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On a Conjecture of Goresky, Kottwitz and MacPherson

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Abstract. We settle a conjecture of Goresky, Kottwitz and MacPherson related to Koszul duality, *i.e.*, to the correspondence between differential graded modules over the exterior algebra and those over the symmetric algebra.

We show that a variant of Conjecture (13.9) in [GKM] holds whereas the original form of the conjecture does not. We assume that the reader is familiar with [GKM] and we apply the notation used therein.

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1 The Minimal Hirsch-Brown Model

We begin by recalling some results of [AP, Appendix B] concerning the so-called minimal Hirsch-Brown model of the Borel construction. We concentrate on the case where the connected, graded commutative algebra called R in [AP] is a polynomial ring S in finitely many variables ξ_i, $i = 1, \dots, r$ of even degrees $|\xi_i|$ over a field k.

Let $\tilde{K} = S \tilde{\otimes} K$ be a "twisted tensor product", *i.e.*, K here is a graded k-vector space and $\tilde{K}$ a differential graded S-module, which is equal to $S \otimes K$ as an S-module (disregarding the differential); in other words, $\tilde{K}$ is a differential graded S-module, which is free as an S-module (the differential on S is trivial). The S-linear differential δ of $S \tilde{\otimes} K$ induces a k-linear differential $\delta_1 = \mathrm{id}_k \otimes_S \tilde{\delta}$ on K, where k is considered an S-module via the standard augmentation $\epsilon \colon S \to k$. The differential graded S-module $(\tilde{K}, \tilde{\delta})$ can be viewed as obtained from the differential graded k-module (K, δ_1) by "twisting" the differential of the usual tensor product $S \otimes K$ with respect to the "parameter space" S (*cf.* [AP]).

Proposition 1.1 *Let $\tilde{f} \colon \tilde{K} \to \tilde{L}$ be a morphism of twisted tensor products. Assume that $\tilde{K}$ and $\tilde{L}$ are bounded from below. Then the following statements are equivalent:*

*($\tilde{a}$) $\tilde{f}$ is a homotopy equivalence in the category δgS-*Mod *of differential graded S-modules*
($\tilde{b}$) $H(\tilde{f})$ is an isomorphism
*(a) $f := \mathrm{id}_k \otimes_S \tilde{f} \colon k \otimes_S (S \tilde{\otimes} K) \cong K \to L \cong k \otimes_S (S \tilde{\otimes} L)$ is a homotopy equivalence in the category δgk-*Mod *of differential graded k-modules*
(b) $H(f)$ is an isomorphism.

For the proof of this proposition *cf.* [AP, (B.2)], in particular (B.2.1) and (B.2.2).

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Since the differential $\tilde{\delta}$ of a twisted tensor product $\tilde{K} = S \tilde{\otimes} K$ is S-linear, it is determined by the k-linear restriction $\tilde{\delta}|_K \colon K \cong 1 \otimes K \subset S \tilde{\otimes} K \to S \tilde{\otimes} K$.

The map $\tilde{\delta}|_K$ corresponds to a family of k-linear maps $\{\delta_s \colon K^q \to K^{q+1-|s|}, s = \xi_1^{n_1} \cdots \xi_r^{n_r}, |s| = \sum_{i=1}^r n_i|\xi_i|\}$, *i.e.*, $\tilde{\delta}(1 \otimes x) = \tilde{\delta}|_K(x) = \sum_s s \otimes \delta_s(x)$ for $x \in K$, where the sum is taken over the set $\mathcal{M}(S)$ of all monomials in S.

The property $\tilde{\delta} \circ \tilde{\delta} \equiv 0$ corresponds to a family of relations

$$\sum_{s=s's''} \delta_{s'} \circ \delta_{s''} \equiv 0 \quad \text{for all } s \in \mathcal{M}(S),$$

where the sum for a fixed $s \in \mathcal{M}(S)$ is taken over all product decompositions of s into two factors, in particular

$$\delta_1 \circ \delta_1 \equiv 0 \quad \big(1 \in \mathcal{M}(S)\big)$$

$$\delta_1 \circ \delta_{\xi_i} + \delta_{\xi_i} \circ \delta_1 \equiv 0 \quad \big(\xi_i \in \mathcal{M}(S)\big).$$

Let $H(K)$ denote the homology of K with respect to the differential $\delta_1 = \mathrm{id}_k \otimes_S \tilde{\delta}$.

