Ergod. Th. & Dynam. Sys., (2023), **43**, 2228–2257
doi:10.1017/etds.2022.46

Directed harmonic currents near non-hyperbolic linearizable singularities

ZHANGCHI CHEN

Morningside Center of Mathematics,
Chinese Academy of Science, Beijing, China,
http://www.mcm.ac.cn/people/postdocs/202110/t20211022_666685.html
(e-mail: zhangchi.chen@amss.ac.cn)

(*Received* 29 *November* 2020 *and accepted in revised form* 14 *May* 2022)

Abstract. Let $(\mathbb{D}^2, \mathscr{F}, \{0\})$ be a singular holomorphic foliation on the unit bidisc $\mathbb{D}^2$ defined by the linear vector field

$$z\frac{\partial}{\partial z} + \lambda w\frac{\partial}{\partial w},$$

where $\lambda \in \mathbb{C}^*$. Such a foliation has a non-degenerate singularity at the origin $0 := (0, 0) \in \mathbb{C}^2$. Let T be a harmonic current directed by $\mathscr{F}$ which does not give mass to any of the two separatrices $(z = 0)$ and $(w = 0)$. Assume $T \neq 0$. The Lelong number of T at 0 describes the mass distribution on the foliated space. In 2014 Nguyên (see [16]) proved that when $\lambda \notin \mathbb{R}$, that is, when 0 is a hyperbolic singularity, the Lelong number at 0 vanishes. Suppose the trivial extension $\tilde{T}$ across 0 is dd^c-closed. For the non-hyperbolic case $\lambda \in \mathbb{R}^*$, we prove that the Lelong number at 0:

(1) is strictly positive if $\lambda > 0$;
(2) vanishes if $\lambda \in \mathbb{Q}_{<0}$;
(3) vanishes if $\lambda < 0$ and T is invariant under the action of some cofinite subgroup of the monodromy group.

Key words: holomorphic foliation, harmonic current, non-hyperbolic linearizable singularity, Lelong number
2020 Mathematics Subject Classification: 32M25 (Primary); 32S65, 32C30, 53C65 (Secondary)

1. *Introduction*

The dynamical properties of singular holomorphic foliations have recently drawn a great deal of attention; see the discussions in [9, 11, 13, 15, 17, 18]. Let us mention one of the remarkable results which establishes the unique ergodicity for general singular holomorphic foliations on compact Kähler surfaces.

THEOREM 1.1. (Dinh, Nguyên and Sibony [7]) *Let $\mathscr{F}$ be a holomorphic foliation with only hyperbolic singularities in a compact Kähler surface (X, ω). Assume that $\mathscr{F}$ admits no directed positive closed current. Then there exists a unique positive dd^c-closed current T of mass 1 directed by $\mathscr{F}$.*

The first version was stated for $X = \mathbb{P}^2$ and proved by Fornæss and Sibony [12]. Later Dinh and Sibony proved the unique ergodicity for foliations in $\mathbb{P}^2$ with an invariant curve [8]. So one may expect to describe recurrence properties of leaves by studying the density distribution of directed harmonic currents. One has the following result about leaves.

THEOREM 1.2. (Fornæss and Sibony [12]) *Let $(X, \mathscr{F}, E)$ be a holomorphic foliation on a compact complex surface X with singular set E. Assume that:*

(1) *there is no invariant analytic curve;*
(2) *all the singularities are hyperbolic;*
(3) *there is no non-constant holomorphic map $\mathbb{C} \to X$ such that out of E the image of $\mathbb{C}$ is locally contained in a leaf.*

Then every harmonic current T directed by $\mathscr{F}$ gives no mass to each single leaf.

A practical way to measure the density of harmonic currents is to use the notion of Lelong number introduced by Skoda [22]. Indeed Theorem 1.2 above is equivalent to the statement that the Lelong number of T vanishes everywhere outside E. Another result holds near hyperbolic singularities.

