Canad. Math. Bull. Vol. **57** (4), 2014 pp. 845–852
http://dx.doi.org/10.4153/CMB-2013-044-2

Factorisation of Two-variable p-adic L-functions

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Abstract. Let f be a modular form that is non-ordinary at p. Loeffler has recently constructed four two-variable p-adic L-functions associated with f. In the case where $a_p = 0$, he showed that, as in the one-variable case, Pollack's plus and minus splitting applies to these new objects. In this article, we show that such a splitting can be generalised to the case where $a_p \neq 0$ using Sprung's logarithmic matrix.

1 p-adic Logarithmic Matrices

We first review the theory of Sprung's factorisation of one-variable p-adic L-functions in [Spr12b, Spr12a], which is a generalisation of Pollack's work [Pol03].

Let $f = \sum_{n\geq 1} a_n q^n$ be a normalised eigen-newform of weight 2 and level N with nebentypus ϵ. Fix an odd[1] prime p that does not divide N and $v_p(a_p) \neq 0$. Here, v_p is the normalised p-adic valuation with $v_p(p) = 1$. Let α and β be the two roots to

$$X^2 - a_p X + \epsilon(p)p = 0$$

with $r = v_p(\alpha)$ and $s = v_p(\beta)$. Note in particular that $0 < r, s < 1$.

Let G be a one-dimensional p-adic Lie group, which is of the form $\Delta \times \langle \gamma_p \rangle$, where Δ is a finite abelian group and $\langle \gamma_p \rangle \cong \mathbb{Z}_p$. If H is a subset of G, we write 1_H for the indicator function of H on G. Let F be a finite extension of $\mathbb{Q}_p$ that contains $\mu_{|\Delta|}$, a_n and $\epsilon(n)$ for all $n \geq 1$. For a real number $u \geq 0$, we define $D^{(u)}(G, F)$ to be the set of distributions μ on G such that for a fixed integer $n \geq 0$,

$$\inf_{g\in G} v_p\left(\mu\left(1_{g\langle\gamma_p\rangle^{p^n}}\right)\right) \geq R - un$$

for some constant $R \in \mathbb{R}$ that only depends on μ. Note that we can identify $D^{(u)}(G, F)$ with the set of power series

$$\sum_{n\geq 0}\sum_{\sigma\in\Delta} c_{\sigma,n}\sigma(\gamma_p - 1)^n,$$

where $c_{\sigma,n} \in F$ and $\sup_{n>0} (|c_{\sigma,n}|_p)/n^u < \infty$ for all $\sigma \in \Delta$ (here $|\cdot|_p$ denotes the p-adic norm with $|p|_p = p^{-1}$). Let $X = \gamma_p - 1$. If η is a character on Δ, we write

Received by the editors April 29, 2013; revised September 10, 2013.
Published electronically November 26, 2013.
The author is supported by a CRM-ISM fellowship.
AMS subject classification: **11S40**, 11S80.
Keywords: modular forms, p-adic L-functions, supersingular primes.

[1]Our results in fact hold for $p = 2$. Since the interpolation formulae of p-adic L-functions are slightly different from the other cases, we assume that $p \neq 2$ for notational simplicity.

$e_\eta\mu$ for the η-isotypical component of μ, namely, the power series

$$\sum_{n\geq 0}\sum_{\sigma\in\Delta} c_{\sigma,n}\eta(\sigma)(\gamma_p-1)^n \in F[[X]].$$

For $\mu_1 \in D^{(u)}(\langle\gamma_p\rangle, F)$ and $\mu_2 \in D^{(u)}(G,F)$, we say that μ_1 divides μ_2 over $D^{(u)}(G,F)$ if μ_1 divides all isotypical components of μ_2 as elements in $F[[X]]$.

Definition 1.1 We say that $(\mu_\alpha,\mu_\beta) \in D^{(r)}(G,F)\oplus D^{(s)}(G,F)$ is a pair of interpolating functions for f if for all nontrivial characters ω on G that send γ_p to a primitive p^{n-1}-st root of unity for some $n\geq 1$, there exists a constant $C_\omega \in \overline{F}$ such that

$$\mu_\alpha(\omega) = \alpha^{-n}C_\omega \quad\text{and}\quad \mu_\beta(\omega) = \beta^{-n}C_\omega.$$

Remark 1.2 The p-adic L-functions L_α, L_β of Amice–Vélu [AV75] and Višik [Viš76] associated with f satisfy the property stated above, with G being the Galois group $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ and C_ω being the algebraic part of the complex L-value $L(f,\omega^{-1},1)$ multiplied by some fudge factor.

