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ABOUT THE ZEROS OF SOME ENTIRE FUNCTIONS AND THEIR DERIVATIVES

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Abstract

In this article we localize the zeros of some polynomials and the derivatives of some entire functions of finite genus. If we put $m = 1$ in the condition of Theorem 1 we obtain the famous Obreshkoff Theorem which can be regarded as a 'complex version' of a well-known theorem due to Laguerre. The nonreal zeros of the derivative of the real entire function of Theorem 3 must belong to circles V_k which are similar to the Jensen circles for polynomials.

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DEFINITION. By 'real entire function' will be meant an entire function whose Maclaurin-series expansion has only real coefficients.

Further all sequences $\{z_k\}_{k=1}^{\infty}$ will satisfy $\lim_{k\to\infty} |z_k| = \infty$.

DEFINITION. We will write ${}_nP_k = n!/(n-k)!$, $\binom{n}{k} = {}_nP_k/k!$.

THEOREM 1. *Let the zeros a_k, $k = 1, \dots, n$, of the polynomial $p(z)$ satisfy $|a_k| \le 1$. Then the zeros z of the polynomial $q(z) = \gamma p(z) + \sum_{k=1}^{m} z^k p^{(k)}(z)/{}_nP_k$ satisfy $|z| \le \theta$, where* $\operatorname{Re}\gamma \ge -m/2$, $1 \le m \le n$, *and* $\theta = (2^{1/m} - 1)^{-1}$.

PROOF. Let z be such that $q(z) = 0$ and $p(z) \ne 0$. Then

$$A = \frac{q(z)}{p(z)} = \gamma + \frac{1}{n}\left[\frac{z}{z-a_1} + \cdots + \frac{z}{z-a_n}\right] + \cdots + \frac{m!}{{}_nP_m}\left[\frac{z^m}{(z-a_1)\cdots(z-a_m)} + \cdots\right] = 0.$$

Let us consider the last term s_m in this sum. The estimates for the other terms are analogous. Define

$$A = \gamma + s_1 + \cdots + s_m = 0,$$
$$B = \binom{n}{m} s_m = \frac{z^m}{(z-a_1)\cdots(z-a_m)} + \cdots + \frac{z^m}{(z-a_{n-m+1})\cdots(z-a_n)},$$
$$a = z^m - (z-a_1)\cdots(z-a_m).$$

Then

$$C = \frac{z^m}{(z-a_1)\cdots(z-a_m)} = \frac{z^m}{(z^m - a)} = \frac{1}{2} + \frac{z^m + a}{2(z^m - a)},$$
$$D = \operatorname{Re}\frac{z^m + a}{z^m - a} = \frac{|z^m|^2 - |a|^2}{|z^m - a|^2} = \frac{\big(|z|^m - |a|\big)\big(|z|^m + |a|\big)}{|z^m - a|^2},$$
$$|a| \le |a_1 + \cdots + a_m|\,|z|^{m-1} + \cdots + |a_1 \cdots a_m| \le m|z|^{m-1} + \binom{m}{2}|z|^{m-2} + \cdots + 1.$$

Thus

$$|z|^m - |a| \ge 2|z|^m - \left(|z|^m + m|z|^{m-1} + \binom{m}{2}|z|^{m-2} + \cdots + 1\right)$$
$$= 2|z|^m - \big(|z| + 1\big)^m.$$

If $|z| > \theta = (2^{1/m} - 1)^{-1}$ we obtain $D > 0$, that is $\operatorname{Re} s_m > 1/2$. Finally if we note that $\theta_m = (2^{1/m} - 1)^{-1}$, then obviously $1 = \theta_1 < \theta_2 < \cdots < \theta_m$ and therefore $\operatorname{Re} A = \operatorname{Re}(\gamma + s_1 + \cdots + s_m) > 0$ which proves the theorem. □

REMARK. We can formulate Theorem 1 in the following form: Let the zeros a_k, $k = 1, \dots, n$, of the polynomial $p(z)$ satisfy $|a_k| \le 1$. Then the zeros z of the polynomial $q(z) = \gamma p(z) + \sum_{k=1}^{m} z^k p^{(k)}(z)$ satisfy $|z| \le \theta$, where $\operatorname{Re}\gamma \ge -1/2 \sum_{k=1}^{m} {}_nP_k$, $1 \le m \le n$, and $\theta = (2^{1/m} - 1)^{-1}$.

