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HARDY INEQUALITIES WITH MIXED NORMS

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ABSTRACT. We give a necessary and sufficient condition on weight functions u and v such that for $1 \le p \le q \le \infty$ there exists a constant C for which

$$\left(\int_0^\infty \left|u(x)\int_0^x f(t)\,dt\right|^q dx\right)^{1/q} \le C\left(\int_0^\infty |f(x)v(x)|^p\,dx\right)^{1/p}.$$

A corresponding dual result is also given. This extends a result of B. Muckenhoupt which appeared in Studia Math., 34 (1972).

1. **Introduction.** The classical Hardy inequality ([1], [2]) states that for $f(x) \ge 0$ and $p > 1$

$$\int_0^\infty \left[\frac{1}{x}\int_0^x f(t)\,dt\right]^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p\,dx. \tag{1.1}$$

Muckenhoupt, in [3], showed that the more general inequality

$$\left(\int_0^\infty \left|u(x)\int_0^x f(t)\,dt\right|^p dx\right)^{1/p} \le C\left(\int_0^\infty |f(x)v(x)|^p\,dx\right)^{1/p} \tag{1.2}$$

holds if and only if

$$\sup_{r>0}\left(\int_r^\infty |u(x)|^p\,dx\right)^{1/p}\left(\int_0^r |v(x)|^{-p'}\,dx\right)^{1/p'} = K < \infty \tag{1.3}$$

and $K \le C \le K(p)^{1/p}(p')^{1/p'}$. A similar result for the dual inequality

$$\left(\int_0^\infty \left|u(x)\int_x^\infty f(t)\,dt\right|^p dx\right)^{1/p} \le C\left(\int_0^\infty |f(x)v(x)|^p\,dx\right)^{1/p} \tag{1.4}$$

was also obtained.

2. **Generalized Hardy inequalities.** Our results are the following:

THEOREM 1. *Let $1 \le p \le q \le \infty$. Suppose u and v are non-negative. Then*

$$\left(\int_0^\infty \left[u(x)\int_0^x f(t)\,dt\right]^q dx\right)^{1/q} \le C\left(\int_0^\infty [f(x)v(x)]^p\,dx\right)^{1/p} \tag{2.1}$$

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holds for non-negative f if and only if

$$\sup_{r>0}\left(\int_r^\infty u(x)^q\,dx\right)^{1/q}\left(\int_0^r v(x)^{-p'}\,dx\right)^{1/p'}=K<\infty. \tag{2.2}$$

Furthermore $K\le C\le K(p)^{1/q}(p')^{1/p'}$ *for* $1<p<q<\infty$ *and* $K=C$ *if* $p=1$ *or* $q=\infty$.

The result on the constants is best possible since $K=C=1$ if u is the characteristic function of $[1,2]$ and v is 1 on $[0,1]$ and ∞ elsewhere while, if $p=q$, $C=Kp^{1/p}(p')^{1/p'}$ in the classical case.

Proof. To prove the theorem for $1<p\le q<\infty$ we first suppose that (2.1) holds. A reduction in the intervals of integration yields

$$\left(\int_r^\infty u(x)^q\,dx\right)^{1/q}\left(\int_0^r f(x)\,dx\right)\le C\left(\int_0^r [f(x)v(x)]^p\,dx\right)^{1/p}$$

and choosing $f(x)=v(x)-p'$ gives (2.2) with $K\le C$. To prove that (2.2) implies (2.1) we define $h(t)=(\int_0^t v(s)^{-p'}\,ds)^{1/pp'}$. Then by Hölder's inequality and Minkowski's integral inequality [4] we see that

$$\begin{aligned} I&\equiv\int_0^\infty\left[u(x)\int_0^x f(t)\,dt\right]^q dx\\ &\le\int_0^\infty u(x)^q\left(\int_0^\infty [f(t)v(t)h(t)\chi_{\{0\le t\le x\}}(x,t)]^p\,dt\right)^{q/p}\\ &\qquad\qquad\times\left(\int_0^x [v(s)h(s)]^{-p'}\,ds\right)^{q/p'}dx\\ &\le\left\{\int_0^\infty [f(t)v(t)h(t)]^p\left(\int_t^\infty u(x)^q\left(\int_0^x [v(s)h(s)]^{-p'}\,ds\right)^{q/p'}dx\right)^{p/q}dt\right\}^{q/p}. \end{aligned}$$