Definition 1.3 A matrix

$$M_p = \begin{pmatrix} m_{1,1} & m_{1,2}\\ m_{2,1} & m_{2,2}\end{pmatrix}$$

with $m_{1,1}, m_{2,1} \in D^{(r)}(\langle\gamma_p\rangle, F)$ and $m_{1,2}, m_{2,2} \in D^{(s)}(\langle\gamma_p\rangle, F)$, is called a p-adic logarithmic matrix associated with f if $\det(M_p)$ is, up to a constant in $F^\times$, equal to $\log_p(\gamma_p)/(\gamma_p-1)$, and $\det(M_p)$ divides both $m_{2,2}\mu_\alpha - m_{2,1}\mu_\beta$ and $-m_{1,2}\mu_\alpha + m_{1,1}\mu_\beta$ over $D^{(1)}(G,F)$ for all interpolating functions μ_α, μ_β for f.

Lemma 1.4 *Let μ_α, μ_β be a pair of interpolating functions for f. If M_p is a p-adic logarithmic matrix associated with f, then there exist $\mu_\#, \mu_\flat \in D^{(0)}(G,F)$ such that*

$$\begin{pmatrix}\mu_\alpha & \mu_\beta\end{pmatrix} = \begin{pmatrix}\mu_\# & \mu_\flat\end{pmatrix} M_p.$$

Proof Let

$$\mu_\# := \frac{m_{2,2}\mu_\alpha - m_{2,1}\mu_\beta}{\det(M_p)} \quad\text{and}\quad \mu_\flat := \frac{-m_{1,2}\mu_\alpha + m_{1,1}\mu_\beta}{\det(M_p)}.$$

By definition, the numerators lie inside $D^{(1)}(G,F)$, and the coefficients of $\det(M_p)$ have the same growth rate as those of $\log_p(\gamma_p)$, so $\mu_\#$ and $\mu_\flat$ lie inside $D^{(0)}(G,F)$. The factorisation follows from the fact that

$$\begin{pmatrix} m_{2,2} & -m_{1,2}\\ -m_{2,1} & m_{1,1}\end{pmatrix} M_p = \begin{pmatrix}\det(M_p) & 0\\ 0 & \det(M_p)\end{pmatrix}. \qquad \blacksquare$$

We now recall the construction of Sprung's canonical p-adic logarithmic matrix associated with f.

Let

$$C_n = \begin{pmatrix} a_p & 1\\ -\epsilon(p)\Phi_{p^n}(\gamma_p) & 0\end{pmatrix},$$

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where Φ_{p^n} denotes the p^n-th cyclotomic polynomial for $n \geq 1$,

$$C = \begin{pmatrix} a_p & 1 \\ -\epsilon(p)p & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -1 & -1 \\ \beta & \alpha \end{pmatrix}.$$

Define $M_p^{(n)} := C_1 \cdots C_n C^{-n-2} A$.

Theorem 1.5 (Sprung) *The entries of the sequence of matrices $M_p^{(n)}$ converge in $D^{(1)}(\langle \gamma_p \rangle, F)$ as $n \to \infty$ (under the standard sup-norm on p-adic power series), and the limit $\lim_{n\to\infty} M_p^{(n)}$ is a p-adic logarithmic matrix associated with f.*

Proof We only sketch our proof here, since this is merely a slight generalisation of Sprung's results in [Spr12a, Spr12b].

Since $C_{n+1} \equiv C \mod (X+1)^{p^n} - 1$, we have

$$M_p^{(n+1)} \equiv M_p^{(n)} \mod (X+1)^{p^n} - 1.$$

Note that $A^{-1}CA = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$, which implies that

$$C^{-n-2} A = A \begin{pmatrix} \alpha^{-n-2} & 0 \\ 0 & \beta^{-n-2} \end{pmatrix} = \begin{pmatrix} -\alpha^{-n-2} & -\beta^{-n-2} \\ \beta\alpha^{-n-2} & \alpha\beta^{-n-2} \end{pmatrix}. \tag{1.1}$$

Since all the entries in $C_1 \cdots C_n$ are integral, the coefficients of the first (resp., second) row of $M_p^{(n)}$ grow like $O(p^{-rn})$ (resp., $O(p^{-sn})$) as $n \to \infty$. Therefore, by [PR94, §1.2.1], the entries of the first (resp., second) row of $M_p^{(n)}$ converge to elements in $D^{(r)}(\langle \gamma_p \rangle, F)$ (resp., $D^{(s)}(\langle \gamma_p \rangle, F)$).