Proposition 1.2 *Let $\tilde{K} = S \tilde{\otimes} K$ be a twisted tensor product, bounded from below. There exists a differential $\tilde{\delta}^H$ on $S \otimes H(K)$, which gives a twisted tensor product structure $S \tilde{\otimes} H(K)$, called minimal Hirsch-Brown model, such that* $\mathrm{id}_k \otimes_S \tilde{\delta}^H \equiv 0$*, and $S \tilde{\otimes} K$ and $S \tilde{\otimes} H(K)$ are homotopy equivalent in δgS-*Mod*. By these properties $S \tilde{\otimes} H(K)$ is uniquely determined up to isomorphism in δgS-*Mod.

The differential $\tilde{\delta}^H$ can be given by

$$\tilde{\delta}^H\big(1 \otimes [x]\big) = \sum_{i=1}^r \xi_i \otimes \big[\delta_{\xi_i}(x)\big] + \sum_{\ell(s) \geq 2} s \otimes \delta_s^H[x]$$

where [ ] denotes the class in $H(K)$ of a cycle in K with respect to the differential δ_1; $\ell(s)$ denotes the length of $s \in \mathcal{M}(S)$ as a product.

The proof of Proposition 1.2 is essentially given in [AP, pp. 451–454], in particular (B.2.4), Exercise (B.7).

The Exercise (B.7), *i.e.*, the uniqueness of the minimal Hirsch-Brown model up to isomorphism, is easily proved using the fact that the functor $k \otimes_S - \colon \delta gS\text{-Mod} \to \delta gk\text{-Mod}$ preserves homotopies, and that a morphism $\tilde{f} \colon S \tilde{\otimes} K \to S \tilde{\otimes} L$ is an isomorphism if (and only if) $f := \mathrm{id}_k \otimes_S \tilde{f} \colon K \to L$ is an isomorphism (*cf.* [AP, (A.7.3)]). In [AP] the above expression for $\tilde{\delta}^H(1 \otimes [x])$ is verified only in case $r = 1$, but the argument for the general case is analogous.

Remark 1.3 The differential $\tilde{\delta}^H$ of $S \tilde{\otimes} H(K)$ corresponds to a family of "cohomology operations"

$$\{\delta_s^H \colon H^q(K) \to H^{q+1-|s|}(K), s \in \mathcal{M}(S)\}.$$

The property $\tilde{\delta}^H \circ \tilde{\delta}^H \equiv 0$ corresponds to

$$\sum_{s=s's''} \delta_{s'}^H \circ \delta_{s''}^H = 0 \quad \text{for all } s \in \mathcal{M}(S).$$

Since $\delta_1^H \equiv 0$ one has in particular

$$\delta^H_{\xi_i} \circ \delta^H_{\xi_j} + \delta^H_{\xi_j} \circ \delta^H_{\xi_i} \equiv 0 \quad \text{for } i, j = 1 \dots r.$$

Since the minimal Hirsch-Brown model is only unique up to isomorphism in δgS-Mod, the family of cohomology operations $\{\delta^H_s, s \in \mathcal{M}(S)\}$ is only unique up to the corresponding equivalence. Clearly $H(K)$ together with $\{\delta^H_s, s \in \mathcal{M}(S)\}$ determine $S \tilde{\otimes} H(K)$ up to isomorphism in δgS-Mod and hence, by Proposition 1.2, $S\tilde{\otimes}K$ up to homotopy equivalence.

We can now apply the above results to obtain a variant of Conjecture (13.9) in [GKM]. Using the notation of [GKM], let $N \in D_+(\Lambda_\bullet)$, where $D_+(\Lambda_\bullet)$ is the derived (with respect to quasi-isomorphism) category of the homotopy category of differential graded $\Lambda_\bullet$-modules, which are bounded from below; $\Lambda_\bullet$ is the exterior algebra on a graded vector space $P = \oplus_{j>0} P_j$ with $P_{2j} = 0$ for all j.