THEOREM 2. *Let* $f(z) = \exp(d + cz^m + bz^{m+1} + az^{m+2}) \prod_{k=1}^{\infty} E_m(z/z_k)$ *be a real entire function, where m is a positive integer,* $a, b, c, d \in \mathbb{R}$, $a \ge 0$, $b \le 0$, $c \ge 0$, *and the Weierstrass factors are*

$$E_m(\zeta) = (1 - \zeta)\exp\big(\zeta + \zeta^2/2 + \cdots + \zeta^m/m\big).$$

Let all zeros z_k *of* $f(z)$ *satisfy* $\operatorname{Arg}(z_k) \in \big(-\pi/[2m+2], \pi/[2m+2]\big)$. *Then all zeros* z *of* $f'(z)$ *satisfy* $\operatorname{Re}(z) \ge 0$.

PROOF. Let z be such that $f'(z) = 0$, and write $z = x + iy$, where $x, y \in \mathbb{R}, x < 0$. Then

$$\begin{aligned} A &= (\log f(z))' = \frac{f'(z)}{f(z)} \\ &= mcz^{m-1} + (m+1)bz^m + (m+2)az^{m+1} \\ &\quad + \sum_{k=1}^{\infty} \left[\frac{1}{z - z_k} + \frac{1}{z_k} + \frac{z}{z_k^2} + \cdots + \frac{1}{z_k} \left(\frac{z}{z_k} \right)^{m-1} \right] \\ &= \frac{mcz^m \bar{z}}{|z|^2} + (m+1)bz^m + (m+2)az^m z + \sum_{k=1}^{\infty} \frac{(z/z_k)^m}{z - z_k} \\ &= z^m \left(\frac{mc\bar{z}}{|z|^2} + (m+1)b + (m+2)az + \sum_{k=1}^{\infty} \frac{\left(\overline{z - z_k}\right)/z_k^m}{|z - z_k|^2} \right) = 0. \end{aligned}$$

Let

$$B = \frac{mc\bar{z}}{|z|^2} + (m+1)b + (m+2)az + \sum_{k=1}^{\infty} \frac{\left(\overline{z - z_k}\right)/z_k^m}{|z - z_k|^2}.$$

If $f(z_k) = 0$ then $f(\bar{z}_k) = 0$, since $f(z)$ is real entire function. Let $z_k = x_k + iy_k$, $\bar{z}_k = x_k - iy_k$, where $x_k, y_k \in \mathbb{R}$. Then

$$\begin{aligned} C &= 2B - \left[\frac{2mc\bar{z}}{|z|^2} + 2(m+1)b + 2(m+2)az \right] \\ &= \sum_{k=1}^{\infty} \left\{ \frac{\left(\overline{z - z_k}\right)/z_k^m}{|z - z_k|^2} + \frac{(\bar{z} - z_k)/\bar{z}_k^m}{|z - \bar{z}_k|^2} \right\} \\ &= \sum_{k=1}^{\infty} \left[\bar{z}_k^m \left(\overline{z - z_k}\right) |z - \bar{z}_k|^2 + z_k^m (\bar{z} - z_k) |z - z_k|^2 \right] / D_k, \end{aligned}$$

where $D_k = |z_k^m|^2 |z - \bar{z}_k|^2 |z - z_k|^2$.