Let

$$M_p = \begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{pmatrix}$$

be the limit $\lim_{n\to\infty} M_p^{(n)}$. If ω is a character that sends γ_p to a primitive p^{n-1}-st root of unity, then

$$M_p(\omega) = C_1(\omega) \cdots C_{n-1}(\omega) C^{-n-1} A.$$

Note that $C_{n-1}(\omega) = \left(\begin{smallmatrix} a_p & 1 \\ 0 & 0 \end{smallmatrix}\right)$, so from (1.1), we see that there exist two constants $A_\omega, B_\omega \in \overline{F}$ such that

$$M_p(\omega) = \begin{pmatrix} a_p A_\omega & A_\omega \\ a_p B_\omega & B_\omega \end{pmatrix} \begin{pmatrix} -\alpha^{-n-1} & -\beta^{-n-1} \\ \beta\alpha^{-n-1} & \alpha\beta^{-n-1} \end{pmatrix} = \begin{pmatrix} -\alpha^{-n} A_\omega & -\beta^{-n} A_\omega \\ -\alpha^{-n} B_\omega & -\beta^{-n} B_\omega \end{pmatrix}.$$

In particular, if μ_α, μ_β is a pair of interpolating functions for f,

$$m_{2,2}(\omega)\mu_\alpha(\omega) - m_{2,1}(\omega)\mu_\beta(\omega) = -m_{1,2}(\omega)\mu_\alpha(\omega) + m_{1,1}(\omega)\mu_\beta(\omega) = 0.$$

By [Spr12a, Remark 2.19],

$$\det(M_p) = \frac{\log_p(1+X)}{X} \times \frac{\beta - \alpha}{(\alpha\beta)^2}.$$

But $\alpha \neq \beta$ by [CB98, Theorem 2.1], hence the result. ∎

Remark 1.6 Similar logarithmic matrices have been constructed in [LLZ10] using the theory of Wach modules, but they are not canonical.

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2 Two-variable p-adic L-functions

2.1 Setup for Two-variable Distributions

We now fix an imaginary quadratic field K in which p splits into $\mathfrak{p}\overline{\mathfrak{p}}$. If $\mathfrak{I}$ is an ideal of K, we write $G_{\mathfrak{I}}$ for the ray class group of K modulo $\mathfrak{I}$. Define

$$G_{p^\infty} = \varprojlim G_{p^n}, \quad G_{\mathfrak{p}^\infty} = \varprojlim G_{\mathfrak{p}^n}, \quad G_{\overline{\mathfrak{p}}^\infty} = \varprojlim G_{\overline{\mathfrak{p}}^n}.$$

These are the Galois groups of the ray class fields $K(p^\infty)$, $K(\mathfrak{p}^\infty)$, and $K(\overline{\mathfrak{p}}^\infty)$ respectively. Fix topological generators $\gamma_{\mathfrak{p}}$ and $\gamma_{\overline{\mathfrak{p}}}$ of the $\mathbb{Z}_p$-parts of $G_{\mathfrak{p}^\infty}$ and $G_{\overline{\mathfrak{p}}^\infty}$ respectively. We have an isomorphism

$$G_{p^\infty} \cong \Delta \times \langle\gamma_{\mathfrak{p}}\rangle \times \langle\gamma_{\overline{\mathfrak{p}}}\rangle,$$

where Δ is a finite abelian group. For real numbers $u, v \geq 0$, we define $D^{(u,v)}(G_{p^\infty}, F)$ to be the set of distributions μ of G_{p^∞} such that for fixed integers $m, n \geq 0$,

$$\inf_{g \in G_{p^\infty}} v_p\Big(\mu\big(1_{g\langle\gamma_{\mathfrak{p}}\rangle^{p^m}\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}}\big)\Big) \geq R - um - vn$$

for some constant $R \in \mathbb{R}$ that only depends on μ.

Let $X = \gamma_{\mathfrak{p}} - 1$ and $Y = \gamma_{\overline{\mathfrak{p}}} - 1$. We can identify an element of $D^{(u,v)}(G_{p^\infty}, F)$ with a power series

$$\sum_{i,j \geq 0} \sum_{\sigma \in \Delta} c_{\sigma,i,j} \sigma X^i Y^j,$$

where $c_{\sigma,i,j} \in F$. Upon identifying each Δ-isotypical component of μ with a power series in X and Y, we have the notion of divisibility, as in the one-dimensional case. We define the operator $\partial_{\mathfrak{p}}$ to be the partial derivative $\frac{\partial}{\partial X}$.

