OSCILLATION CRITERIA FOR MATRIX DIFFERENTIAL EQUATIONS: CORRIGENDUM

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Professor W. T. Reid in a recent review (**1**) pointed out that the proof of Lemma 2 was incorrect and the validity of Theorem 6 was therefore in doubt. In this correction modified versions of Lemma 2 and Theorem 6 are proved. In the original paper, replace § 5 from Lemma 2 to the end of the section by the following material.

LEMMA 2. *Suppose that*

(1) *T is a real, continuous, symmetric solution of the equation*

$$T' = G^{-1}(T + H)^2$$

in $[a, b)$ such that $T(a) \geqq -t_0 I$, $t_0 \geqq 0$, with H real, continuous and symmetric there, $H \geqq -h_0 I$ in $[a, b]$, $h_0 \geqq 0$, and $G = gI$, a positive scalar matrix with $g \in C'[a, b]$,

(2) *$\Phi = \phi I$, $L = lI$ are real scalar matrices belonging to $C'[a, b]$ and $C[a, b]$, respectively, such that $\Phi' > G^{-1}(\Phi + L)^2$ and $L \geqq H$, in $[a, b]$, $\Phi(a) > T(a)$, and $\Phi + L \geqq (t_0 + h_0)I$ in $[a, b]$.*

Then $\Phi > T$ in $[a, b)$.

Proof. Suppose the contrary. Then there exists a point c in (a, b) and a non-empty set S of unit vectors such that $\xi^*\Phi(c)\xi = \xi^*T(c)\xi$ for $\xi \in S$. Thus, $\Phi \geqq T$ and $\Phi + L \geqq T + H$ in $[a, c]$. In $[a, c]$, $T + H \geqq -(t_0 + h_0)I \equiv -k_0 I$ and $\Phi + L \geqq k_0 I$, hence $\Phi + L \geqq -(T + H)$. Note that the matrices $A\pm \equiv \Phi + L \pm (T + H)$ are commuting symmetric, positive semi-definite matrices since $\Phi + L$ is a scalar matrix. Thus, $A+ \cdot A_- \geqq 0$ and

$$G^{-1}(\Phi + L)^2 \geqq G^{-1}(T + H)^2 \quad \text{in } [a, c]. \tag{5.6}$$

Now we know that

$$\xi^*\Phi'(c)\xi \leqq \xi^*T'(c)\xi, \qquad \xi \in S, \tag{5.7}$$

for if not (i.e., $\xi^*\Phi'(x)\xi > \xi^*T'(x)\xi$) in a neighbourhood including c as an interior point (by continuity) and by integrating from x to c, $x < c$, one has $\xi^*(\Phi(c) - T(c))\xi > \xi^*(\Phi(x) - T(x))\xi > 0$ (for $x < c$ recall that $\Phi > T$), a contradiction to the definition of c. If (5.7) holds, we have, for $\xi \in S$, $\xi^*G^{-1}(c)(\Phi(c) + L(c))^2\xi < \xi^*\Phi'(c)\xi \leqq \xi^*T'(c)\xi = \xi^*G^{-1}(c)(T(c) + H(c))^2\xi$, which contradicts (5.6), and the result follows.

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We now have the following non-oscillation theorem.

THEOREM 6. *Suppose that*

(1) *Y is a solution of* (1.1) *with* $|Y(a)| \neq 0$, $a > 0$, *and satisfying hypothesis* (1) *of Theorem* 1,

(2) $G = gI$ *is a positive scalar matrix with* $g \in C'[a, \infty)$,

(3) *for every* $b > a$, *the matrix*

$$H(x) = \int_a^x \{G(t)P(t) - \tfrac{1}{4}(G'(t))^2 G^{-1}(t)\}\, dt + \tfrac{1}{2}G'(x), \qquad a \leqq x \leqq b,$$

is such that there exist scalar matrices Φ, Ψ, *and* L *of class* C', C, *and* C, *respectively, in* $[a, b]$ *with the properties for* x *in* $[a, b]$, $\Psi^2 \geqq (\Phi + L)^2$, $\Phi' > G^{-1}\Psi^2$, $L \geqq H$, *and* $\Phi(a) \geqq kI$ (*where* $k = t_0 + 2h_0$, t_0 *and* h_0 *are numbers* $\geqq 0$ *such that* $(t_0 + 2h_0)I > -G(a)Y'(a)Y^{-1}(a) \geqq -t_0 I$ *and* $H \geqq -h_0 I$ *in* $[a, b]$).

Then Y is a non-oscillatory solution.

Proof. Suppose the contrary. Then there exists a first zero of the equation $|Y(x)| = 0$ after the point a mentioned in hypothesis (1), at $x = b$, say. For x in $[a, b)$ we can transform (1.1) into

$$T' = G^{-1}(T + H)^2 \tag{5.8}$$

by means of the transformation

$$G^{-1}(x)\left(T(x) + \int_a^x \{GP - \tfrac{1}{4}(G')^2 G^{-1}\}\, dt\right) = -Y'(x)Y^{-1}(x). \tag{5.9}$$

By Lemma 1, $|\xi^* Y' Y^{-1} \xi|$, and hence $|\xi^* G Y' Y^{-1} \xi|$, must assume arbitrarily large positive values in $[a, b)$, for at least one properly chosen unit vector ξ, say ξ_0. The same must be true of $|\xi_0^* T \xi_0|$, by an inspection of the transformation (5.9) and noting that $H \in C[a, b]$. From hypothesis (3), we have the existence of a matrix Φ such that in $[a, b)$, $\Phi' > G^{-1}(\Phi + L)^2$, and such that Lemma 2 is valid. Hence, $\Phi > T$ in $[a, b)$. However, for $|\xi_0^* T \xi_0|$ to assume arbitrarily large values in $[a, b)$ is an impossibility since $\Phi \in C'[a, b]$. This contradiction proves the theorem.

REFERENCE

1. H. C. Howard, *Oscillation criteria for matrix differential equations*, Can. J. Math. *19* (1967), 184–199; reviewed by W. T. Reid, MR *35*, #3126.

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