By the Koszul duality theorem (8.4) in [GKM] there are equivalences of categories $t\colon D_+(\Lambda_\bullet) \to D_+(S)$, $h\colon D_+(S) \to D_+(\Lambda_\bullet)$ which are quasi-inverse to each other; $D_+(S)$ is the derived category of the homotopy category of differential graded S-modules, bounded from below.

In [GKM, (8.3)], $t(N)$ is defined as a twisted tensor product $S \tilde{\otimes} N$ with differential

$$d(s \otimes n) = s \otimes d_N n + \sum_{i=1}^{r} \xi_i s \otimes \lambda_i n$$

where $\lambda_1, \dots, \lambda_r$ generate $\Lambda_\bullet$ and $\xi_1, \dots, \xi_r$ generate S.

By Proposition 1.2, $t(N)$ is equivalent to its minimal Hirsch-Brown model in δgS-Mod, which is in turn determined (up to isomorphism) by $H(N)$ and the family $\{\delta^H_s, s \in \mathcal{M}(S)\}$; in particular, $t(N) \cong S \tilde{\otimes} H(N)$ in $D_+(S)$. The maps $\delta^H_{\xi_i}\colon H(N) \to H(N)$, $i = 1, \dots, r$ coincide with the action induced by $\lambda_i\colon N \to N$ in cohomology. Since h is quasi-inverse to t, one has $N \cong h \circ t(N)$ in $D_+(\Lambda_\bullet)$. Hence one gets the following corollary.

Corollary 1.4 *$N \in D_+(\Lambda_\bullet)$ is determined by $H(N)$ and $\{\delta^H_s, s \in \mathcal{M}(S)\}$.*

Let $K_+^{\mathrm{fr}}(S)$ denote the homotopy category of differential graded S-modules, which are bounded from below and free over S. Let M_1 and M_2 be objects in $K_+^{\mathrm{fr}}(S)$. The group $\mathrm{Hom}_{D_+(S)}(M_1, M_2)$ is defined as the direct limit of the system $\{\mathrm{Hom}_{K_+(S)}(M_1', M_2);\ f\colon M_1' \to M_1$ quasi-isomorphism in $K_+(S)\}$. For any M_1' in $K_+(S)$ the map $\Theta\colon th(M_1') \to M_1'$ (see [GKM, (16.6)]) is a quasi-isomorphism and $M_1'' := th(M_1')$ is in $K_+^{\mathrm{fr}}(S)$. Therefore the above limit can be restricted to the cofinal subsystem, where M_1' is in $K_+^{\mathrm{fr}}(S)$. By Proposition 1.1 any quasi-isomorphism in $K_+^{\mathrm{fr}}(S)$ is an isomorphism in $K_+(S)$. Hence $\mathrm{Hom}_{K_+(S)}(M_1', M_2) \cong \mathrm{Hom}_{K_+(S)}(M_1, M_2)$ and taking the direct limit gives:

$$\mathrm{Hom}_{K_+^{\mathrm{fr}}(S)}(M_1, M_2) \cong \mathrm{Hom}_{D_+(S)}(M_1 M_2).$$

One therefore gets the following proposition.

Proposition 1.5 *The canonical functor $i\colon K_+^{\mathrm{fr}}(S) \to K_+(S)$ induces an equivalence of categories $\bar{\imath}\colon K_+^{\mathrm{fr}}(S) \to D_+(S)$.*

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Let $H_+^{\mathrm{fr}}(S)$ denote the homotopy category of differential graded S-modules, which are bounded from below, free over S and minimal in the sense that $\mathrm{id}_k \otimes_S d_M \equiv 0$ for M in $H_+^{\mathrm{fr}}(S)$. Then Proposition 1.2 implies that the canonical functor $m\colon H_+^{\mathrm{fr}}(S) \to K_+^{\mathrm{fr}}(S)$ is an equivalence. As a consequence one has

Corollary 1.6 *All functors in the following diagram are equivalences of triangulated categories.*

$$\begin{array}{ccccc} D_+(\Lambda_\bullet) & \xrightarrow{t'} & K_+^{\mathrm{fr}}(S) & \xrightarrow{\bar{\imath}} & D_+(S) \\ & & m\uparrow & & \\ & & H_+^{\mathrm{fr}}(S) & & \end{array}$$

with $\bar{\imath}t' = t$.