For $\star \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, we let $\Omega_\star$ be the set of characters on G_{p^∞} with conductor $(\star)^n$ for some integer $n \geq 1$.

Let $\mu \in D^{(u,v)}(G_{p^\infty}, F)$, where $u, v \geq 0$. If $\omega_{\mathfrak{p}} \in \Omega_{\mathfrak{p}}$, then we define a distribution $\mu^{(\omega_{\mathfrak{p}})}$ by $\mu^{(\omega_{\mathfrak{p}})}(\omega_{\overline{\mathfrak{p}}}) = \mu(\omega_{\mathfrak{p}}\omega_{\overline{\mathfrak{p}}})$.

Lemma 2.1 *The distribution $\mu^{(\omega_{\mathfrak{p}})}$ lies inside $D^{(v)}(G_{\overline{\mathfrak{p}}^\infty}, F')$, where F' is the extension $F(\omega_{\mathfrak{p}}(\gamma_{\mathfrak{p}}))$.*

Proof By definition, for any integers $m, n \geq 0$, we have

$$\inf_{g \in G_{p^\infty}} v_p\Big(\mu\big(1_{g\langle\gamma_{\mathfrak{p}}\rangle^{p^m}\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}}\big)\Big) \geq R - um - vn \tag{2.1}$$

for some R. Since $\omega_{\mathfrak{p}}$ is of finite order, we have

$$\omega_{\mathfrak{p}} = \sum_{h \in G_{p^\infty}/\ker(\omega_{\mathfrak{p}})} \omega_{\mathfrak{p}}(h) 1_{h\ker(\omega_{\mathfrak{p}})}.$$

Moreover,

$$v_p\big(\omega_{\mathfrak{p}}(h)\big) = 0 \tag{2.2}$$

for all h and $\ker(\omega_{\mathfrak{p}}) \cap \langle\gamma_{\mathfrak{p}}\rangle\langle\gamma_{\overline{\mathfrak{p}}}\rangle = \langle\gamma_{\mathfrak{p}}\rangle^{p^m}\langle\gamma_{\overline{\mathfrak{p}}}\rangle$ for some integer m.

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If $g \in G_{\overline{\mathfrak{p}}^\infty}$ and $n \geq 0$ is an integer, we can lift the coset $g\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}$ in $G_{\overline{\mathfrak{p}}^\infty}$ to one in G_{p^∞} that is of the form $g'\langle\gamma_{\mathfrak{p}}\rangle\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}$. Therefore,

$$\begin{aligned}\mu^{(\omega_{\mathfrak{p}})}\big(1_{g\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}}\big) &= \mu\Big(\sum_{h\in G_{p^\infty}/\ker(\omega_{\mathfrak{p}})} \omega_{\mathfrak{p}}(h)1_{h\ker(\omega_{\mathfrak{p}})}1_{g'\langle\gamma_{\mathfrak{p}}\rangle\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}}\Big)\\ &= \sum_{h\in G_{p^\infty}/\ker(\omega_{\mathfrak{p}})} \omega_{\mathfrak{p}}(h)\mu\big(1_{h\ker(\omega_{\mathfrak{p}})\cap g'\langle\gamma_{\mathfrak{p}}\rangle\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}}\big)\\ &= \sum_{\substack{h\in G_{p^\infty}/\ker(\omega_{\mathfrak{p}})\\ h\in g'\langle\gamma_{\mathfrak{p}}\rangle\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}}} \omega_{\mathfrak{p}}(h)\mu\big(1_{h\langle\gamma_{\mathfrak{p}}\rangle^{p^m}\langle\gamma_{\overline{\mathfrak{p}}}\rangle^{p^n}}\big).\end{aligned}$$

Therefore, by (2.1) and (2.2), we have

$$v_p\Big(\mu^{(\omega_{\mathfrak{p}})}\big(1_{g\langle\gamma_{\overline{\mathfrak{p}}}\rangle^n}\big)\Big) \geq (R-um)-vn,$$

as required. ∎

Similarly, for $\omega_{\overline{\mathfrak{p}}} \in \Omega_{\overline{\mathfrak{p}}}$, we can define a distribution $\mu^{(\omega_{\overline{\mathfrak{p}}})} \in D^{(u)}(G_{\mathfrak{p}^\infty}, F')$, where $F' = F(\omega_{\overline{\mathfrak{p}}}(\gamma_{\overline{\mathfrak{p}}}))$.