THEOREM 1.3. (Nguyên [16]) *Let $(\mathbb{D}^2, \mathscr{F}, \{0\})$ be a holomorphic foliation on the unit bidisc $\mathbb{D}^2$ defined by the linear vector field $Z(z, w) = z(\partial/\partial z) + \lambda w(\partial/\partial w)$, where $\lambda \in \mathbb{C}\backslash\mathbb{R}$, that is to say, 0 is a hyperbolic singularity. Let T be a harmonic current directed by $\mathscr{F}$ which does not give mass to any of the two separatrices $(z = 0)$ and $(w = 0)$. Then the Lelong number of T at 0 vanishes.*

Next, Nguyên applies this result to prove the existence of Lyapunov exponents for singular holomorphic foliations on compact projective surfaces [20]. Very recently he has proved in [19] that for every $n \geqslant 2$, the Lelong numbers of any directed harmonic current which gives no mass to invariant hyperplanes vanishes near *weakly hyperbolic* singularities in $\mathbb{C}^n$. This result is optimal; see [10]. The mass-distribution problem would be completed once we could understand the behaviour of harmonic currents near non-hyperbolic non-degenerate singularities, and near degenerate singularities.

The present paper answers (partly) the problem in the non-hyperbolic linearizable singularity case. Here is our first main result.

THEOREM 1.4. *Let $(\mathbb{D}^2, \mathscr{F}, \{0\})$ be a holomorphic foliation on the unit bidisc $\mathbb{D}^2$ defined by the linear vector field $Z(z, w) = z(\partial/\partial z) + \lambda w(\partial/\partial w)$, where $\lambda \in \mathbb{R}^*$. Let T be a harmonic current directed by $\mathscr{F}$ which does not give mass to any of the two separatrices $(z = 0)$ and $(w = 0)$. Assume $T \neq 0$. Then the Lelong number of T at 0:*

- *is strictly positive and could be infinite if $\lambda > 0$;*
- *vanishes if $\lambda \in \mathbb{Q}_{<0}$.*

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

For the foliation concerned $(\mathbb{D}^2, \mathscr{F}, \{0\})$, a local leaf P_α, with $\alpha \in \mathbb{C}^*$, can be parametrized by $(z, w) = (e^{-v+iu}, \alpha e^{-\lambda v+i\lambda u})$, with $u, v \in \mathbb{R}$. See the parametrization (1) for details. The *monodromy group* around the singularity is generated by $(z, w) \mapsto (z, e^{2\pi i\lambda}w)$. It is a cyclic group of finite order when $\lambda \in \mathbb{Q}^*$, of infinite order when $\lambda \notin \mathbb{Q}$.

We are now ready to introduce the notion of *periodic current*, an essential tool in this paper. A directed harmonic current T is called *periodic* if it is invariant under some cofinite subgroup of the monodromy group, that is, under the action of $(z, w) \mapsto (z, e^{2k\pi i\lambda}w)$ for some $k \in \mathbb{Z}_{>0}$.

Observe that if $\lambda = (a/b) \in \mathbb{Q}^*$ with $a \in \mathbb{Z}^*$, $b \in \mathbb{Z}_{>0}$, then any directed harmonic current is invariant under the action of $(z, w) \mapsto (z, e^{2b\pi i\lambda}w)$, hence is periodic. But when $\lambda \notin \mathbb{Q}^*$, the periodicity is a non-trivial assumption. It does not follow from the ergodicity of irrational rotation because the current is only continuous on leaf parameters (u, v) for each fixed α. It may not be continuous in variables (z, w).

We are in a position to state our second main result.

THEOREM 1.5. *Using the same notation as above, the Lelong number of T at the singularity is* 0 *when* $\lambda < 0$ *and the current is periodic, in particular, when* $\lambda \in \mathbb{Q}_{<0}$.

It remains open to determine the possible Lelong number values of non-periodic T when $\lambda < 0$ is irrational.

Section 2 reviews the definition of singular holomorphic foliations, directed harmonic currents, the mass and the Lelong number. Section 3 describes the topology of leaves near linearizable non-hyperbolic singularities, resolves the ambiguity of normalizing harmonic functions on the leaves and provides practical formulas for the mass and the Lelong number. Section 4 calculates the Lelong number when $\lambda \in \mathbb{Q}_{>0}$. Section 5 calculates the Lelong number when $\lambda \in \mathbb{R}_{>0}\backslash\mathbb{Q}$, with an analysis on Poisson integrals of non-periodic currents. Section 6 calculates the Lelong number when $\lambda < 0$, assuming that the currents are periodic.

2. *Background*

2.1. *Singularities of holomorphic foliations.* To start with, recall the definition of singular holomorphic foliation on a complex surface M.

Definition 2.1. Let $E \subset M$ be some closed subset, possibly empty, such that $\overline{M\backslash E} = M$. A *singular holomorphic foliation* $(M, E, \mathscr{F})$ consists of a holomorphic *atlas* $\{(\mathbb{U}_i, \Phi_i)\}_{i\in I}$ on $M\backslash E$ which satisfies the following conditions.

