*

(c) *if* G *is a* $\sigma(E, V)$-*closed linear subspace of* E *and* F *a finite-dimensional subspace of* E *such that* $G \cap F = \{0\}$, *then* $G \oplus F$ *is* $\sigma(E, V)$-*closed.*

The following application of Theorem 1 may be also regarded as a sharpening of part of Lemma 1 *(a)*.

THEOREM 2. *Let* E *be a normed linear space,* V *a total linear* *subspace of* E^* , *and* G *a* $\sigma(E, V)$*-closed linear subspace of* E . *Then for every* $x \in E$ *with* $\operatorname{dist}(x, G) = d > 0$ *and every* $\varepsilon > 0$ *there exists an element* $f_\varepsilon \in G^\perp \cap V$ *satisfying* $f_\varepsilon(x) = 1$ *and*

$$\|f_\varepsilon\| \leq \frac{1}{r'_{\widetilde{u(G\oplus[x])}}\left(u(G\oplus[x])\right)r(V)d} + \varepsilon , \tag{10}$$

where u *is the canonical mapping of* E *into* V^* .

Proof. If either $r'_{\widetilde{u(G\oplus[x])}}\left(u(G \oplus [x])\right) = 0$ or $r(V) = 0$, then condition (10) is void, so we may take f_ε to be any f as in Lemma 1 *(a)*.

Assume now that both $r'_{\widetilde{u(G\oplus[x])}}\left(u(G \oplus [x])\right) > 0$ and $r(V) > 0$. By Lemma 1 *(a)* let $f \in G^\perp \cap V$, $f(x) = 1$. Then

$$\|y+\lambda x\| = |\lambda|\left\|\frac{1}{\lambda} y+x\right\| \geq |\lambda|d = |f(y+\lambda x)|d \quad (y+\lambda x \in G \oplus [x]) ,$$

whence $\|f|_{G\oplus[x]}\| \leq \frac{1}{d}$. Consequently, by Theorem 1 (applied to $G \oplus [x]$), there exists $f_\varepsilon \in G^\perp \cap V$ satisfying $f_\varepsilon(x) = 1$ and (10), which completes the proof.

REMARK 4. In general, $r'_{\widetilde{u(G\oplus[x])}}\left(u(G \oplus [x])\right) \neq r'_{\widetilde{u(G)}}\left(u(G)\right)$.

Indeed, for example, let $E = c_0^* \equiv l^1$, $V = \pi_{c_0}\left(c_0\right) \subset l^\infty$ and let G be a $\sigma\left(c_0^*, c_0\right)$-dense norm-closed hyperplane in c_0^* . Then $r_{c_0^*}(G) \leq \frac{1}{2}$ (by [4]), but for any $x \in c_0^*\backslash G$ we have $G \oplus [x] = c_0^*$, whence $r'_{\widetilde{G\oplus[x]}}(G \oplus [x]) = 1$. Consequently, as in Remarks 1 and 2 above, we have

$$r'_{\widetilde{u(G\oplus[x])}}\left(u(G \oplus [x])\right) = 1 \neq \tfrac{1}{2} = r'_{\widetilde{u(G)}}\left(u(G)\right) .$$

REMARK 5. (a) In the particular case when $V = E^*$, we have $r(V) = 1$ and $u = \pi$, the canonical embedding of E into E^{**} , whence $r'_{\widetilde{u(G)}}\left(u(G)\right) = 1$ for every subspace G of E . Consequently, Theorem 2 yields that for every $\sigma(E, E^*)$-closed (and hence for every) linear subspace G of E, every $x \in E$ with $\operatorname{dist}(x, G) = d > 0$ and every $\varepsilon > 0$ there exists $f_\varepsilon \in G^\perp$ satisfying $f_\varepsilon(x) = 1$ and $\|f_\varepsilon\| \le \frac{1}{d} + \varepsilon$. Combining this result with the $\sigma(E^*, E)$-compactness of balls in E^* we obtain that there also exists $f_0 \in G^\perp$ satisfying $f_0(x) = 1$ and $\|f_0\| \le \frac{1}{d}$, which is a well known corollary of the Hahn-Banach Theorem.