2.2 Sprung-type Factorisation

Let $L_{\alpha,\alpha}$, $L_{\alpha,\beta}$, $L_{\beta,\alpha}$, $L_{\beta,\beta}$ be the two-variable p-adic L-functions constructed in [Loe13] (note that $L_{\alpha,\alpha}$ and $L_{\beta,\beta}$ have also be constructed in [Kim11]). By [Loe13, Theorem 4.7], $L_{\star,\bullet}$ is an element of $D^{(v_p(\star),v_p(\bullet))}(G_{p^\infty}, F)$ for $\star,\bullet \in \{\alpha,\beta\}$. Moreover, if ω is a character on G_{p^∞} of conductor $\mathfrak{p}^{n_{\mathfrak{p}}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$ with $n_{\mathfrak{p}}, n_{\overline{\mathfrak{p}}} \geq 1$, we have

(2.3) $$L_{\alpha,\alpha}(\omega) = \alpha^{-n_{\mathfrak{p}}}\alpha^{-n_{\overline{\mathfrak{p}}}}C_\omega$$

(2.4) $$L_{\alpha,\beta}(\omega) = \alpha^{-n_{\mathfrak{p}}}\beta^{-n_{\overline{\mathfrak{p}}}}C_\omega$$

(2.5) $$L_{\beta,\alpha}(\omega) = \beta^{-n_{\mathfrak{p}}}\alpha^{-n_{\overline{\mathfrak{p}}}}C_\omega$$

(2.6) $$L_{\beta,\beta}(\omega) = \beta^{-n_{\mathfrak{p}}}\beta^{-n_{\overline{\mathfrak{p}}}}C_\omega$$

for some $C_\omega \in \overline{F}$ that is independent of α and β.

Let M_p be the logarithmic matrix given by Theorem 1.5. On replacing γ_p by $\gamma_{\mathfrak{p}}$ and $\gamma_{\overline{\mathfrak{p}}}$, respectively, we have two logarithmic matrices

$$M_{\mathfrak{p}} = \begin{pmatrix} m_{1,1}^{\mathfrak{p}} & m_{1,2}^{\mathfrak{p}}\\ m_{2,1}^{\mathfrak{p}} & m_{2,2}^{\mathfrak{p}}\end{pmatrix} \quad\text{and}\quad M_{\overline{\mathfrak{p}}} = \begin{pmatrix} m_{1,1}^{\overline{\mathfrak{p}}} & m_{1,2}^{\overline{\mathfrak{p}}}\\ m_{2,1}^{\overline{\mathfrak{p}}} & m_{2,2}^{\overline{\mathfrak{p}}}\end{pmatrix}$$

defined over $D^{(1)}(\langle\gamma_{\mathfrak{p}}\rangle, F)$ and $D^{(1)}(\langle\gamma_{\overline{\mathfrak{p}}}\rangle, F)$ respectively.

Our goal is to prove the following generalisation of [Loe13, Corollary 5.4].

Theorem 2.2 *There exist $L_{\#,\#}, L_{\flat,\#}, L_{\#,\flat}, L_{\flat,\flat} \in D^{(0,0)}(G_{p^\infty}, F)$ such that*

$$\begin{pmatrix} L_{\alpha,\alpha} & L_{\beta,\alpha} & L_{\alpha,\beta} & L_{\beta,\beta}\end{pmatrix} = \begin{pmatrix} L_{\#,\#} & L_{\flat,\#} & L_{\#,\flat} & L_{\flat,\flat}\end{pmatrix} M_{\mathfrak{p}} \otimes M_{\overline{\mathfrak{p}}}.$$

We shall prove this theorem in two steps, namely, to show that we can first factor out $M_{\mathfrak{p}}$, then $M_{\overline{\mathfrak{p}}}$.