Let $r = x - x_k$, $q = y - y_k$, $s = y + y_k$ and $z_k = p_k(\cos\varphi_k + i \sin\varphi_k)$, and

$$\begin{aligned} \Delta_k &= \left[\bar{z}_k^m \left(\overline{z - z_k}\right) |z - \bar{z}_k|^2 + z_k^m (\bar{z} - z_k) |z - z_k|^2 \right] / p_k^m \\ &= [\cos(m\varphi_k) - i \sin(m\varphi_k)](r - iq) \left(r^2 + q^2\right) \\ &\quad + [\cos(m\varphi_k) + i \sin(m\varphi_k)](r - is) \left(r^2 + s^2\right), \end{aligned}$$

and

$$\begin{aligned} \operatorname{Re} \Delta_k &= [r \cos(m\varphi_k) - q \sin(m\varphi_k)] \left(r^2 + s^2\right) + [r \cos(m\varphi_k) + s \sin(m\varphi_k)] \left(r^2 + q^2\right) \\ &= r \cos(m\varphi_k) \left(2r^2 + s^2 + q^2\right) + \sin(m\varphi_k) \left(r^2 s + q^2 s - r^2 q - s^2 q\right) \end{aligned}$$

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$$= 2r\cos(m\varphi_k)\left(r^2+y^2+y_k^2\right) + 2y_k\sin(m\varphi_k)\left(r^2-y^2+y_k^2\right)$$
$$= 2\left(r^2+y_k^2\right)[r\cos(m\varphi_k)+y_k\sin(m\varphi_k)]+2y^2[r\cos(m\varphi_k)-y_k\sin(m\varphi_k)].$$

Since, by hypothesis, $\varphi_k \in (-\pi/[2m+2], \pi/[2m+2])$, we have $\cos(m\varphi_k) > 0$, and if we assume that $[r\cos(m\varphi_k) - y_k\sin(m\varphi_k)] \geq 0$, then $(x - x_k)\cos(m\varphi_k) + y_k\sin(m\varphi_k) \leq 0$. Since $x < 0$, $x_k = p_k\cos\varphi_k$ and $y_k = p_k\sin\varphi_k$, we obtain

$$x_k\cos(m\varphi_k) + y_k\sin(m\varphi_k) \leq 0, \qquad \text{or}$$
$$\cos\varphi_k\cos(m\varphi_k) + \sin\varphi_k\sin(m\varphi_k) \leq 0,$$

that is $\cos[(m-1)\varphi_k] \leq 0$, which is impossible since $\varphi_k \in (-\pi/[2m+2], \pi/[2m+2])$. Therefore, $[r\cos(m\varphi_k) - y_k\sin(m\varphi_k)] < 0$. If we assume that $[r\cos(m\varphi_k) + y_k\sin(m\varphi_k)] \geq 0$, by the same way we obtain $\cos[(m+1)\varphi_k] \leq 0$, which is impossible since $\varphi_k \in (-\pi/[2m+2], \pi/[2m+2])$. Therefore, $[r\cos(m\varphi_k) + y_k\sin(m\varphi_k)] < 0$ and $\operatorname{Re}\Delta_k < 0$, that is $\operatorname{Re} B < 0$, because $a \geq 0$, $b \leq 0$, $c \geq 0$. But $A = z^m B = 0$, that is $B = 0$. The contradiction completes the proof. □

THEOREM 3. *Let $f(z) = \exp(d + cz^m + bz^{m+1} + az^{m+2})\prod_{k=1}^{\infty} E_m(z/z_k)$ be a real entire function, where m is a positive integer and $a, b, c, d \in \mathbb{R}$, $a \leq 0$, $c \geq 0$. Let all zeros z_k of $f(z)$ satisfy $\varphi_k \in (-\pi/[2m], \pi/[2m])$, where $\varphi_k = \operatorname{Arg}(z_k)$. Let V_k be the disk*

$$V_k = \left\{ |z - \operatorname{Re} z_k| \leq |\operatorname{Im} z_k| \frac{1 + |\sin(m\varphi_k)|}{\cos(m\varphi_k)} \right\},$$

and let $M = \bigcup_{k=1}^{\infty} V_k$. Then if $f'(z)$ has nonreal roots, they must belong to M.

