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INEQUALITIES FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

BY

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ABSTRACT. Bernstein's inequality says that if f is an entire function of exponential type τ which is bounded on the real axis then

$$\max_{-\infty<x<\infty} |f'(x)| \le \tau \max_{-\infty<x<\infty} |f(x)|.$$

Genchev has proved that if, in addition, $h_f(\pi/2) \le 0$, where h_f is the indicator function of f, then

$$\max_{-\infty<x<\infty} |f'(x)| \le \tau \max_{-\infty<x<\infty} |\mathrm{Re}\, f(x)|.$$

Using a method of approximation due to Lewitan, in a form given by Hörmander, we obtain, to begin, a generalization and a refinement of Genchev's result. Also, we extend to entire functions of exponential type two results first proved for polynomials by Rahman. Finally, we generalize a theorem of Boas concerning trigonometric polynomials vanishing at the origin.

1. **Introduction and statement of results.** Let B_τ be the class of entire functions of exponential type τ which are bounded on the real axis. A result of S. N. Bernstein says that if $f \in B_\tau$ then [3]:

$$|f'(x)| \le \tau \max_{-\infty<t<\infty} |f(t)|, \qquad -\infty < x < \infty. \tag{1}$$

Equality in (1) holds only if

$$f(z) = ae^{-i\tau z} + be^{i\tau z}, \qquad a, b \in \mathbb{C}.$$

Genchev [9] has proved that if $f \in B_\tau$ and $h_f(\pi/2) \le 0$, where

$$h_f(\theta) := \overline{\lim_{r\to\infty}} \frac{\log |f(re^{i\theta})|}{r}$$

is the indicator function of f, then

$$|f'(x)| \le \tau \max_{-\infty<t<\infty} |\mathrm{Re}\, f(t)|, \qquad -\infty < x < \infty. \tag{2}$$

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The inequality (2) extends a result of Szegö: if $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $\leq n$ then the function $f(z)=P(e^{iz})$ satisfies the hypothesis of Genchev's result and (2) becomes (see [15]):

$$|P'(z)| \leq n \max_{|\xi|=1} |\operatorname{Re} P(\xi)|, \qquad |z| \leq 1. \tag{3}$$

In this article we first obtain a generalization of (2).

THEOREM 1. *Let $f \in B_\tau$ such that $h_f(\pi/2) \leq 0$ and $|\operatorname{Re} f(x)| \leq 1$ for $-\infty < x < \infty$. Then*

$$|f(x+iy)-f(x)| \leq (e^{-\tau y}-1), \qquad y \leq 0, \qquad -\infty < x < \infty.$$

We shall also prove the following theorem which is stronger than Genchev's result.

THEOREM 2. *Let $f \in B_\tau$ such that $h_f(\pi/2) \leq 0$ and $|\operatorname{Re} f(x)| \leq 1$ for $-\infty < x < \infty$. Then*

$$|\operatorname{Re}(\tau f(x)+if'(x))|+|f'(x)| \leq \tau, \qquad -\infty < x < \infty.$$

It is to be noted that the case $f(z)=P(e^{iz})$, where P is a polynomial of degree $\leq n$, of Theorem 2 gives us a refinement of (3).

To prove Theorems 1 and 2 we use a method of approximation due to Lewitan [13] in a form given by Hörmander [11]. This method turns out to be very useful and we use it to obtain the next results.

THEOREM 3. *Let α, β, γ be complex numbers such that the roots of the polynomial*

$$u(z):=(\alpha\tau^2+i\beta\tau-\gamma)+2i\tau(2i\alpha\tau-\beta)z+4\alpha\tau^2z^2$$

lie in $\operatorname{Re}(z) \leq \frac{1}{2}$. If $f \in B_\tau$ and $|f(x)| \leq 1$ for $-\infty < x < \infty$ then

$$|\alpha f''(x+iy)+\beta f'(x+iy)+\gamma f(x+iy)| \leq |-\alpha\tau^2+i\beta\tau+\gamma|\, e^{-\tau y},$$
$$y \leq 0, \qquad -\infty < x < \infty.$$