(1) For each $i \in I$, $\Phi_i : \mathbb{U}_i \to \mathbb{B}_i \times \mathbb{T}_i$ is a biholomorphism, where $\mathbb{B}_i$ and $\mathbb{T}_i$ are domains in $\mathbb{C}$.

(2) For each pair $(\mathbb{U}_i, \Phi_i)$ and $(\mathbb{U}_j, \Phi_j)$ with $\mathbb{U}_i \cap \mathbb{U}_j \neq \emptyset$, the transition map

$$\Phi_{ij} := \Phi_i \circ \Phi_j^{-1} : \Phi_j(\mathbb{U}_i \cap \mathbb{U}_j) \to \Phi_i(\mathbb{U}_i \cap \mathbb{U}_j)$$

has the form

$$\Phi_{ij}(b, t) = (\Omega(b, t), \Lambda(t)),$$

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

where (b, t) are the coordinates on $\mathbb{B}_j \times \mathbb{T}_j$, and the functions Ω, Λ are holomorphic, with Λ independent of b.

Each open set $\mathbb{U}_i$ is called a *flow box*. For each $c \in \mathbb{T}_i$, the Riemann surface $\Phi_i^{-1}\{t = c\}$ in $\mathbb{U}_i$ is called a *plaque*. Property (2) above ensures that in the intersection of two flow boxes, plaques are mapped to plaques.

A *leaf* L is a minimal connected subset of M such that if L intersects a plaque, it contains that plaque. A *transversal* is a Riemann surface immersed in M which is transverse to each leaf of M.

The local theory of singular holomorphic foliations is closely related to holomorphic vector fields. One recalls some basic concepts in $\mathbb{C}^2$; see [**5**, **11**, **17**, **18**].

Definition 2.2. Let $Z = P(z, w)\partial/\partial z + Q(z, w)\partial/\partial w$ be a holomorphic vector field defined in a neighbourhood $\mathbb{U}$ of $(0, 0) \in \mathbb{C}^2$. One says that Z is:

(1) *singular* at $(0, 0)$ if $P(0, 0) = Q(0, 0) = 0$;

(2) *linear* if it can be written as

$$Z = \lambda_1 z \frac{\partial}{\partial z} + \lambda_2 w \frac{\partial}{\partial w}$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$ are not simultaneously zero;

(3) *linearizable* if it is linear after a biholomorphic change of coordinates.

Suppose the holomorphic vector field $Z = P(\partial/\partial z) + Q(\partial/\partial w)$ admits a singularity at the origin. Let λ_1, λ_2 be the eigenvalues of the Jacobian matrix $\left(\begin{smallmatrix} P_z & P_w \\ Q_z & Q_w \end{smallmatrix}\right)$ at the origin.

Definition 2.3. The singularity is *non-degenerate* if both λ_1, λ_2 are non-zero. This condition is biholomorphically invariant.

In this paper, all singularities are assumed to be non-degenerate. Then the foliation defined by integral curves of Z has an isolated singularity at 0. Degenerate singularities are studied in [**5**]. Seidenberg's reduction theorem [**21**] shows that degenerate singularities can be resolved into non-degenerate ones after finitely many blow-ups.

Definition 2.4. A singularity of Z is *hyperbolic* if the quotient $\lambda := (\lambda_1/\lambda_2) \in \mathbb{C}\backslash\mathbb{R}$. It is *non-hyperbolic* if $\lambda \in \mathbb{R}^*$. It is in the *Poincaré domain* if $\lambda \in \mathbb{C}\backslash\mathbb{R}_{\leqslant 0}$. It is in the *Siegel domain* if $\lambda \in \mathbb{R}_{<0}$.

One can verify that the quotient is unchanged by multiplication of Z by any non-vanishing holomorphic function.

One could consider $\lambda^{-1} = \lambda_2/\lambda_1$ instead of λ, but then $\lambda \notin \mathbb{R}$ if and only if $\lambda^{-1} \notin \mathbb{R}$. Thus, the notion of hyperbolicity is well defined. Also, being non-hyperbolic, in the Poincaré domain or Siegel domain, is well defined. The complex number λ will be called an *eigenvalue* of Z at the singularity, with an inessential abuse due to this exchange $\lambda \leftrightarrow \lambda^{-1}$. The unordered pair $\{\lambda, \lambda^{-1}\}$ is invariant under local biholomorphic changes of coordinates.

