A REMARK ON FINITELY GENERATED MODULES

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Theorem 5 of Azumaya's recent article[1] can be formulated in the following generalized form:

I. Let R be a ring. Let $\mathfrak{m}$ be a finitely generated right-module of R such that $\mathfrak{m}R = \mathfrak{m}$. Assume that $\mathfrak{m} = u_1\mathfrak{r} + u_2\mathfrak{r} + \ldots + u_m\mathfrak{r}$ for every generating system $u_1, u_2, \ldots, u_m$ of $\mathfrak{m}$ and for every maximal right-ideal $\mathfrak{r}$ of R. Then $\mathfrak{m} = 0$.

For the proof, we first consider the case where R possesses a unit element 1. Then the assertion can be proved quite similarly as in Azumaya, l. c. Let namely $u_1, u_2, \ldots, u_m$ be any finite generating system of $\mathfrak{m}$; $\mathfrak{m} = u_1R + u_2R + \ldots + u_mR$. Let $\mathfrak{r}_0$ be the right-ideal of R consisting of all elements x of R such that

$$u_1x \in u_2R + \ldots + u_mR.$$

Suppose $\mathfrak{r}_0 \neq R$. There exists a maximal right-ideal $\mathfrak{r}$ which contains $\mathfrak{r}_0$, and we have $\mathfrak{m} = u_1\mathfrak{r} + u_2\mathfrak{r} + \ldots + u_m\mathfrak{r}$, whence $= u_1\mathfrak{r} + u_2R + \ldots + u_mR$ much the more, by our assumption. There is an element a in $\mathfrak{r}$ such that $1 - a \in \mathfrak{r}_0 \subseteq \mathfrak{r}$, which is a contradiction. Hence necessarily $\mathfrak{r}_0 = R$ and $\mathfrak{m} = u_2R + \ldots + u_mR$. Now the assertion can be proved by an induction with respect to the minimal number of generating elements.

Let next R be general. Let R^* be the ring which is as module a direct sum of R and the ring of rational integers and in which $1x = x1 = x$ $(x \in R)$. If $\mathfrak{r}^*$ is a maximal right-ideal of R^*, then $\mathfrak{r}^* \cap R$ is either R or a maximal right-ideal of R. Thus $\mathfrak{m} = u_1\mathfrak{r}^* + u_2\mathfrak{r}^* + \ldots + u_m\mathfrak{r}^*$, much the more, and the assertion can be reduced to the above case.

It is perhaps of interest to observe that from this generalization Jacobson's theorem[2] may be derived:

Received May 15, 1951.

[1] G. Azumaya, On maximally central algebras, Nagoya Math. J. **2** (1951).

[2] N. Jacobson, The radical and semi-simplicity for arbitrary rings, Amer. J. Math. **67** (1945), Theorem 10; the present formulation is in 1), l. c.

II. Let R be a ring and N be its radical. If $\mathfrak{m}$ is a finitely generated right-module of R and if $\mathfrak{m}=\mathfrak{m}N$, then $\mathfrak{m}=0$.

For, the radical N^* of R^*, as above, contains (coincides with, as a matter of fact) N. Hence $\mathfrak{m}N^*=\mathfrak{m}$. Then $\mathfrak{m}=u_1N^*+u_2N^*+\ldots+u_mN^*$ for any generating system $u_1, u_2, \ldots, u_m$. Since every maximal right-ideal of R^* contains N^*, the assertion is an immediate consequence of I.

Next we want to observe:

III. In I we may restrict ourselves to those maximal right-ideals $\mathfrak{r}$ which contain the radical N.

We consider namely the residue-module $\mathfrak{m}/\mathfrak{m}N$ and find, by virtue of I, $\mathfrak{m}=\mathfrak{m}N$. Then $\mathfrak{m}=0$ because of II. (It would also be possible to prove III directly.)

Now, it is clear that the radical is the largest two-sided ideal possessing the property of II. Namely:

IV. If M is a two-sided ideal and if $\mathfrak{m}=0$ is the only finitely generated module with $\mathfrak{m}=\mathfrak{m}M$, then $M\subseteqq N$.

For, if $M\not\subseteqq N$, there exists a maximal right-ideal $\mathfrak{r}$ with left modulo-unit a, such that $\mathfrak{r}\not\supseteqq M$. The right-module $\mathfrak{m}=R/\mathfrak{r}$ of R, generated by a (mod $\mathfrak{r}$), satisfies $\mathfrak{m}=\mathfrak{m}M$, since $aM\equiv M\equiv R$ mod $\mathfrak{r}$.

Similary, the family of right-ideals of III gives the "natural boundary," i.e.:

V. *A family of right-ideals $\{\mathfrak{r}'\}$ of R possesses the property of* I *(with $\{\mathfrak{r}'\}$ in place of $\{\mathfrak{r}\}$) if and only if for every maximal right-ideal $\mathfrak{r}$ with left modulo-unit there exists an $\mathfrak{r}'$ in the family such that $\mathfrak{r}'\subseteqq\mathfrak{r}$.*

It is evident that if this is the case then $\{\mathfrak{r}'\}$ possesses the required property. On the other hand, if there exists a maximal right-ideal $\mathfrak{r}$ with left modulo-unit a such that $\mathfrak{r}\not\supseteqq\mathfrak{r}'$ for every $\mathfrak{r}'\in\{\mathfrak{r}'\}$, then $\mathfrak{m}=R/\mathfrak{r}$ satisfies $\mathfrak{m}=\mathfrak{m}\mathfrak{r}'$ for every $\mathfrak{r}'$, as in IV.

Needless to say that I, II, III and IV are special cases of V. Further, we may replace "finitely generated module" by ("cyclic module," or) "minimal module" in V (and IV) (and then the statement includes also a well known characterization of the radical).

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