This theorem extends a result of Rahman on trigonometric polynomials [14, Theorem 5]. Also, suppose that $f \in B_\tau$ is real on the real axis and that $|f(x)| \leq 1$ for $-\infty < x < \infty$. Choosing $\alpha=0$, $\beta=f'(x)$, $\gamma=\tau^2 f(x)$ and taking $y=0$ we see that Theorem 3 contains an inequality of Duffin and Schaeffer [8]:

$$(f'(x))^2+(\tau f(x))^2 \leq \tau^2, \qquad -\infty < x < \infty. \tag{4}$$

THEOREM 4. *Let α, β, γ be complex numbers such that the roots of the polynomial*

$$v(z):=\gamma+i\beta\tau z-\alpha\tau^2z^2$$

lie in $\operatorname{Re}(z) \leq \frac{1}{2}$. If $f \in B_\tau$, $h_f(\pi/2) \leq 0$ and $|f(x)| \leq 1$ for $-\infty < x < \infty$ then

$$|\alpha f''(x+iy)+\beta f'(x+iy)+\gamma f(x+iy)| \leq |\alpha^2+i\beta\tau+\gamma| e^{-\tau y},$$
$$y \leq 0, \qquad -\infty < x < \infty.$$

Like Theorem 3 this theorem extends a result of Rahman [14, Theorem 4]. It is readily seen that the condition on the roots of the polynomial $v(z)$ in Theorem 4 is less restrictive than the corresponding condition on $u(z)$ in Theorem 3; this latter is already satisfied if α, β, γ are reals and $\beta^2 \geq 4\alpha\gamma$.

THEOREM 5. *Let $f \in B_\tau$ such that $|f(x)| \leq 1$ for $-\infty < x < \infty$ and $f(0) = 0$. Then*

$$|f(x)| \leq |\sin \tau x| \quad \textit{for} \quad |x| \leq \frac{\pi}{2\tau}.$$

If, in addition, $h_f(\pi/2) \leq 0$ then

$$|f(x)| \leq \left|\sin \frac{\tau x}{2}\right| \quad \textit{for} \quad |x| \leq \frac{\pi}{\tau}.$$

Theorem 5 generalizes a result of Boas [6] according to which the inequality

$$(5) \qquad |S(x)| \leq |\sin nx|, \qquad |x| \leq \frac{\pi}{2n},$$

holds for all trigonometric polynomials $S(x) = \sum_{m=-n}^{n} b_m e^{imx}$ satisfying $S(0) = 0$ and $\max_{0 \leq x < 2\pi} |S(x)| \leq 1$. It is also an amelioration of a result of Giroux and Rahman [10]: let $f \in B_\tau$ such that $h_f(\pi/2) \leq 0$, $f(0) = 0$ and $|f(x)| \leq 1$ for $-\infty < x < \infty$; we have then $|f(x)| \leq \tau/2\,|x|$ for $|x| \leq 2/\tau$.

2. **The method of approximation.** Let $f \in B_\tau$ such that $|f(x)| \leq 1$ for $-\infty < x < \infty$. Put $\varphi(x) = (\sin \pi x/\pi x)^2$ and

$$(6) \qquad f_h(x) = \sum_{k=-\infty}^{\infty} \varphi(hx+k) f\left(x + \frac{k}{h}\right), \qquad h > 0.$$

LEMMA 1. *The functions f_h defined by (6) are trigonometric polynomials with period $1/h$ and degree less or equal to $N := 1 + [\tau/2\pi h]$. When x is real we have $|f_h(x)| \leq 1$, and $f_h(z) \to f(z)$ uniformly in every bounded set when $h \to 0$.*

In view of Lemma 1 we may write

$$(7) \qquad f_h(x) = \sum_{m=-N}^{N} C_m(h) e^{2\pi i m h x}$$

where

$$C_m(h) = h \int_0^{1/h} f_h(x) e^{-2\pi i m h x}\, dx.$$

LEMMA 2. *If $h_f(\pi/2) \leq 0$ then*

$$C_m(h) = 0 \quad \textit{for} \quad -N \leq m \leq -1.$$

Proof. Proceeding as in [11, p. 22] we have

$$(8) \qquad C_m(h) = h \int_{-\infty}^{\infty} \varphi(h(x+iy)) f(x+iy) e^{-2\pi i m h(x+iy)}\, dx,$$

