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OSCILLATIONS OF DELAY DIFFERENTIAL EQUATIONS

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Abstract

Sufficient conditions are established for all solutions of the linear system

$$\frac{dy_i(t)}{dt} + \sum_{j=1}^{n} q_{ij} y_j(t - \tau_{ij}) = 0, \qquad i = 1, 2, \dots, n,$$

to be oscillatory, where $q_{ij} \in (-\infty, \infty)$, $\tau_{jj} \in (0, \infty)$, $i, j = 1, 2, \dots, n$.

1. Introduction

Consider the system of delay differential equations

$$\frac{dy_i(t)}{dt} + \sum_{j=1}^{n} q_{ij} y_j(t - \tau_{ij}) = 0, \qquad i = 1, 2, \dots, n \tag{1}$$

where the coefficients are real numbers and the delays are positive real numbers. We say that a solution

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} \tag{2}$$

of (1) oscillates if for some $i \in (1, 2, \dots, n)$, $y_i(t)$ has arbitrarily large zeros. A solution $y(t)$ of (1) is said to be nonoscillatory if there exists a $t_0 \geq 0$ such that for each $i = 1, 2, \dots, n$, $y_i(t) \neq 0$ for $t \geq t_0$. The aim of this brief paper is to derive a set of sufficient conditions for all solutions

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of (1) to oscillate. Our result is an extension of a result of Gopalsamy in [2], where only bounded solutions of systems like (1) have been considered. For references concerning the oscillation of systems, the reader is referred to the references in [2].

2. Sufficient conditions for oscillation

The following lemma will be useful in the proof of our theorem below.

LEMMA 1. *Assume that* (1) *has a nonoscillatory solution* (2). *Then there are numbers*

$$\delta_i \in \{-1, 1\} \quad \textit{for } i = 1, 2, \dots, n$$

such that the system

$$\frac{dz_i(t)}{dt} + \sum_{j=1}^{n} p_{ij} z_j(t - \tau_{jj}) = 0 \tag{3}$$

where

$$p_{ij} = \frac{\delta_j}{\delta_i} q_{ij} \quad \textit{for } i, j = 1, 2, \dots, n \tag{4}$$

has a nonoscillatory solution $[z_1(t), z_2(t), \dots, z_n(t)]^{\mathsf{T}}$ *with eventually positive components* $z_i(t)$, $i = 1, 2, \dots, n$.

PROOF. The components $y(t)$ of (2) are positive or negative eventually. That is, there exists a $T \geq 0$ such that $y_i(t) \neq 0$ for $t \geq T$ and $i = 1, 2, \dots, n$. Set $\delta_i = \operatorname{sign}[y_i(t)]$, $i = 1, 2, \dots, n$ and $t \geq T$. It is now easy to see that

$$z(t) = [\delta_1 y_1(t), \delta_2 y_2(t), \dots, \delta_n y_n(t)]^{\mathsf{T}} \tag{5}$$

satisfies (3) and $\delta_i y_i(t) > 0$ for $i = 1, 2, \dots, n$ and $t \geq T$.

The next result is concerned with the asymptotic behaviour of nonoscillatory solutions of (1).

LEMMA 2. *Consider the system* (1) *and suppose that the constant coefficients of* (1) *satisfy*

$$q = \min_{1 \leq i \leq n} \left[q_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |q_{ji}| \right] > 0. \tag{6}$$

Then every nonoscillatory solution $y(t) = (y_1, y_2, \dots, y_n)$ *satisfies*

$$\lim_{t \to \infty} y_i(t) = 0.$$

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PROOF. Clearly (6) is also satisfied with the q_{ij} replaced by the respective p_{ij} of (4). From this and (5) it suffices to prove the lemma for nonoscillatory solutions of (2) with eventually positive components. Let us assume that there is a $t_0 \geq 0$ such that $y_i(t) > 0$ for $t \geq t_0$, $i = 1, 2, \ldots, n$. If we let

$$w(t) = \sum_{j=1}^{n} y_j(t), \quad t \geq t_0 \tag{7}$$

then

$$\frac{dw(t)}{dt} + \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} y_j(t - \tau_{jj}) = 0$$

or

$$\frac{dw(t)}{dt} + \sum_{i=1}^{n} q_{ii} y_i(t - \tau_{ii}) = -\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} q_{ij} y_j(t - \tau_{jj})$$

$$\leq \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} |q_{ji}| y_i(t - \tau_{ii}). \tag{8}$$

