A NECESSARY AND SUFFICIENT CONDITION FOR SIMULTANEOUS DIAGONALIZATION OF TWO HERMITIAN MATRICES AND ITS APPLICATION

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1. Introduction and statement of the theorems. We denote by F the field R of real numbers, the field C of complex numbers, or the skew field H of real quaternions, and by F^n an n-dimensional left vector space over F. If A is a matrix with elements in F, we denote by A^* its conjugate transpose. In all three cases of F, an $n \times n$ matrix A is said to be *hermitian* if $A = A^*$, and we say that two $n \times n$ hermitian matrices A and B with elements in F can be diagonalized simultaneously if there exists a non singular matrix U with elements in F such that UAU^* and UBU^* are diagonal matrices. We shall regard a vector $u \in F^n$ as a $1 \times n$ matrix and identify a 1×1 matrix with its single element, and we shall denote by diag $\{A_1, \ldots, A_m\}$ a diagonal block matrix with the square matrices $A_1, \ldots, A_m$ lying on its diagonal.

Let $A = \text{diag}\{A_1, \ldots, A_m\}$ and $B = \text{diag}\{B_1, \ldots, B_m\}$ be any two hermitian block matrices such that, for each $k = 1, \ldots, m$, A_k and B_k are of the same size. Then it is obvious that, if each pair A_k and B_k can be diagonalized simultaneously, so also can the pair A and B. Whether the converse is true or not is not at all obvious. In this note the author gives a simple proof of the converse (Theorem 2) by first proving the following theorem on a necessary and sufficient condition for simultaneous diagonalization of two hermitian matrices.

THEOREM 1. *Let A and B be two $n \times n$ hermitian matrices with elements in F. Then A and B can be diagonalized simultaneously if and only if there exists a basis $\{u_1, \ldots, u_n\}$ of F^n such that, for each $i = 1, \ldots, n$, the two vectors u_iA and u_iB are linearly dependent over R.*

THEOREM 2. *Let $A = \text{diag}\{A_1, \ldots, A_m\}$ and $B = \text{diag}\{B_1, \ldots, B_m\}$ be two hermitian diagonal block matrices with elements in F such that, for each $k = 1, \ldots, m$, A_k and B_k are of the same size. If A and B can be diagonalized simultaneously, then so also can the pair A_k and B_k for each k.*

A theorem similar to Theorem 1, on the simultaneous diagonalization of two nondegenerate symmetric bilinear forms over a field of characteristic not equal to 2, has been established by M. J. Wonenburger [3, Theorem 1, p. 617].

2. Proof of Theorem 1. Suppose that A and B can be diagonalized simultaneously. Then there exists a basis $\{u_1, \ldots, u_n\}$ such that $u_iAu_j^* = u_iBu_j^* = 0$ for all $i \neq j$ $(i, j = 1, \ldots, n)$. Now, for each fixed i, if $u_iAu_i^* = u_iBu_i^* = 0$, then $u_iA = 0 = u_iB$, while if $u_iAu_i^*$ and $u_iBu_i^*$ are not both zero, then $(u_iBu_i^*)u_iA - (u_iAu_i^*)u_iB = 0$. Hence in both cases u_iA and u_iB are linearly dependent over R.

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To prove the sufficiency of the condition, suppose that there exists a basis $X = \{u_1, \ldots, u_n\}$ of F^n such that, for each $i = 1, \ldots, n$, $u_i A$ and $u_i B$ are linearly dependent over R. Then, for each i, there exist α_i, $\beta_i \in R$, not both zero, such that

$$\alpha_i u_i A + \beta_i u_i B = 0.$$

Now in the set $X = \{u_1, \ldots, u_n\}$ we define a relation $\sim$ by setting $u_i \sim u_j$ if $\alpha_i \beta_j - \alpha_j \beta_i = 0$. Obviously this is an equivalence relation. Let

$$X = X_1 \cup X_2 \cup \ldots \cup X_m$$

be the partition defined by this relation. Then, for each $k = 1, \ldots, m$, there exist $a_k, b_k \in R$, not both zero, such that

$$a_k uA + b_k uB = 0, \quad \text{for all} \quad u \in X_k; \tag{1}$$

$$a_k b_l - a_l b_k \neq 0, \quad \text{for all} \quad k \neq l \; (k, l = 1, \ldots, m). \tag{2}$$

From these two properties and $(uAv^*)^* = vAu^*$, it follows immediately that

$$uAv^* = uBv^* = 0, \quad \text{for all} \quad u \in X_k \quad \text{and} \quad v \in X_l \quad \text{with} \quad k \neq l. \tag{3}$$

Without loss of generality we may assume that $u_1, \ldots, u_{n_1} \in X_1$, $u_{n_1+1}, \ldots, u_{n_1+n_2} \in X_2, \ldots$, $u_{n_1+n_2+\cdots+n_{m-1}+1}, \ldots, u_n \in X_m$. Let U be the matrix whose elements in ith row are the components of u_i. Then U is non singular and, by (3), we have

$$UAU^* = \operatorname{diag}\{A_1, \ldots, A_m\},$$

$$UBU^* = \operatorname{diag}\{B_1, \ldots, B_m\},$$

where A_k and B_k are hermitian matrices of size n_k and, by (1), we have

$$a_k uAv + b_k uBv = 0 \quad \text{for all} \quad u, v \in X_k.$$

Hence

$$a_k A_k + b_k B_k = 0 \quad \text{for each} \quad k = 1, \ldots, m.$$

Since any hermitian matrix can be diagonalized (for $F = R$ or C, this is well-known; for $F = H$, see [1] or [2]) and a_k, b_k are not both zero, A_k and B_k can be diagonalized simultaneously for each k. Hence A and B can be diagonalized simultaneously.

3. Proof of Theorem 2. It suffices to prove the theorem for $m = 2$. Let $A = \operatorname{diag}\{A_1, A_2\}$ and $B = \operatorname{diag}\{B_1, B_2\}$, where A_1 and B_1 are of size n_1 and A_2 and B_2 are of size n_2, and let $n = n_1 + n_2$. If A and B can be diagonalized simultaneously, then, by Theorem 1, there exists a basis $\{u_1, \ldots, u_n\}$ of F^n such that, for each $i = 1, \ldots, n$, $u_i A$ and $u_i B$ are linearly dependent over R.

Let $u_i = (x_i, y_i)$, where $x_i \in F^{n_1}$ and $y_i \in F^{n_2}$. Then $(x_i A_1, y_i A_2)$ and $(x_i B_1, y_i B_2)$ are linearly dependent over R for each i. Hence $x_i A_1$ and $x_i B_1$ are linearly dependent over R for each i. Since $\{u_1, \ldots, u_n\}$ is a basis of F^n, there exists $\{x_{i_1}, \ldots, x_{i_{n_1}}\}$ which forms a basis of F^{n_1}. By Theorem 1, A_1 and B_1 can be diagonalized simultaneously. Similarly, A_2 and B_2 can be diagonalized simultaneously. This completes the proof.

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