PROOF. Let z be such that $f'(z) = 0$ and $z \notin \mathbb{R}$, where $z = x + iy$, $x, y \in \mathbb{R}$, $y > 0$, $z \neq z_k$ and $z \notin M$. Let $A = (\log f(z))' = f'(z)/f(z)$. As in Theorem 2 we obtain:

$$\Delta_k = \left[\bar{z}_k^m \left(\overline{z - z_k}\right)|z - \bar{z}_k|^2 + z_k^m\left(\bar{z} - z_k\right)|z - z_k|^2\right]/p_k^m$$
$$= [\cos(m\varphi_k) - i\sin(m\varphi_k)](r - iq)\left(r^2 + s^2\right)$$
$$+ [\cos(m\varphi_k) + i\sin(m\varphi_k)](r - is)\left(r^2 + q^2\right).$$

Thus

$$-\operatorname{Im}\Delta_k = [q\cos(m\varphi_k) + r\sin(m\varphi_k)]\left(r^2+s^2\right) + [s\cos(m\varphi_k) - r\sin(m\varphi_k)]\left(r^2+q^2\right)$$
$$= 2y\left(r^2 + y^2 - y_k^2\right)\cos(m\varphi_k) + 4yy_k r\sin(m\varphi_k).$$

Let $C_k = \cos(m\varphi_k)\left(r^2 + y^2 - y_k^2\right) + 2\sin(m\varphi_k)y_k r$ and $R = |z - \operatorname{Re} z_k|$, so that $r^2 + y^2 = R^2$, and therefore $r = \varepsilon\sqrt{R^2 - y^2}$, where $\varepsilon = \pm 1$. We have

$$C_k \geq \cos(m\varphi_k)\left(R^2 - y_k^2\right) - 2|\sin(m\varphi_k)||y_k|R$$

$$= \cos(m\varphi_k)R^2 - 2|\sin(m\varphi_k)||y_k|R - \cos(m\varphi_k)y_k^2$$
$$= \cos(m\varphi_k)(R - R_1)(R - R_2),$$

where

$$R_1 = |y_k|\frac{|\sin(m\varphi_k)| - 1}{\cos(m\varphi_k)}, \qquad R_2 = |y_k|\frac{|\sin(m\varphi_k)| + 1}{\cos(m\varphi_k)}.$$

Because $\varphi_k \in (-\pi/[2m], \pi/[2m])$, we have $\cos(m\varphi_k) > 0$. Then if $R > |\operatorname{Im} z_k|\big[1 + |\sin(m\varphi_k)|\big]/\cos(m\varphi_k)$, we obtain $C_k > 0$. Hence $\operatorname{Im}\Delta_k = -2yC_k < 0$, that is $\operatorname{Im} B < 0$, because $a \leq 0$, $c \geq 0$, and from proof of Theorem 2 we know that

$$C = 2B - \left[\frac{2mc\bar{z}}{|z|^2} + 2(m+1)b + 2(m+2)az\right]$$
$$= \sum_{k=1}^{\infty}\left\{\frac{(\overline{z - z_k})/z_k^m}{|z - z_k|^2} + \frac{(\bar{z} - z_k)/\bar{z}_k^m}{|z - \bar{z}_k|^2}\right\}$$
$$= \sum_{k=1}^{\infty}\left[\bar{z}_k^m(\overline{z - z_k})|z - \bar{z}_k|^2 + z_k^m(\bar{z} - z_k)|z - z_k|^2\right]/D_k = \sum_{k=1}^{\infty}\Delta_k/D_k,$$

where $D_k = |z_k^m|^2|z - \bar{z}_k|^2|z - z_k|^2$.

But $A = z^m B = 0$, that is $B = 0$. This contradiction proves the theorem. □

REMARK. Theorem 3 remains true if we change the condition $\varphi_k \in (-\pi/[2m], \pi/[2m])$ to the condition $\cos(m\varphi_k) > 0$.

COROLLARY 1. *Let $f(z) = \exp(d + cz^m + bz^{m+1} + az^{m+2})\prod_{k=1}^{\infty} E_m(z/z_k)$ be a real entire function, where m is a positive integer and $a, b, c, d \in \mathbb{R}$, $a \leq 0$, $c \geq 0$ and all zeros z_k of $f(z)$ are real and positive. Then all zeros of $f'(z)$ are real.*

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