Proposition 2.3 *For $\star \in \{\alpha, \beta\}$, there exist $L_{\sharp,\star}, L_{\flat,\star} \in D^{(0,v_p(\star))}(G_{p^\infty}, F)$ such that*

$$\begin{pmatrix} L_{\alpha,\star} & L_{\beta,\star} \end{pmatrix} = \begin{pmatrix} L_{\sharp,\star} & L_{\flat,\star} \end{pmatrix} M_{\mathfrak{p}}. \tag{2.7}$$

Proof We take $\star = \alpha$ (since the proof for the case $\star = \beta$ is identical). Let $\omega_{\mathfrak{p}} \in \Omega_{\mathfrak{p}}$ and $\omega_{\overline{\mathfrak{p}}} \in \Omega_{\overline{\mathfrak{p}}}$ and write $\omega = \omega_{\mathfrak{p}}\omega_{\overline{\mathfrak{p}}}$.

By (2.3) and (2.5), $L_{\alpha,\alpha}^{(\omega_{\overline{\mathfrak{p}}})}$ and $L_{\beta,\alpha}^{(\omega_{\overline{\mathfrak{p}}})}$ is a pair of interpolating functions for f. In particular, $\det(M_{\mathfrak{p}})$ divides both

$$m_{2,2}^{\mathfrak{p}} L_{\alpha,\alpha}^{(\omega_{\overline{\mathfrak{p}}})} - m_{2,1}^{\mathfrak{p}} L_{\beta,\alpha}^{(\omega_{\overline{\mathfrak{p}}})} \quad \text{and} \quad -m_{1,2}^{\mathfrak{p}} L_{\alpha,\alpha}^{(\omega_{\overline{\mathfrak{p}}})} + m_{1,1}^{\mathfrak{p}} L_{\beta,\alpha}^{(\omega_{\overline{\mathfrak{p}}})}$$

over $D^{(1)}(G_{\mathfrak{p}^\infty}, F)$. Therefore, the distributions

$$m_{2,2}^{\mathfrak{p}} L_{\alpha,\alpha} - m_{2,1}^{\mathfrak{p}} L_{\beta,\alpha} \quad \text{and} \quad -m_{1,2}^{\mathfrak{p}} L_{\alpha,\alpha} + m_{1,1}^{\mathfrak{p}} L_{\beta,\alpha}$$

vanish at all characters of the form $\omega = \omega_{\mathfrak{p}}\omega_{\overline{\mathfrak{p}}}$. This implies that

$$\left(m_{2,2}^{\mathfrak{p}} L_{\alpha,\alpha} - m_{2,1}^{\mathfrak{p}} L_{\beta,\alpha}\right)^{(\omega_{\mathfrak{p}})} = \left(-m_{1,2}^{\mathfrak{p}} L_{\alpha,\alpha} + m_{1,1}^{\mathfrak{p}} L_{\beta,\alpha}\right)^{(\omega_{\mathfrak{p}})} = 0,$$

since these two distributions lie inside $D^{(r)}(G_{\overline{\mathfrak{p}}^\infty}, F')$ for some F' with $r < 1$, and they vanish at an infinite number of characters for each of their isotypical components. Hence, $\det(M_{\mathfrak{p}})$ divides

$$m_{2,2}^{\mathfrak{p}} L_{\alpha,\alpha} - m_{2,1}^{\mathfrak{p}} L_{\beta,\alpha} \quad \text{and} \quad -m_{1,2}^{\mathfrak{p}} L_{\alpha,\alpha} + m_{1,1}^{\mathfrak{p}} L_{\beta,\alpha}$$

over $D^{(1,r)}(G_{p^\infty}, F)$. Let

$$L_{\sharp,\alpha} := \frac{m_{2,2}^{\mathfrak{p}} L_{\alpha,\alpha} - m_{2,1}^{\mathfrak{p}} L_{\beta,\alpha}}{\det(M_{\mathfrak{p}})} \quad \text{and} \quad L_{\flat,\alpha} := \frac{-m_{1,2}^{\mathfrak{p}} L_{\alpha,\alpha} + m_{1,1}^{\mathfrak{p}} L_{\beta,\alpha}}{\det(M_{\mathfrak{p}})}.$$

We can then conclude as in the proof of Lemma 1.4. ∎

Lemma 2.4 *Let ω be a character of G_{p^∞} of conductor $\mathfrak{p}^{n_{\mathfrak{p}}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$ with $n_{\mathfrak{p}}, n_{\overline{\mathfrak{p}}} \geq 1$. There exist constants D_ω and E_ω in $\overline{F}$ such that*

$$\begin{aligned} \partial_{\mathfrak{p}} L_{\alpha,\alpha}(\omega) &= \alpha^{-n_{\overline{\mathfrak{p}}}} D_\omega, & \partial_{\mathfrak{p}} L_{\alpha,\beta}(\omega) &= \beta^{-n_{\overline{\mathfrak{p}}}} D_\omega, \\ \partial_{\mathfrak{p}} L_{\beta,\alpha}(\omega) &= \alpha^{-n_{\overline{\mathfrak{p}}}} E_\omega, & \partial_{\mathfrak{p}} L_{\beta,\beta}(\omega) &= \beta^{-n_{\overline{\mathfrak{p}}}} E_\omega. \end{aligned}$$