It follows from (8) that

$$\frac{dw(t)}{dt} + \sum_{i=1}^{n}\left\{ q_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} |q_{ji}| \right\} y_i(t - \tau_{ii}) \leq 0. \tag{9}$$

An integration of both sides of (9) leads to

$$w(t) + q\int_{t_0+\tau}^{t} \sum_{i=1}^{n} y_i(s - \tau_{ii})\, ds \leq w(t_0 + \tau) \tag{10}$$

where $\tau = \max_{1\leq j\leq n} \tau_{jj}$. A consequence of (10) is that w is bounded and $y_i \in L_1(t_0 + \tau, \infty)$ for $i = 1, 2, \ldots, n$. From the boundedness of w one can conclude that of y_i since $w(t) = \sum_{i=1}^{n} y_i(t)$ and $y_i(t) > 0$ eventually. It will now follow from (1) that $\dot{y}_i$ is bounded for $t \geq \tau$, and therefore y_i is uniformly continuous on $[0, \infty)$. The uniform continuity of y_i on $[0, \infty)$, the eventual positivity of y_i and the integrability of y_i on a half-line together with a lemma of Barbalat [1], will imply that $\lim_{t\to\infty} y_i(t) = 0$, $i = 1, 2, \ldots, n$ and this completes the proof.

THEOREM. *Let* $q_{ij} \in (-\infty, \infty)$, $\tau_{jj} \in (0, \infty)$, $i, j = 1, 2, \ldots, n$. *If*

$$q\tau_* > \frac{1}{e} \quad \textit{where } q = \min_{1\leq i\leq n}\left(q_{ii} \sum_{\substack{j=1 \\ j\neq i}}^{n} |q_{ji}| \right), \quad \tau_* = \min_{1\leq i\leq n} \tau_{ii} \tag{11}$$

then every solution of (1) *oscillates.*

PROOF. Assume for the sake of a contradiction that (1) has a nonoscillatory solution (2). In view of Lemma 1 we can assume that the components of $y_i(t)$ are eventually positive for $i = 1, 2, \ldots, n$. We have directly from (1) that

$$\sum_{i=1}^{n} \frac{dy_i(t)}{dt} + \sum_{j=1}^{n}\sum_{i=1}^{n} q_{ij} y_j(t-\tau_{jj}) = 0$$

which satisfies

$$\sum_{i=1}^{n}\left[\frac{dy_i(t)}{dt}\right] + \sum_{i=1}^{n}\left(q_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n} |q_{ji}|\right) y_i(t-\tau_{ii}) \leq 0. \tag{12}$$

We have from (12) that $w(t) = \sum_{i=1}^{n} y_i(t)$ satisfies

$$\frac{dw(t)}{dt} + \sum_{i=1}^{n}\left[q_{ii} - \sum_{\substack{j=1\\ j\neq i}} |q_{ji}|\right] y_i(t-\tau_{ii}) \leq 0. \tag{13}$$

Integrating both sides of (13) over (t, ∞) and using the fact

$$w(t) \to 0 \quad \text{as } t\to\infty \quad (\text{since } y_i(t)\to 0,\ i = 1, 2, \ldots, n)$$

we derive that

$$-w(t) + q\int_t^{\infty} \sum_{i=1}^{n} y_i(s-\tau_{ii}) \leq 0 \tag{14}$$

and this leads to

$$w(t) \geq q\int_t^{\infty} \sum_{i=1}^{n} y_i(s-\tau_{ii})\,ds. \tag{15}$$

It is found from (15) that

$$w(t) \geq q\int_{t-\tau_*}^{\infty} \sum_{i=1}^{n} y_i(s)\,ds, \qquad \tau_* = \min_{1\leq i\leq n} \tau_{ii} \tag{16}$$

or

$$w(t) \geq q\int_{t-\tau_*}^{\infty} w(s)\,ds. \tag{17}$$

Now we let

$$F(t) = \int_{t-\tau_*}^{\infty} w(s)\,ds \tag{18}$$

and derive from (17) and (18) that

$$\begin{aligned}\frac{dF(t)}{dt} &= -w(t-\tau_*) \\ &\leq -qF(t-\tau_*); \qquad t > 2\tau_*. \end{aligned} \tag{19}$$

It follows from (19) that F is an eventually positive solution of

$$\frac{dF(t)}{dt} + qF(t-\tau_*) \leq 0; \qquad t > 2\tau_*. \tag{20}$$

But it is well known (from Ladas and Stavroulakis [3]) that when (11) holds, (20) cannot have an eventually positive solution and this contradiction completes the proof.

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