(b) If B is a normed linear space and G a $\sigma(E^*, E)$-closed subspace of B^* and if we take E and V as in Remark 1, then Theorem 2 yields that for every $x \in B^*$ with $\operatorname{dist}(x, G) = d > 0$ and every $\varepsilon > 0$ there exists $b_\varepsilon \in G_\perp \subset B$ satisfying $x(b_\varepsilon) = 1$ and

$$\|b_\varepsilon\| \le \frac{1}{d} + \varepsilon .$$

This result is nothing else than a well known theorem of Banach ([2], p. 122, Theorem 1), which has been reproved also in [8], [9] as a consequence of Theorem DHB.

THEOREM 3. *Let E be a normed linear space, V a linear subspace of E^* with $r(V) = 1$, A a set in E such that $G = [A]$ satisfies $r'_{\widetilde{u(G)}}\big(u(G)\big) = 1$ (where $[A]$ is the closed linear subspace spanned by A and u is the canonical mapping of E into V^*), $f \in V$, and $M > 0$. In order that for every $\varepsilon > 0$ there exist an $f_\varepsilon \in V$ satisfying*

$$f_\varepsilon(x) = f(x) \quad (x \in A) , \tag{11}$$

$$\|f_\varepsilon\| \le M + \varepsilon , \tag{12}$$

it is necessary and sufficient that we have

$$\left|\sum_{i=1}^{n} \lambda_i f(x_i)\right| \le M \left\|\sum_{i=1}^{n} \lambda_i x_i\right\| \tag{13}$$

for every finite collection of scalars $\lambda_1, \ldots, \lambda_n$ and of elements $x_1, \ldots, x_n \in A$.

Proof. If for every $\varepsilon > 0$ there exists an $f_\varepsilon \in V$ satisfying (11),

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(12), then for every $\lambda_1, \ldots, \lambda_n$ and $x_1, \ldots, x_n \in A$ and every $\varepsilon > 0$, we have

$$\left|\sum_{i=1}^{n} \lambda_i f(x_i)\right| = \left|\sum_{i=1}^{n} \lambda_i f_\varepsilon(x_i)\right| \le \|f_\varepsilon\| \left\|\sum_{i=1}^{n} \lambda_i x_i\right\| \le (M+\varepsilon)\left\|\sum_{i=1}^{n} \lambda_i x_i\right\| ,$$

whence, since $\varepsilon > 0$ was arbitrary, we obtain (13).

Conversely, if we have (13), then $\|f|_G\| \le M$ and hence, by Theorem 1, for every $\varepsilon > 0$ there exists an $f_\varepsilon \in V$ satisfying (11), (12), which completes the proof.

REMARK 6. If B is a normed linear space and if we take E and V as in Remark 1, then Theorem 3 yields the extension of a "dual" theorem of Helly, which was obtained in [8], [9] (as a consequence of Theorem DHB).

LEMMA 2. *Let E be a normed linear space, V a total linear subspace of E^*, $x_1, \ldots, x_n \in E$ and let $c_1, \ldots, c_n$ be scalars. In order that there exist an $f \in V$ satisfying*

(14) $$f(x_i) = c_i \quad (i = 1, \ldots, n) ,$$

it is necessary and sufficient that for any scalars $\lambda_1, \ldots, \lambda_n$ the relations $\sum_{i=1}^{n} \lambda_i x_i = 0$ imply $\sum_{i=1}^{n} \lambda_i c_i = 0$.

Proof. The necessity is an immediate consequence of the linearity of f.

We shall prove the sufficiency by induction on n.

Let $n = 1$ and assume that for any scalar λ_1 the relation $\lambda_1 x_1 = 0$ implies $\lambda_1 c_1 = 0$. If $x_1 \ne 0$, then, since V is total on E, there exists $f_0 \in V$ such that $f_0(x_1) = 1$. Hence $f = c_1 f_0 \in V$ and satisfies (14). If $x_1 = 0$, then, by our assumption, $\lambda_1 c_1 = 0$ for all scalars λ_1, whence $c_1 = 0$; so any $f \in V$ satisfies (14).

Assume now that the sufficiency part is true for $n - 1$ and that the condition is satisfied (for n). Then the condition is also satisfied for

$n - 1$ and hence, by the induction hypothesis, there exist $f_j' \in V$ $(j = 1, \ldots, n)$ such that $f_j'(x_i) = c_i$ $(i \neq j;\ i, j = 1, \ldots, n)$.