Remark 1.7 Koszul duality can be viewed as an algebraic analogue of the following topological situation (*cf. e.g.* [tD, I.8], in particular (8.18)). Let hGTop be the homotopy category of G-spaces and G-maps, G a topological group, $\mathrm{hG^{fr}\,Top}$ the full subcategory, whose objects are numerable free G-spaces; let $\mathrm{hTop_{BG}}$ the homotopy category of spaces and maps over BG and $\mathrm{hFib_{BG}}$ the full subcategory of numerable fibre bundles over BG. Let $i\colon \mathrm{hG^{fr}\,Top} \to \mathrm{hGTop}$ and $j\colon \mathrm{hFib_{BG}} \to \mathrm{hTop_{BG}}$ be the canonical inclusion functors. Define $t'\colon \mathrm{hGTop} \to \mathrm{hFib_{BG}}$ by the Borel construction, *i.e.*, $t'(X) := \{X \times_G \mathrm{EG} \to \mathrm{BG}\}$, and $h'\colon \mathrm{hTop_{BG}} \to \mathrm{hG^{fr}\,Top}$ by $h'(\{Y \to \mathrm{BG}\}) := Y \times_{\mathrm{BG}} \mathrm{EG}$, *i.e.*, the total space of the principal G-bundle, classified by $\{Y \to \mathrm{BG}\}$.

The composition $h' \circ j \circ t'\colon \mathrm{hGTop} \to \mathrm{hG^{fr}\,Top}$ is equivalent to $\mathrm{fr}\colon \mathrm{hGTop} \to \mathrm{hG^{fr}\,Top}$, where $\mathrm{fr}(X) := X \times \mathrm{EG}$ (with diagonal G-action), since

$$(X \times_G \mathrm{EG}) \times_{\mathrm{BG}} \mathrm{EG} \cong X \times \mathrm{EG} \;\text{ as } G\text{-spaces}$$
$$\big([x,e], ge\big) \to (gx, ge), \quad \big([x,e], e\big) \leftarrow (x,e).$$

Since $\mathrm{fr} \circ i$ is equivalent to the identity (for $X \in \mathrm{hG^{fr}\,Top}$, $pr\colon X \times \mathrm{EG} \to X$, $(x,e) \to x$ is an isomorphism in $\mathrm{hG^{fr}\,Top}$) one gets that $h' \circ j \circ t' \circ i$ is equivalent to the identity on $\mathrm{hG^{fr}\,Top}$. In a similar way one gets that $t' \circ i \circ h' \circ j$ is equivalent to the identity on $\mathrm{hFib_{BG}}$. (Note that $\{Y \to \mathrm{BG}\} \in \mathrm{hFib_{BG}}$ is isomorphic to $\{(Y \times_{\mathrm{BG}} \mathrm{EG}) \times_G \mathrm{EG} \to \mathrm{BG}\}$ in $\mathrm{hFib_{BG}}$: The G-map $(y,e) \to (y,e,e)$ from $(Y \times_{\mathrm{BG}} \mathrm{EG})$ to $(Y \times_{\mathrm{BG}} \mathrm{EG}) \times \mathrm{EG}$ induces a homotopy equivalence on the quotients,

$$Y \cong (Y \times_{\mathrm{BG}} \mathrm{EG})/G \to (Y \times_{\mathrm{BG}} \mathrm{EG}) \times_G \mathrm{EG},$$
$$y \leftarrow [y,e] \to \big[(y,e), e\big]$$

which is a map over BG; and since $\{Y \to \mathrm{BG}\}$ and $\{(Y \times_{\mathrm{BG}} \mathrm{EG}) \times_G \mathrm{EG} \to \mathrm{BG}\}$ fulfil the homotopy lifting property this map is even a homotopy equivalence over BG.)

