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ORTHOGONAL FAMILIES OF SETS

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In this note we answer in the affirmative the following question.

If S is a finite family of sets such that each subset of a member of S also belongs to S, is it possible to arrange the sets in pairs so that the first member of the pair is contained in the complement of the second member?

This question was posed by Erdös and Schönheim and an independent solution with a generalization to samples will be published by Herzog and Schönheim.

The affirmative answer is an immediate corollary of the following somewhat sharper theorem.

THEOREM. *If* $S=\{S_i : i \in N, N=\{1, \ldots, n\}\}$ *is a finite family of sets with the property that the difference of any pair of members of S belongs to S; then there is a permutation* σ *of N such that* $S_i \cap S_{\sigma(i)}=\phi$ *for each i in N.*

Proof. Let S_i^* consist of those members of S which are disjoint from S_i and for each $M\subseteq N$ let $M^*=\bigcup\{S_i^* : i \in M\}$. Let ΔM be the set of differences S_i-S_j; $i, j \in M$, which by hypothesis also belong to S. The existence of the required permutation is equivalent to the existence of a set of distinct representatives of $\{S_i^*, i \in N\}$. By a well known theorem of Hall such a set of distinct representatives exists if and only if, for each M, $|M^*|\geq|M|$. By the theorem of [1] we have $|\Delta M|\geq|M|$ and since $\Delta M\subseteq M^*$ we obtain $|M^*|\geq M$. The theorem follows.

REFERENCE

1. J. Marica and J. Schönheim, *Differences of sets and a problem of Graham*, Canad. Math. Bull. (5), **12** (1969), 635–637.

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