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FINITE *p*-GROUPS AND COCLASS THEORY

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A finite p-group is a group whose order is a power of a prime p. The classification of finite p-groups by order is a difficult problem mainly because the number of groups (up to isomorphism) grows rapidly with order. Leedham-Green and Newman [10] suggested classifying finite p-groups by coclass. The coclass of a finite p-group of order p^n and nilpotency class c is $n - c$. Blackburn [1] classified the 2- and 3-groups of coclass 1, but a classification for primes greater than 3 is significantly more difficult (see [2, 3] for recent work). However, many structural results were obtained from the proofs of the famous coclass conjectures A–E (see, for example, [9, 14]). A useful feature of coclass theory is that finite p-groups of coclass r can be visualised by a graph, the so-called coclass graph $\mathcal{G}(p, r)$.

The coclass graph. The vertices of $\mathcal{G}(p, r)$ are the isomorphism types of finite p-groups of coclass r. Two vertices are connected, say $G \to H$, if and only if $G \cong H/\gamma(H)$, where $\gamma(H)$ is the last nontrivial term of the lower central series of H. It is a deep result that $\mathcal{G}(p, r)$ consists of finitely many infinite trees, so-called coclass trees, and finitely many groups outside these trees. Each coclass tree contains a unique infinite path (mainline) starting at its root. Explicit computations have revealed surprising patterns in these trees, which in some cases have been proved on a group-theoretic level [2, 7, 8, 11]. This has led to a number of open conjectures about the structure of $\mathcal{G}(p, r)$. A recent description can be found in [3, Appendix A]. Since coclass trees are the building blocks of coclass graphs, they have become one of the main interests in coclass theory. In the thesis [13], we concentrate on two avenues for obtaining more insight into their structure.

Uniserial *p*-adic space groups. The inverse limit of the groups on the mainline of a coclass tree in $\mathcal{G}(p, r)$ is an infinite pro-p-group of coclass r and, conversely, any

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such pro-p-group defines an infinite path in $\mathcal{G}(p, r)$. As a result, these pro-p-groups, especially the uniserial p-adic space groups, play an important role in coclass theory (see [6, 9, 10]). For odd primes, a constructive classification of such space groups was given by Eick [6]; however; the prime 2 poses severe complications. In [4, 13], we provide the theoretical description for determining all uniserial 2-adic space groups up to isomorphism.

Skeleton groups. Coclass graphs have in general a very intricate structure. One possible way to look into these graphs is to restrict attention to the subgraph spanned by the so-called skeleton groups. Unlike other groups in $\mathcal{G}(p, r)$, skeleton groups can be conveniently parametrised by certain homomorphisms. For some special cases, this structure has been used in the literature [2, 3, 8]. In the thesis [13] we develop a systematic treatment of skeleton groups. It turns out that almost every group in $\mathcal{G}(p, r)$ is close to a skeleton group (see [5]). Since the groups are parametrised by certain homomorphisms, a crucial problem is to decide when two homomorphisms define isomorphic groups. We investigate this problem and provide complete solutions for two important cases. We utilise our results to derive some new periodicity results for coclass trees. Some of this research has been published in [4, 5, 12], where [4, 12] are in preparation.

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