Now if $x_1, \ldots, x_n$ are linearly independent, then, since by Lemma 1 *(b)*, $G_j = [x_i]_{i \neq j}$ is $\sigma(E, V)$-closed $(j = 1, \ldots, n)$, there exist, by Lemma 1 *(a)*, $f_j \in V$ satisfying $f_j(x_i) = \delta_{ij}$ $(i, j = 1, \ldots, n)$. Consequently, $f = \sum_{j=1}^{n} c_j f_j \in V$ and satisfies (14).

On the other hand, if $x_1, \ldots, x_n$ are linearly dependent, say $x_j - \sum_{i \neq j} \alpha_i x_i = 0$ for some j , then by our assumption $c_j - \sum_{i \neq j} \alpha_i c_i = 0$, whence

$$f_j'(x_j) = \sum_{i \neq j} \alpha_i f_j'(x_i) = \sum_{i \neq j} \alpha_i c_i = c_j ,$$

so $f = f_j' \in V$ satisfies (14), which completes the proof.

REMARK 7. More generally, both Lemma 1 and Lemma 2 are valid, with the same proof, for E^* replaced by $E^{\#}$, the linear space of all linear functionals on E . In the particular case when $V = E^{\#}$, Lemma 2 has been given in [3], Chapter I, §2, Lemma 1, and in the particular case when $V \subset E^{\#}$ is total, but $x_1, \ldots, x_n$ are linearly independent (whence the condition of the lemma is satisfied), it can be found in [3], Chapter I, §2, Corollary 1.

The following "Helly type" application may be also regarded as a sharpening of Lemma 2.

THEOREM 4. *Let E be a normed linear space, V a linear subspace of E^* with $r(V) = 1$, $x_1, \ldots, x_n \in E$, and let $c_1, \ldots, c_n$ be scalars and $M > 0$. In order that for every $\varepsilon > 0$ there exist an $f_\varepsilon \in V$ satisfying*

(15) $$f_\varepsilon(x_i) = c_i \quad (i = 1, \ldots, n) ,$$

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and (12), *it is necessary and sufficient that we have*

$$(16) \qquad \left|\sum_{i=1}^{n} \lambda_i c_i\right| \leq M \left\|\sum_{i=1}^{n} \lambda_i x_i\right\|$$

for all scalars $\lambda_1, \ldots, \lambda_n$.

Proof. The necessity is obvious.

Conversely, assume now that we have (16) for all scalars $\lambda_1, \ldots, \lambda_n$. Then the condition of Lemma 2 is satisfied and hence there exists an $f \in V$ satisfying (14).

Now let $A = \{x_1, \ldots, x_n\}$ and $G = [A]$, and let u be the canonical linear mapping of E into V^* . Then $\dim u(G) \leq \dim G < \infty$, whence $u(G) = \widetilde{u(G)}$ and thus $r'_{\widetilde{u(G)}}(u(G)) = 1$. Consequently, by Theorem 3, there exists an $f_\varepsilon \in V$ satisfying $f_\varepsilon(x_i) = f(x_i) = c_i$ $(i = 1, \ldots, n)$ and (12), which completes the proof.

REMARK 8. (a) In the particular case when $V = E^*$ (hence $r(V) = 1$) , Theorem 4, combined with the $\sigma(E^*, E)$-compactness of balls in E^* , yields the classical theorem of Helly (see, for example, [3], Chapter II, §5, Corollary 1), with $M + \varepsilon$ replaced by M in (12).

(b) If B is a normed linear space and if we take E and V as in Remark 1, then Theorem 4 yields the "dual" Helly Theorem (see, for example, [3], Chapter II, §5, Theorem 3), which gives a necessary and sufficient condition in order that for $x_1, \ldots, x_n \in B^*$, $c_1, \ldots, c_n$ scalars, $M > 0$ and $\varepsilon > 0$, there exist an element $b_\varepsilon \in B$ satisfying $x_i(b_\varepsilon) = c_i$ $(i = 1, \ldots, n)$ and $\|b_\varepsilon\| \leq M + \varepsilon$.

3.

For the finite-dimensional subspaces G (and hence, more generally, for the subspaces G with $r'_{\widetilde{u(G)}}\, u(G) = 1$), the constant in formula (8) of Theorem 1 is the "best possible", in the following sense.