Proof We only prove the result concerning $\partial_{\mathfrak{p}} L_{\alpha,\alpha}$ and $\partial_{\mathfrak{p}} L_{\alpha,\beta}$. Fix an $\omega_{\overline{\mathfrak{p}}} \in \Omega_{\overline{\mathfrak{p}}}$. By (2.7) and (2.8), we have

$$\beta^{n_{\overline{\mathfrak{p}}}} L_{\alpha,\beta}^{(\omega_{\overline{\mathfrak{p}}})}(\omega_{\mathfrak{p}}) = \alpha^{n_{\overline{\mathfrak{p}}}} L_{\alpha,\alpha}^{(\omega_{\overline{\mathfrak{p}}})}(\omega_{\mathfrak{p}})$$

for all $\omega_{\mathfrak{p}} \in \Omega_{\mathfrak{p}}$. But $L_{\alpha,\beta}^{(\omega_{\overline{\mathfrak{p}}})}, L_{\alpha,\alpha}^{(\omega_{\overline{\mathfrak{p}}})} \in D^{(r)}(G_{\mathfrak{p}^\infty}, F')$ for some F'. As $r < 1$, this implies that

$$\beta^{n_{\overline{\mathfrak{p}}}} L_{\alpha,\beta}^{(\omega_{\overline{\mathfrak{p}}})} = \alpha^{n_{\overline{\mathfrak{p}}}} L_{\alpha,\alpha}^{(\omega_{\overline{\mathfrak{p}}})}.$$

In particular, their derivatives agree, that is

$$\beta^{n_{\overline{\mathfrak{p}}}} \partial_{\mathfrak{p}} L_{\alpha,\beta}^{(\omega_{\overline{\mathfrak{p}}})} = \alpha^{n_{\overline{\mathfrak{p}}}} \partial_{\mathfrak{p}} L_{\alpha,\alpha}^{(\omega_{\overline{\mathfrak{p}}})}.$$

But for a general $\mu \in D^{(r,s)}(G_{p^\infty}, F)$, we have

$$\partial_{\mathfrak{p}} \left(\mu^{(\omega_{\overline{\mathfrak{p}}})}\right)(\omega_{\mathfrak{p}}) = \partial_{\mathfrak{p}} \mu(\omega_{\mathfrak{p}}\omega_{\overline{\mathfrak{p}}})$$

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for all $\omega_{\mathfrak{p}} \in \Omega_{\mathfrak{p}}$, hence

$$\beta^{n_{\overline{\mathfrak{p}}}} \partial_{\mathfrak{p}} L_{\alpha,\beta}(\omega) = \alpha^{n_{\overline{\mathfrak{p}}}} \partial_{\mathfrak{p}} L_{\alpha,\alpha}(\omega)$$

as required. ■

Proposition 2.5 *For $\star \in \{\#, \flat\}$, there exist $L_{\star,\#}, L_{\star,\flat} \in D^{(0,0)}(G_{p^\infty}, F)$ such that*

$$\begin{pmatrix} L_{\star,\alpha} & L_{\star,\beta} \end{pmatrix} = \begin{pmatrix} L_{\star,\#} & L_{\star,\flat} \end{pmatrix} M_{\overline{\mathfrak{p}}}. \tag{2.8}$$

Proof Let us prove the proposition for $\star = \#$. Let $\omega_{\mathfrak{p}} \in \Omega_{\mathfrak{p}}$ and $\omega_{\overline{\mathfrak{p}}} \in \Omega_{\overline{\mathfrak{p}}}$ and write $\omega = \omega_{\mathfrak{p}}\omega_{\overline{\mathfrak{p}}}$. Recall that

$$L_{\#,\bullet} \det(M_{\mathfrak{p}}) = m_{2,2}^{\mathfrak{p}} L_{\alpha,\bullet} - m_{2,1}^{\mathfrak{p}} L_{\beta,\bullet}$$