If DGTop denotes the "derived" category of hGTop, obtained by inverting morphisms $\alpha\colon X_1 \to X_2$ such that $(h' \circ j \circ t')(\alpha)$ are isomorphisms in $\mathrm{hG^{fr}\,Top}$ (which is equivalent

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to α being the equivalence class of a G-map that is a homotopy equivalence in Top), then i induces an equivalence of categories $\bar{\imath}\colon \mathrm{hG}^{\mathrm{fr}}\,\mathrm{Top} \to \mathrm{DGTop}$.

Similarly, if $\mathrm{DTop}_{\mathrm{BG}}$ denotes the "derived" category of $\mathrm{hTop}_{\mathrm{BG}}$ obtained by inverting morphisms (over BG) $\beta\colon Y_1 \to Y_2$ such that $(t' \circ i \circ h')(\beta)$ are isomorphisms in $\mathrm{hFib}_{\mathrm{BG}}$ (which is equivalent to β being the equivalence class of a map over BG that is a homotopy equivalence in Top), then j induces an equivalence of categories $\bar{\jmath}\colon \mathrm{hFib}_{\mathrm{BG}} \to \mathrm{DTop}_{\mathrm{BG}}$.

Altogether one has the following diagram

$$\begin{array}{ccccccc}
\mathrm{DGTop} & \longleftarrow & \mathrm{hGTop} & \xrightarrow{t'} & \mathrm{hFib}_{\mathrm{BG}} & & \\
 & {}^{\bar{\imath}}\nwarrow & i\uparrow & & j\downarrow & \searrow^{\bar{\jmath}} & \\
 & & \mathrm{hG}^{\mathrm{fr}}\,\mathrm{Top} & \xleftarrow{h'} & \mathrm{hTop}_{\mathrm{BG}} & \longrightarrow & \mathrm{DTop}_{\mathrm{BG}}
\end{array}$$

The functors $\bar{\imath}, \bar{\jmath}$ are equivalences; $h' \circ j \circ t' \circ i$ and $t' \circ i \circ h' \circ j$ are equivalent to the respective identity functors: so $(t' \circ i)$ and $(h' \circ j)$ are quasi-inverse equivalences, which induce quasi-inverse equivalences

$$t\colon \mathrm{DGTop} \to \mathrm{DTop}_{\mathrm{BG}} \quad \text{and} \quad h\colon \mathrm{DTop}_{\mathrm{BG}} \to \mathrm{DGTop}\,.$$

Work of Eilenberg-Moore and many other mathematicians is concerned with translating the above (and more general topological) situation into an algebraic set up which can be related to Koszul duality (*cf.* [McC, Chapters 7 and 8] for a comprehensive presentation of this translation and for detailed references).

2 Examples

For $N \in D_+(\Lambda_\bullet)$ a family of "higher cohomology operations" $\{\lambda_s\}$ on $H^*(N)$ is defined in [GKM, Section 13]. They can be considered as the differentials which start at $E_n^{o,*}$ in the spectral sequence which converges to $H^*(tN)$ having $E_2^{*,*} \cong S \otimes H^*(N)$. This spectral sequence can be obtained by filtering $tN = S \tilde{\otimes} N$ according to degree in S, *i.e.*, $\mathcal{F}^p(tN)$ is generated by elements $s \otimes n \in S \tilde{\otimes} N$ with $|s| \geq p$. Since $S \tilde{\otimes} N$ is homotopy equivalent to $S \tilde{\otimes} H(N)$ one obtains the same spectral sequence by filtering the latter complex in an analogous way.

Proposition 2.1

(a) The family $\{\delta_s^H, s \in \mathcal{M}(S)\}$ determines the family $\{\lambda_s\}$.
(b) $\{\delta_s^H, s \in \mathcal{M}(S)\}$ is trivial if and only if $\{\lambda_s\}$ is trivial.