Performing the innermost integration yields

$$\left(\int_0^x [v(s)h(s)]^{-p'}\,ds\right)^{q/p'}=(p')^{q/p'}\left[\left(\int_0^x v(u)^{-p'}\,du\right)^{1/p'}\right]^{q/p'},$$

which by (2.2) is bounded by

$$K^{q/p'}(p')^{q/p'}\left(\int_x^\infty u(s)^q\,ds\right)^{-1/q}.$$

Hence

$$I\le(Kp')^{q/p'}\left\{\int_0^\infty [f(t)v(t)h(t)]^p\left(\int_t^\infty u(x)^q\left(\int_x^\infty u(s)^q\,ds\right)^{-1/p'}dx\right)^{p/q}dt\right\}^{q/p}$$

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and again evaluating the inner integral and applying (2.2) we obtain

$$\left(\int_t^\infty u(x)^q\left(\int_x^\infty u(s)^q\,ds\right)^{-1/p'}dx\right)^{p/q}=p^{p/q}\left(\int_t^\infty u(s)^q\,ds\right)^{1/q}$$
$$\leq Kp^{p/q}\left(\int_0^t v(s)^{-p'}\,ds\right)^{-1/p'}=Kp^{p/q}h(t)^{-p}.$$

Consequently

$$I\leq K^q p(p')^{q/p'}\left\{\int_0^\infty [f(t)v(t)]^p\,dt\right\}^{q/p},$$

which proves (2.1) with $C\leq K(p)^{1/q}(p')^{1/p'}$.

If $p=1$ and/or $q=\infty$ we show that (2.1) implies (2.2) by an argument which is essentially the same as the corresponding one used in [3] for the cases $p=1$ and ∞, and is hence omitted. To prove the reverse implication, if $p=1$ and $q<\infty$ we apply Minkowski's inequality to the left side of (2.1) while, if $1\leq p\leq q=\infty$ we use Hölder's inequality. The result follows immediately.

The following dual result is obtained analogously.

THEOREM 2. *Suppose that $1\leq p\leq q\leq\infty$ and that u and v are non-negative. Then*

$$\left(\int_0^\infty\left(u(x)\int_x^\infty f(t)\,dt\right)^q dx\right)^{1/q}\leq C\left(\int_0^\infty [f(x)v(x)]^p\,dx\right)^{1/p} \tag{2.3}$$

if and only if

$$\sup_{r>0}\left(\int_0^r u(x)^q\,dx\right)^{1/q}\left(\int_r^\infty u(x)^{-p'}\,dx\right)^{1/p'}=K<\infty. \tag{2.4}$$

In addition $K\leq C\leq K(p)^{1/q}(p')^{1/p'}$.

We single out the following specific case:

COROLLARY 1. *Let $1<p\leq q<\infty$. Then if $aq>1$, $bp<1$ and f is non-negative*

$$\left(\int_0^\infty\left[x^{-a}\int_0^x f(t)\,dt\right]^q dx\right)^{1/q}\leq C\left(\int_0^\infty\left[x^{-a+1/q+1/p'}f(x)\right]^p dx\right)^{1/p} \tag{2.5}$$

and

$$\left(\int_0^\infty\left[x^{-b}\int_x^\infty f(t)\,dt\right]^q dx\right)^{1/q}\leq C\left(\int_0^\infty\left[x^{-b+1/q+1/p'}f(x)\right]^p dx\right)^{1/p}. \tag{2.6}$$

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