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for all real values of y, and the estimate

$$|\varphi(h(x+iy))| \le \frac{e^{2\pi h|y|}}{(\pi h)^2(x^2+y^2)}. \tag{9}$$

Suppose that $y>0$. The hypothesis $|f(x)|\le 1$, $-\infty<x<\infty$, and $h_f(\pi/2)\le 0$ imply [4, p. 82, Theorem 6.2.4] that

$$|f(x+iy)|\le 1, \qquad y\ge 0, \qquad -\infty<x<\infty. \tag{10}$$

Using (8), (9) and (10) we obtain

$$|C_m(h)| \le \frac{e^{2\pi hy(m+1)}}{\pi^2 h}\int_{-\infty}^{\infty}\frac{dx}{(x^2+y^2)} = \frac{e^{2\pi hy(m+1)}}{\pi hy}.$$

Letting $y\to\infty$ we get $C_m(h)=0$, $m=-1,-2,\ldots$

3. Proofs of the theorems

Proof of Theorem 1. Consider the trigonometric polynomials (7), $h>0$. In view of Lemma 2 we have $C_m(h)=0$, $-N\le m\le -1$. Thus, we may write

$$f_h\left(\frac{x}{2\pi h}\right)=P_h(e^{ix})$$

where P_h is an algebraic polynomial of degree $\le N$. Applying a result of Szegö [15, p. 68] we have

$$|P_h(Re^{ix})-P_h(e^{ix})| \le (R^N-1)\max_{|\xi|=1}|\operatorname{Re} P_h(\xi)|, \qquad -\infty<x<\infty, \qquad R\ge 1. \tag{11}$$

If we change x to $2\pi hx$ and R to $R^{2\pi h}$ then (11) becomes

$$|f_h(x-i\log R)-f_h(x)| \le (R^{2\pi hN}-1), \qquad -\infty<x<\infty, \qquad R\ge 1, \tag{12}$$

since $\max_{-\infty<x<\infty}|\operatorname{Re} f_h(x)|\le 1$ whenever $\max_{-\infty<x<\infty}|\operatorname{Re} f(x)|\le 1$. Observe that

$$\lim_{h\to 0} 2\pi hN=\tau. \tag{13}$$

By Lemma 1, $f_h(z)\to f(z)$ uniformly in every bounded set, when $h\to 0$, and the result then follows from (13) if we let $h\to 0$ in (12).

Proof of Theorem 2. It is known [7, Theorem 4] that if P is a polynomial of degree $\le n$ such that $|\operatorname{Re} P(z)|\le 1$ for $|z|\le 1$ then

$$|\operatorname{Re}((\xi-z)P'(z)+nP(z))|\le n, \qquad |\xi|\le 1, \qquad |z|\le 1. \tag{14}$$

Write $P'(z)=a_1+ia_2$, $nP(z)-zP'(z)=b_1+ib_2$ $(a_1, a_2, b_1, b_2\in\mathbb{R})$ and take $\xi=e^{i\omega}$, $\omega\in\mathbb{R}$ in (14); we obtain

$$-n\le a_1\cos\omega - a_2\sin\omega + b_1\le n, \qquad \omega\in\mathbb{R}. \tag{15}$$

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The choice

$$\sin\omega=\frac{-a_2}{\sqrt{(a_1^2+a_2^2)}},\qquad \cos\omega=\frac{a_1}{\sqrt{(a_1^2+a_2^2)}},$$

in (15), gives us the inequality $\sqrt{(a_1^2+a_2^2)}+b_1\le n$ while the choice

$$\sin\omega=\frac{a_2}{\sqrt{(a_1^2+a_2^2)}},\qquad \cos\omega=\frac{-a_1}{\sqrt{(a_1^2+a_2^2)}}$$

gives $-n\le-\sqrt{(a_1^2+a_2^2)}+b_1$; combining these inequalities we obtain

$$(16)\qquad |\mathrm{Re}(nP(z)-zP'(z))|+|P'(z)|\le n,\qquad |z|\le 1.$$

Consider now the trigonometric polynomials (7), $h>0$. In view of Lemma 2 we may write $f_h(x/2\pi h)=P_h(e^{ix})$ where P_h is a polynomial of degree $\le N$. Furthermore $|\mathrm{Re}\,P_h(z)|\le 1$, $|z|\le 1$, and so, applying (16), we get

$$(17)\qquad |\mathrm{Re}(NP_h(z)-zP_h'(z))|+|P_h'(z)|\le N,\qquad |z|\le 1,$$

which is equivalent to

$$(18)\qquad |\mathrm{Re}(2\pi hNf_h(x)+if_h'(x)|+|f_h'(x)|\le 2\pi hN,\qquad -\infty<x<\infty.$$