Proof Part (a) is immediate since $\{\delta_s^H, s \in \mathcal{M}(S)\}$ determines the differential in $S \tilde{\otimes} H(N)$. For part (b) let

$$\tilde{\delta}^H\big(1 \otimes [x]\big) = \sum_{|s|=\gamma} s \otimes \delta_s^H\big([x]\big) + \sum_{|s|>\gamma} s \otimes \delta_s^H\big([x]\big)$$

for $[x] \in H(K)$. Then

$$d_n\big(1 \otimes [x]\big) = \begin{cases} 0 & \text{for } n < \gamma \\ \sum_{|s|=\gamma} \big[s \otimes \delta_s^H([x])\big] & \text{for } n = \gamma, \end{cases}$$

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with $1 \otimes [x] \in E_\gamma^{o,*} \subseteq \cdots \subseteq E_2^{o,*}$, where $\left[s \otimes \delta_s^H([x])\right]$ denotes the equivalence class in $E_\gamma^{*,*}$, *i.e.*, $\lambda_s([x]) = \left[\delta_s^H([x])\right]$ for $|s| = \gamma$, where $\left[\delta_s^H([x])\right]$ denotes the equivalence class in the appropriate quotient of $H(N)$, dividing out the indeterminancy of the cohomology operation. Hence part (b) follows. ■

Remarks 2.2

(a) With the notation of Proposition 2.1, if $\delta_s^H([x]) \neq 0$ but $\left[\delta_s^H([x])\right] = 0$ then an operation $\lambda_{s'}$ with $|s'| > |s|$ may be defined on $[x]$, but may not be determined solely by $\tilde{\delta}^H(1 \otimes [x])$; the values of $\tilde{\delta}^H$ on other elements could play a role.

(b) It follows from [CS] that in case $r = 1$, *i.e.*, $\Lambda_\bullet = \Lambda(\lambda_1)$, $|\lambda_1| = 1$, all differentials in the above spectral sequence are determined by those starting from $E_n^{o,*}$, but this does not hold for $r > 1$.

Example 2.3 Let $r = 2$, $S = \mathbb{Q}[\xi_1, \xi_2]$, H a $\mathbb{Q}$-vector space with basis $\{a, b, u, v\}$; $|a| = 2$, $|b| = 4$, $|u| = |v| = 5$. We define two differentials on $S \otimes H$, *i.e.*, two objects $M_1, M_2 \in D_+(S)$, such that the respective cohomology modules are not isomorphic, but the "higher cohomology operations" in the sense of [GKM, (13.8), (13.9)] coincide. Since, by Koszul duality, $M_i \in D_+(S)$ can be considered as $t(N_i)$ for $N_i \in D_+(\Lambda_\bullet)$ and $H(N_i) = H$, this shows that N_i is not determined by the $\Lambda_\bullet$-module $H(N_i)$ and the collection of higher cohomology operations, in particular, Conjecture (13.9) in its original form, which states that the triangulated category $D_+(\Lambda_\bullet)$ is equivalent to the category of graded $\Lambda_\bullet$-modules together with the collection of higher cohomology operations, does not hold.

The non-zero terms of the differential $\tilde{\delta}|_H$ are given by:

(1) $\tilde{\delta}(1 \otimes u) = \xi_1 \otimes b$; $\tilde{\delta}(1 \otimes v) = \xi_2 \otimes b$ for M_1

(2) $\tilde{\delta}(1 \otimes u) = \xi_1 \otimes b + \xi_1^2 \otimes a$; $\tilde{\delta}(1 \otimes v) = \xi_2 \otimes b + \xi_2^2 \otimes a$ for M_2.

$H^*(M_1)$ and $H^*(M_2)$ are not isomorphic as graded S-modules, in particular M_1 and M_2 are not isomorphic in δgS-Mod. The differentials in the two spectral sequences which start from $E_n^{o,*}$ coincide, *i.e.*, the only non-zero differentials of this kind are $d_2(1 \otimes u) = \xi_1 \otimes b$ and $d_2(1 \otimes v) = \xi_2 \otimes b$ in both cases. But for M_2 there is another non-zero differential, namely $d_4([\xi_2 \otimes u - \xi_1 \otimes v]) = [\xi_1^2\xi_2 \otimes a - \xi_1\xi_2^2 \otimes a]$. Note that while $M_1 = t(H)$ for an appropriate action of $\Lambda_\bullet$ on H, it is not possible to get M_2 as $t(H)$ for H equipped with a graded $\Lambda_\bullet$-module structure.