Consider a holomorphic foliation $(M, E, \mathscr{F})$ where E is discrete. When one tries to linearize a vector field near an isolated non-degenerate singularity, one has to divide power series coefficients by quantities $m_1 + \lambda m_2 - 1$ and $m_1 + \lambda m_2 - \lambda$ where $m_1, m_2 \in \mathbb{Z}_{\geqslant 0}$

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

with $m_1 + m_2 \geqslant 2$. To ensure convergence, these quantities have to be non-zero and not too close to zero.

These quantities are non-zero if and only if $\lambda \notin \mathbb{Q}_{\neq 1}$. They do not have 0 as a limit if and only if $\lambda \notin \mathbb{R}_{\leqslant 0}$, that is, the singularity is in the Poincaré domain.

We are now ready to state some linearization results in $\mathbb{C}^2$.

THEOREM 2.5. (Poincaré; see [2, Ch. 4, §1.2, pp. 72]) *A singular holomorphic vector field in $\mathbb{C}^2$ is holomorphically equivalent to its linear part if its eigenvalue $\lambda \in (\mathbb{C}\backslash\mathbb{R}_{\leqslant 0})\backslash\mathbb{Q}_{\neq 1}$.*

Remark 2.6. The linear part of a singular holomorphic vector field is

$$(az + bw)\frac{\partial}{\partial z} + (cz + dw)\frac{\partial}{\partial w}$$

for some $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ if the singularity is assumed to be non-degenerate. It is non-linearizable if and only if the Jordan normal form of the Jacobian matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has a rank-2 block $\left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$ with $a \neq 0$. In this case $\lambda = 1$, hence Poincaré's theorem holds. The vector field is holomorphically equivalent to its linear part $(az + w)\partial/\partial z + aw(\partial/\partial w)$, but is not linearizable.

For the resonant case $\lambda \in \mathbb{Q}_{\neq 1}$ and the degenerate case, one may use the Poincaré–Dulac normal form [2, Ch. 3, §3.2, pp. 54].

In particular, all hyperbolic singularities are linearizable.

To get linearization for λ in the Siegel domain, the following result assumes the more advanced *Brjuno condition*.

THEOREM 2.7. (Brjuno [2, 4]) *A singular holomorphic vector field with a non-resonant linear part is holomorphically linearizable if its eigenvalue $\lambda \in \mathbb{R}$ satisfies the condition*

$$\sum_{n \geqslant 1} \frac{\log q_{n+1}}{q_n} < \infty,$$

where p_n/q_n is the nth approximant of the continued fraction expansion of λ.

The golden ratio

$$\frac{\sqrt{5}-1}{2} = 1 + \frac{1}{1 + \frac{1}{1+\cdots}}$$

is a Brjuno number. Indeed, any irrational number whose continued fraction expansion ends with a string of 1s

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{\cdots}} = [a_0, a_1, \ldots, a_k, 1, 1, \ldots] \in \mathbb{R}\backslash\mathbb{Q} \quad (a_0 \in \mathbb{Z}, a_1, \ldots, a_k \in \mathbb{N}),$$

is a Brjuno number. The Brjuno numbers are dense in $\mathbb{R}\backslash\mathbb{Q}$. See [14, Propositions 1.2 and 1.3].

In this paper, all singularities are assumed to be linearizable.

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

2.2. *Directed harmonic currents.* Let $(\mathbb{D}^2, \mathscr{F}, \{0\})$ be a holomorphic foliation on the unit bidisc $\mathbb{D}^2$ defined by the linear vector field $Z = z\partial/\partial z + \lambda w(\partial/\partial w)$ with $\lambda \in \mathbb{R}^*$. One may assume $0 < |\lambda| \leqslant 1$ after switching z and w if necessary. There are always two separatrices $\{z = 0\}$ and $\{w = 0\}$. Other leaves can be parametrized as

$$L_\alpha := \{(z, w) = \psi_\alpha(\zeta) := (e^{i\zeta}, \alpha e^{i\lambda\zeta}) = (e^{-v+iu}, \alpha e^{-\lambda v + i\lambda u})\} \quad (\alpha \neq 0), \tag{1}$$

where $\zeta = u + iv \in \mathbb{C}$. The map

$$\Psi : \mathbb{C} \times \mathbb{C}^* \longrightarrow \mathbb{C}^2$$
$$(\zeta, \alpha) \longmapsto (e^{i\zeta}, \alpha e^{i\lambda\zeta})$$

is locally biholomorphic. Here α is the coordinate on the transversal and ζ is the coordinate on leaves. It is not injective since $\Psi(\zeta + 2\pi, \alpha) = \Psi(\zeta, \alpha e^{2\pi i\lambda})$.