The result then follows from Lemma 1 and (13) if we let $h\to 0$ in (18).

Proof of Theorem 3. Consider the trigonometric polynomials (7), $h>0$. Put

$$S_h(x)=f_h\left(\frac{x}{2\pi h}\right).$$

We have (Lemma 1) $|S_h(x)|\le 1$, $-\infty<x<\infty$. Now, a result of Rahman [14, Theorem 5] says that if $S(\theta)$ is a trigonometric polynomial of degree $\le n$ such that $|S(\theta)|\le 1$ for $0\le\theta<2\pi$ and the roots of the polynomial

$$u_1(z):=\frac{2a(2n-1)}{n}z^2-2\{a(2n-1)+ib\}z+an^2+ibn-c$$

lie in the half plane $|z|\le|z-n|$ then

$$(19)\qquad |aS''(\theta)+bS'(\theta)+cS(\theta)|\le|-an^2+ibn+c|,\qquad \theta\in\mathbb{R}.$$

It is clear that the same line of reasoning used in [14] to prove (19) lead us to the more general conclusion

$$(20)\qquad |aS''(\theta-i\log R)+bS'(\theta-i\log R)+cS(\theta-i\log R)|$$
$$\le|-an^2+ibn+c|\,R^n,\qquad \theta\in\mathbb{R},\qquad R\ge 1.$$

Take $a=(2\pi h)^2\alpha$, $b=2\pi h\beta$, $c=\gamma$, $\theta=2\pi hx$, $y=-2\pi h\log R$ and apply (20) to the trigonometric polynomial S_h (of degree $\le N$). We obtain that if the roots of the polynomial

$$u_1(Nz)=2(2\pi h)^2N(2N-1)\alpha z^2-2\{(2\pi h)^2N(2N-1)\alpha+i2\pi hN\beta\}z$$
$$+(2\pi hN)^2\alpha+i\beta 2\pi hN-\gamma$$

lie in the half plane $|z| \le |z-1|$ then

$$|\alpha f_h''(x+iy)+\beta f_h'(x+iy)+\gamma f_h(x+iy)| \le |-(2\pi hN)^2\alpha + i(2\pi hN)\beta+\gamma| e^{-2\pi hNy}$$

for $-\infty < x < \infty$ and $y \le 0$.

Suppose first that the roots of the polynomial

$$u(z) = (\alpha\tau^2 + i\beta\tau - \gamma) + 2i\tau(2i\alpha\tau - \beta)z + 4\alpha\tau^2 z^2$$

lie in $\mathrm{Re}(z) < \frac{1}{2}$. The result then follows from Lemma 1, (13), the fact that

$$\lim_{h\to 0} u_1(Nz) = u(z)$$

and Hurwitz's theorem (according to which the roots of $u(z)$ are the limits of the roots of the $u_1(Nz)$, when $h \to 0$). If one or two of the roots of $u(z)$ has real part equal to $\frac{1}{2}$ then, putting $\alpha_1 = \alpha$, $\beta_1 = \beta + 4i\alpha\tau\varepsilon$ and $\gamma_1 = \gamma + 2i\beta\tau\varepsilon - 4\alpha\tau^2\varepsilon^2$, where $\varepsilon > 0$, we are led to a new polynomial,

$$U_\varepsilon(z) = (\alpha_1\tau^2 + i\beta_1\tau - \gamma_1) + 2i\tau(2i\alpha_1\tau - \beta_1)z + 4\alpha_1\tau^2 z^2,$$

whose roots have real part $< \frac{1}{2}$, and the result follows by continuity on letting $\varepsilon \to 0$.