Remark 2.4 In view of Corollary 1.6, the Conjecture (13.9) in [GKM] could be rephrased in the following way. Let SP^0 be the category which has as objects, graded k-vector spaces bounded from below, together with the operations in the sense of [GKM, (13.8) (13.9)] and as morphisms, degree preserving homomorphisms of graded k-vector spaces, which commute with the operations. Then the conjecture is equivalent to the statement: The functor $\sigma\colon K_+^{\mathrm{fr}}(S) \to \mathrm{SP}^\circ$, which assigns to each differential graded free S-module, $S \tilde{\otimes} N$, the $E_2^{o,*}$-term of the spectral sequence obtained from the degree filtration on S, together with the differentials starting from $E_n^{o,*}$, considered as higher cohomology operations on $E_2^{o,*} \cong H^*(N)$, is an equivalence of triangulated categories.

Example 2.3 shows that the conjecture fails for $r > 1$ already because $\sigma(S \tilde{\otimes} N)$ does not completely determine the spectral sequence (nor the cohomology $H^*(S \tilde{\otimes} N)$). For

$r = 1$, though, it is true that $\sigma(S \tilde{\otimes} N)$ determines $S \tilde{\otimes} N$ (up to isomorphism in $K_+^{\mathrm{fr}}(S)$). In fact, in this case:

(i) $\sigma(S \tilde{\otimes} N)$ determines the spectral sequence completely (*cf.* Remark 2.2(b)),
(ii) there is no extension problem in calculating $H^*(S \tilde{\otimes} N)$, as an S-module, from the E_∞-term of the spectral sequence,
(iii) $H^*(S \tilde{\otimes} N)$ determines $S \tilde{\otimes} N$ up to homotopy equivalence of differential graded S-modules, *i.e.*, up to isomorphism in $K_+^{\mathrm{fr}}(S)$, since—for $r = 1$—S is a PID.

Yet Conjecture (13.9) in its original form fails even in case $r = 1$: The functor σ is not faithful (for any $r \geq 1$) (s. Example 2.5 below).

Example 2.5 Let N_1 and N_2 be differential graded $\Lambda_\bullet$-modules generated, as k-vector spaces, by $n_1, |n_1| = 0$, and $n_2, |n_2| = 2$, respectively. (Hence the $\Lambda_\bullet$-structures and the differentials on N_1 and N_2 are trivial.) Then $t(N_1) = S \otimes N_1$ and $t(N_2) = S \otimes N_2$ have trivial differentials, too. So $\sigma\big(t(N_1)\big) = N_1$ and $\sigma\big(t(N_2)\big) = N_2$, and the cohomology operations are trivial. Already for degree reasons there is only the trivial morphism from $\sigma\big(t(N_2)\big)$ to $\sigma\big(t(N_1)\big)$ in SP°. But $n_2 \mapsto \xi \otimes n_1$ extends to a morphism $f \colon t(N_2) \to t(N_1)$, which is non trivial in $K_+^{\mathrm{fr}}(S)$. Hence σ is not faithful. Note that although there is only the trivial morphism from N_2 to N_1 in the homotopy category $K_+(\Lambda_\bullet)$, there are non trivial morphisms in the derived category $D_+(\Lambda_\bullet)$:

$$N_2 \xrightarrow{\Phi_2} ht(N_2) = \mathrm{Hom}_k(\Lambda_\bullet, S \otimes N_2) \xrightarrow{h(f)} \mathrm{Hom}_k(\Lambda_\bullet, S \otimes N_1) = ht(N_1) \xleftarrow{\Phi_1} N_1.$$

where $\Phi_i := \Phi(N_i)$, $i = 0, 1$ (s. [GKM, (16.2)], are morphisms in $K_+(\Lambda_\bullet)$. These morphisms induce isomorphisms in homology (s. [GKM, (16.2)(b)], *i.e.*, they become isomorphisms in the derived category $D_+(\Lambda_\bullet)$.

Of course, there are similar examples for $r > 1$.

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