for $\bullet \in \{\alpha, \beta\}$. Since $\det(M_{\mathfrak{p}})$ is, up to a nonzero constant in $F^\times$, equal to $\log_p(1+X)/X$, we have $\det(M_p)(\omega_{\mathfrak{p}}) = 0$ and $\partial_{\mathfrak{p}} \det(M_{\mathfrak{p}})(\omega_{\mathfrak{p}}) \neq 0$. On taking partial derivatives, Lemma 2.4 together with (2.3)–(2.6) imply that

$$L_{\#,\bullet}(\omega) = (\bullet)^{-n_{\overline{\mathfrak{p}}}} \frac{K_\omega}{\partial_{\mathfrak{p}} \det(M_{\mathfrak{p}})(\omega_{\mathfrak{p}})},$$

where K_ω is the constant

$$m_{2,2}^{\mathfrak{p}}(\omega_{\mathfrak{p}}) D_\omega + \partial_{\mathfrak{p}} m_{2,2}^{\mathfrak{p}}(\omega_{\mathfrak{p}}) \alpha^{-n_{\mathfrak{p}}} C_\omega - m_{2,1}^{\mathfrak{p}}(\omega_{\mathfrak{p}}) E_\omega - \partial_{\mathfrak{p}} m_{2,1}^{\mathfrak{p}}(\omega_{\mathfrak{p}}) \beta^{-n_{\mathfrak{p}}} C_\omega.$$

In particular, we see that $L_{\#,\alpha}^{(\omega_{\mathfrak{p}})}$ and $L_{\#,\beta}^{(\omega_{\mathfrak{p}})}$ is a pair of interpolating functions for f, so we can proceed as in the proof of Proposition 2.3 (with the roles of $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ swapped). ■

Combining the factorisations (2.7) and (2.8), we obtain Theorem 2.2. Note that our proof is very different from that of [Loe13, Corollary 5.4]. In fact, it only relies on the properties of logarithmic matrices as specified in Definition 1.3. Therefore, if we replace M_p by any logarithmic matrices, Theorem 2.2 still holds. For example, one can take M_p to be the noncanonical logarithmic matrices mentioned in Remark 1.6.

Acknowledgments The author would like to thank David Loeffler, Henri Darmon, and the anonymous referee for their very helpful comments and suggestions.

References

[AV75] Y. Amice and J. Vélu, *Distributions p-adiques associées aux séries de Hecke*. In: Journées Arithmétiques de Bordeaux (Conf., Univ. Bordeaux, Bordeaux, 1974), Astérisque, 24–25, Soc. Math. France, Paris, 1975, pp. 119–131.

[CB98] R. Coleman and B. Edixhoven, *On the semi-simplicity of the u_p-operator on modular forms*. Math. Ann. **310**(1998), no 1, 119–127. http://dx.doi.org/10.1007/s002080050140

[Kim11] B. D. Kim, *Two-variable p-adic L-functions of modular forms for non-ordinary primes*. Preprint. http://homepages.ecs.vuw.ac.nz/~bdkim/BDKim-2011-1.pdf

[LLZ10] A. Lei, D. Loeffler, and S. L. Zerbes, *Wach modules and Iwasawa theory for modular forms*. Asian J. Math. **14**(2010), no. 4, 475–528. http://dx.doi.org/10.4310/AJM.2010.v14.n4.a2

[Loe13] D. Loeffler, *p-adic integration on ray class groups and non-ordinary p-adic L-functions*. arxiv:1304.4042, 2013.

[PR94] B. Perrin-Riou, *Théorie d'Iwasawa des représentations p-adiques sur un corps local*. Invent. Math. **115**(1994), no. 1, 81–161. http://dx.doi.org/10.1007/BF01231755

[Pol03] R. Pollack, *On the p-adic L-function of a modular form at a supersingular prime*. Duke Math. J. **118**(2003), no. 3, 523–558. http://dx.doi.org/10.1215/S0012-7094-03-11835-9

[Spr12a] F. Sprung, *On pairs of p-adic analogues of the conjectures of Birch and Swinnerton-Dyer* arxiv:1211.1352, 2012.

[Spr12b] ———, *Iwasawa theory for elliptic curves at supersingular primes: a pair of main conjectures*. J. Number Theory **132**(2012), no. 7, 1483–1506. http://dx.doi.org/10.1016/j.jnt.2011.11.003

[Viš76] M. M. Višik, *Nonarchimedean measures associated with Dirichlet series*. Mat. Sb. (N.S.) **99**(**141**)(1976), no. 2, 248–260, 296.

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