Proof of Theorem 4. Since $h_f(\pi/2) \le 0$ we have (Lemma 2)

$$f_h\left(\frac{x}{2\pi h}\right) = P_h(e^{ix})$$

where P_h is a polynomial of degree $\le N$ such that (Lemma 1) $\max_{|z|=1} |P_h(z)| \le 1$. It is known [14, Theorem 4] that if P is a polynomial of degree $\le n$ then $|P(z)| \le 1$, $|z| = 1$ implies

(21) $$|B(P(z))| \le |B(z^n)|, \qquad |z| \ge 1,$$

where B is the operator which carries

$$P(z) = \sum_{j=0}^{n} a_j z^j$$

into

$$B(P(z)) = \lambda_0 P(z) + \lambda_1 \frac{n}{2} zP'(z) + \lambda_2 \frac{n^2}{8} z^2 P''(z)$$

and where λ_0, λ_1, λ_2 are complex numbers such that the roots of

$$v_1(z) := \lambda_0 + \lambda_1 nz + \lambda_2 \frac{n(n-1)}{2} z^2$$

lie in the half plane $|z| \le |z - (n/2)|$.

Put

$$\lambda_2 = \frac{-8(2\pi h)^2\alpha}{N^2}, \qquad \lambda_1 = \frac{2(2\pi h)\beta i - 2(2\pi h)^2\alpha}{N}, \qquad \lambda_0 = \gamma,$$

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change x to $2\pi hx$, R to $R^{2\pi h}$ and apply (21) to the polynomial P_h (of degree $\leq N$); we obtain that if the roots of

$$v_1\left(\frac{Nz}{2}\right)=\gamma+(2\pi hN\beta i-(2\pi h)^2N\alpha)z-(2\pi h)^2N(N-1)\alpha z^2$$

lie in the half plane $|z|\leq|z-1|$ then

$$\begin{aligned}|\alpha f''_h(x-i\log R)+\beta f'_h(x-i\log R)+\gamma f_h(x-i\log R)|\\ \leq|-(2\pi hN)(2\pi h(N-1))\alpha+2\pi hN\beta i+\gamma-(2\pi h)^2N\alpha|\,R^{2\pi hN},\end{aligned}$$

for $-\infty<x<\infty$ and $R\geq 1$.

Suppose first that the two roots of the polynomial $v(z)=\gamma+i\beta\tau z-\alpha\tau^2z^2$ lie in $\text{Re}(z)<\frac{1}{2}$. The result then follows from Lemma 1, (13), the fact that

$$\lim_{h\to 0} v_1\left(\frac{Nz}{2}\right)=v(z)$$

and Hurwitz's theorem. If one or two of the roots of $v(z)$ has real part equal to $\frac{1}{2}$ then, putting $\alpha_1=\alpha$, $\beta_1=\beta+2i\alpha\tau\varepsilon$ and $\gamma_1=\gamma+i\beta\tau\varepsilon-\alpha\tau^2\varepsilon^2$, where $\varepsilon>0$, we are led to a new polynomial, $V_\varepsilon(z)=\gamma_1+i\beta_1\tau z-\alpha_1\tau^2z^2$, whose roots have real part $<\frac{1}{2}$, and the result follows by continuity on letting $\varepsilon\to 0$.

Proof of Theorem 5. Since $f_h(0)=f(0)=0$ the trigonometric polynomial

$$S_h(x)=f_h\left(\frac{x}{2\pi h}\right)$$

satisfies $S_h(0)=0$ and, by Lemma 1, $|S_h(x)|\leq 1$, $0\leq x<2\pi$. Applying (5) to S_h we obtain

$$|S_h(x)|\leq|\sin Nx|,\qquad |x|\leq\frac{\pi}{2N} \tag{22}$$

or, equivalently,

$$|f_h(x)|\leq|\sin 2\pi hNx|,\qquad |x|\leq\frac{\pi}{4\pi hN}. \tag{23}$$

Now let $\varepsilon>0$ and suppose $\tau>0$. If $h>0$ is sufficiently small the interval