PROPOSITION 1. *Let* E *be a normed linear space and* V *a total*

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linear subspace of E^* . *Then* $r(V)$ *is the greatest number* c *such that for every finite-dimensional subspace* G *of* E , *every* $f \in V$ *and every* $\varepsilon > 0$ *there exists an element* $f_\varepsilon \in V$ *satisfying* (7) *and*

(17) $$\|f_\varepsilon\| \le \frac{1}{c} \|f|_G\| + \varepsilon .$$

Proof. If $\dim G < \infty$, then, as we have observed above, for any linear subspace V of E^* we have $r'_{\widetilde{u(G)}}\big(u(G)\big) = 1$. Hence, by Theorem 1, for any c with $0 \le c \le r(V)$ and for every $f \in V$ and $\varepsilon > 0$ there exists $f_\varepsilon \in V$ satisfying (7) and (17).

On the other hand, assume now that $c > r(V)$. We shall show that in this case there exist a one-dimensional subspace G of E and a pair $f \in V$, $\varepsilon > 0$, for which the extension property (7), (17) fails. Since $r(V) < c$, there exists an element $x_0 \in E$, $x_0 \ne 0$, such that

(18) $$\sup_{\substack{f \in V \\ \|f\| \le 1}} |f(x_0)| < c\|x_0\| .$$

Let $G = [x_0]$, the one-dimensional subspace of E spanned by x_0 . Then G is $\sigma(E, V)$-closed (by Lemma 1 *(b)*) and $V \not\subset G^\perp$ (since V is $\sigma(E^*, E)$-dense in E^* and $G^\perp \ne E^*$ is $\sigma(E^*, E)$-closed). Consequently, there exists an $f \in V$ such that $f\left(\frac{x_0}{\|x_0\|}\right) = 1$; clearly, $\|f|_G\| = \sup_{|\alpha| \le 1} \left|f\left(\frac{\alpha x_0}{\|x_0\|}\right)\right| = 1$. Assume that for each $\varepsilon > 0$ there exists an $f_\varepsilon \in V$ satisfying (7) and (17). Then for $h_\varepsilon = \frac{f_\varepsilon}{\|f_\varepsilon\|} \in V$ we have $\|h_\varepsilon\| = 1$ and

$$|h_\varepsilon(x_0)| = \frac{1}{\|f_\varepsilon\|} |f_\varepsilon(x_0)| = \frac{1}{\|f_\varepsilon\|} |f(x_0)| = \frac{\|x_0\|}{\|f_\varepsilon\|} \ge \frac{\|x_0\|}{(1/c)+\varepsilon} = \frac{c}{1+\varepsilon c} \|x_0\| ,$$

which contradicts (18) for $\varepsilon > 0$ sufficiently small. Thus, there exists $\varepsilon > 0$ for which there is no $f_\varepsilon \in V$ satisfying (7) and (17), which completes the proof of Proposition 1.

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Now we shall give some cases in which the factor $\frac{1}{r'r}$ and the $\varepsilon > 0$ in (8) can be replaced by 1 and 0 respectively.

PROPOSITION 2. *Let* E *be a normed linear space,* V *a linear subspace of* E^* *and* G *a linear subspace of* E *, such that* $G^\perp \subset V$ *. Then for every* $f \in V$ *there exists an element* $f_0 \in V$ *such that*

$$f_0(x) = f(x) \quad (x \in G) , \tag{19}$$

$$\|f_0\| = \|f|_G\| . \tag{20}$$

Proof. By the Hahn-Banach Theorem, there exists $f_0 \in E^*$ satisfying (19), (20). But, by (19), $G^\perp \subset V$ and $f \in V$ we have

$$f_0 \in f + G^\perp \subset f + V = V ,$$

which completes the proof. Note that if G is norm-closed, the condition $G^\perp \subset V$ implies that G is $\sigma(E, V)$-closed $\left(\text{since } G = \overline{G} = \bigcap_{f \in G^\perp} \ker f\right)$.

PROPOSITION 3. *Let* E *be a normed linear space,* V *a total linear subspace of* E^* *, and* G *a* $\sigma(E, V)$*-closed subspace of finite codimension in* E *. Then for every* $f \in V$ *there exists an element* $f_0 \in V$ *satisfying* (19), (20).

Proof. Since G is also norm-closed, let $\{x_i\}_{i=1}^n \subset E$ be linearly independent and such that $G \oplus [x_i]_{i=1}^n = E$. Then, since G is $\sigma(E, V)$-closed and since $\dim[x_i]_{i \neq j} < \infty$, the subspaces $G \oplus [x_i]_{i \neq j}$ $(j = 1, \ldots, n)$ are $\sigma(E, V)$-closed (by Lemma 1 *(c)*). Hence, since $x_j \notin G \oplus [x_i]_{i \neq j}$, there exist (by Lemma 1 *(a)*), $f_i \in G^\perp \cap V$ $(i = 1, \ldots, n)$ such that $f_i(x_j) = \delta_{ij}$ $(i, j = 1, \ldots, n)$. But then $f_1, \ldots, f_n$ are linearly independent, so $\dim[f_i]_{i=1}^n = n$, whence, since $[f_i]_{i=1}^n \subset G^\perp$ and $\dim G^\perp = n$, we obtain $[f_i]_{i=1}^n = G^\perp$. Consequently, $G^\perp = [f_i]_{i=1}^n \subset V$, whence, by Proposition 2, the conclusion follows.

If B is a normed linear space and if we take E and V as in

Remark 1, then from Propositions 2 and 3 we obtain

COROLLARY 1. *Let* B *be a normed linear space and* G *a linear subspace of* B^* *such that* $G^\perp \subset \pi(B)$ *, where* $\pi : B \to B^{**}$ *is the canonical embedding. Then for every* $b \in B$ *there exists an element* $b_0 \in B$ *such that*

$$x(b_0) = x(b) \quad (x \in G) , \tag{21}$$

$$\|b_0\| = \sup_{\substack{g \in G \\ \|g\| \le 1}} |g(b)| ; \tag{22}$$

in particular, if G *is a* $\sigma(B^*, B)$*-closed linear subspace of* B^* *, of finite codimension in* B^* *, then for every* $b \in B$ *there exists* $b_0 \in B$ *satisfying* (21), (22).

PROPOSITION 4. *Let* E *be a normed linear space,* V *a linear subspace of* E^* *, and* G *a subspace of* E *such that there exists a (not necessarily linear) projection* p *of* E *onto* G *, of norm* $\|p\| = \sup_{\substack{x \in E \\ x \ne 0}} \frac{\|p(x)\|}{\|x\|} = 1$ *, satisfying*

$$p^*(V) \subset V . \tag{23}$$

Then for every $f \in V$ *there exists an* $f_0 \in V$ *satisfying* (19), (20).

Proof. Let $f_0 = p^*(f)$. Then, by (23), $f_0 \in V$. Furthermore, since $x \in G$ if and only if $x = p(x)$, and since $\|p(x)\| \le \|x\|$ $(x \in E\backslash\{0\})$, we have

$$f_0(x) = p^*(f)(x) = f(p(x)) = f(x) \quad (x \in G) ,$$

$$\|f_0\| = \|p^*(f)\| = \sup_{\substack{x \in E \\ \|x\|=1}} |p^*(f)(x)| \le \sup_{\substack{p(x) \in p(E) \\ \|p(x)\| \le 1}} |f(p(x))| = \|f|_{p(E)}\| = \|f|_G\| = \|f_0|_G\| \le \|f_0\| ,$$

which completes the proof.

COROLLARY 2. *Let* B *be a normed linear space and* G *a linear subspace of* B^* *such that there exists a (not necessarily linear) projection* q *on* B *, of norm* 1 *, satisfying*

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(24) $$q^*(B^*) = G \; .$$

Then for every $b \in B$ *there exists* $b_0 \in B$ *satisfying* (21), (22).

Proof. Take, as in Remark 1, $E = B^*$ and $V = \pi(B)$, where $\pi : B \to B^{**}$ is the canonical embedding, and let $p = q^*$. Then, by our assumption, p is a (not necessarily linear) projection of E onto G, of norm $\|p\| = 1$ and we have

$$p^*(V) = p^*\big(\pi(B)\big) = q^{**}\big(\pi(B)\big) = \pi\big(q(B)\big) \subset \pi(B) = V \; ;$$

that is, (23). Hence, the conclusion follows by applying Proposition 4; the proof also shows that one can take $b_0 = q(b)$.

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