$$\left[-\frac{\pi}{2\tau}+\varepsilon,\frac{\pi}{2\tau}-\varepsilon\right]$$

is contained in

$$\left[\frac{-\pi}{4\pi hN},\frac{\pi}{4\pi hN}\right],$$

whence

$$|f_h(x)|\leq|\sin 2\pi hNx|\quad\text{for}\quad |x|\leq\frac{\pi}{2\tau}-\varepsilon \tag{24}$$

and h sufficiently small. Letting $h\to 0$ we get

$$|f(x)|\le|\sin \tau x|, \qquad |x|\le\frac{\pi}{2\tau}-\varepsilon,$$

and since

$$\bigcup_{\varepsilon>0}\left[\frac{-\pi}{2\tau}+\varepsilon,\frac{\pi}{2\tau}-\varepsilon\right]=\left(\frac{-\pi}{2\tau},\frac{\pi}{2\tau}\right)$$

we obtain the first part of Theorem 5 in the case $\tau>0$. An entire function of exponential type 0 which is bounded on the real axis is a constant so that the conclusion is trivial in the case $\tau=0$.

The second part of Theorem 5 is obtained similarly. We need only to observe that the hypothesis $h_f(\pi/2)\le 0$ implies, in view of Lemma 2, that $f_h(x/2\pi h)=P_h(e^{ix})$, where P_h is a polynomial of degree $\le N$, and apply the preceding reasoning to the trigonometric polynomial (of degree $\le N$)

$$t_h(x)=e^{iNx}P_h(e^{-2ix}).$$

4. **Concluding remarks.** The method described above may be used to prove several well-known results. For example, Bernstein's inequality (1) may be obtained from the corresponding (and previously discovered [2, p. 39]) result on trigonometric polynomials.

As another example, suppose that $f\in B_\tau$ is such that $h_f(\pi/2)\le 0$ and $|f(x)|\le 1$ for $-\infty<x<\infty$. If $f(z)\ne 0$ in $\mathrm{Im}(z)\ge 0$ then there exists a sequence of positive numbers $(h_j)_{j=0}^{\infty}$ such that $\lim_{j\to\infty}h_j=0$ and $f_{h_j}(z)\ne 0$ in $\mathrm{Im}(z)\ge 0$, $j=0,1,2,\cdots$ The polynomial

$$P_{h_j}(z)=\sum_{m=0}^{N_j}C_m(h)z^m,$$

where

$$N_j:=1+\left[\frac{\tau}{2\pi h_j}\right],$$

is then different from 0 in $|z|\le 1$. By the Erdös–Lax Theorem [12] we have $|P'_{h_j}(e^{i\theta})|\le N_j/2$, $0\le\theta<2\pi$, that is

$$|f'_{h_j}(x)|\le\frac{2\pi h_jN_j}{2}, \qquad -\infty<x<\infty.$$

Letting $j\to\infty$ and using Lemma 1 we obtain

$$|f'(x)|\le\frac{\tau}{2}, \qquad -\infty<x<\infty.$$

If $f(z)\ne 0$ only in $\mathrm{Im}(z)>0$ then we may apply the result just proved to the entire function $g(z):=f(z+\varepsilon i)$, $\varepsilon>0$, which is of exponential type τ, satisfies

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$h_g(\pi/2)\le 0$ and is different from 0 in $\mathrm{Im}(z)\ge 0$. We have thus proved a result of Boas [5]: if $f\in B_\tau$, $h_f(\pi/2)\le 0$, $|f(x)|\le 1$ for $-\infty<x<\infty$ and $f(z)\ne 0$ in $\mathrm{Im}(z)>0$ then

$$|f'(x)|\le\frac{\tau}{2},\qquad -\infty<x<\infty. \tag{25}$$

In a similar way we may prove, with the same hypothesis as for (25), that

$$|f(x+iy)|\le\frac{e^{-\tau y}+1}{2},\qquad y\le 0,\qquad -\infty<x<\infty. \tag{26}$$

The inequality (26), also due to Boas [5], is reminiscent to a result of Ankeny and Rivlin [1] according to which the inequality $|P(Re^{i\theta})|\le(R^n+1)/2$, $0\le\theta<2\pi$, $R\ge 1$, holds for all polynomials P not vanishing in the unit disk and satisfying $\max_{|z|=1}|P(z)|\le 1$.

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