Two numbers $\alpha, \beta \in \mathbb{C}^*$ are *equivalent* $\alpha \sim \beta$ if $\beta = e^{2k\pi i\lambda}\alpha$ for some $k \in \mathbb{Z}$. The following statements are equivalent:

- $\alpha \sim \beta$;
- $L_\alpha = L_\beta$;
- $\psi_\alpha = \psi_\beta \circ$ (translation of $2k\pi$) for some $k \in \mathbb{Z}$.

Let $\mathscr{C}_{\mathscr{F}}$ (respectively, $\mathscr{C}^{1,1}_{\mathscr{F}}$) denote the space of functions (respectively, forms of bidegree (1, 1)) defined on leaves of the foliation which are compactly supported on $M \setminus E$, leafwise smooth and transversally continuous. A form $\iota \in \mathscr{C}^{1,1}_{\mathscr{F}}$ is said to be *positive* if its restriction to every plaque is a positive (1,1)-form.

A *directed harmonic current T on $\mathscr{F}$* is a continuous linear form on $\mathscr{C}^{1,1}_{\mathscr{F}}$ satisfying the following two conditions:

(1) $i\partial\bar{\partial}T = 0$ in the weak sense, that is, $T(i\partial\bar{\partial} f) = 0$ for all $f \in \mathscr{C}_{\mathscr{F}}$, where in the expression $i\partial\bar{\partial} f$ one only considers $\partial\bar{\partial}$ along the leaves;
(2) T is positive, that is, $T(\iota) \geqslant 0$ for all positive forms $\iota \in \mathscr{C}^{1,1}_{\mathscr{F}}$.

It is well known (see, for example, [3, 6, 11]) that a directed harmonic current T on a flow box $\mathbb{U} \cong \mathbb{B} \times \mathbb{T}$ can be locally expressed as

$$T = \int_{\alpha \in \mathbb{T}} h_\alpha [P_\alpha]\, d\mu(\alpha). \tag{2}$$

The h_α are non-negative harmonic functions on the local leaves P_α and μ is a Borel measure on the transversal $\mathbb{T}$. If $h_\alpha = 0$ at some point on P_α, then by the mean value theorem $h_\alpha \equiv 0$. For all such $\alpha \in \mathbb{T}$, we replace h_α by the constant function 1 and we set $d\mu(\alpha) = 0$. Thus, we get a new expression of T where $h_\alpha > 0$ for all $\alpha \in \mathbb{T}$.

Such an expression is not unique since $T = \int_{\alpha\in\mathbb{T}} (h_\alpha g(\alpha))[P_\alpha]((1/g(\alpha))\, d\mu(\alpha))$ for any measurable positive function $g : \mathbb{T} \to \mathbb{R}_{>0}$ which is finite and non-zero almost everywhere. The expression is unique after *normalization*, which means that for each $\alpha \in \mathbb{T}$ one fixes $h_\alpha(z_0, w_0) = 1$ at some point $(z_0, w_0) \in P_\alpha$.

Each harmonic function h_α on the leaf V_α can be pulled back by the parametrization Ψ as the harmonic function

$$H_\alpha(u, v) := h_\alpha(e^{-v+iu}, \alpha e^{-\lambda v + i\lambda u}).$$

The domain of definition for u, v will be precisely described later in this section.

In §1 the notion of *periodic current* was introduced. Here is an equivalent characterization.

PROPOSITION 2.8. *A directed harmonic current T is periodic if and only if there exists some $k \in \mathbb{Z}_{>0}$ such that $H_\alpha(u + 2k\pi, v) = H_\alpha(u, v)$ for all u, v and for μ-almost all α.*

Proof. By definition T is invariant under $(z, w) \mapsto (z, e^{2k\pi i\lambda} w)$ for some $k \in \mathbb{Z}_{>0}$, which is equivalent to $H_\alpha(u + 2k\pi, v) = H_\alpha(u, v)$ for all u, v and μ-almost all α. □

A current T of the form (2) is dd^c-closed on $\mathbb{D}^2\backslash\{0\}$. But its trivial extension $\tilde{T}$ across the singularity 0 is not necessarily dd^c-closed on $\mathbb{D}^2$. It is true when T is compactly supported, for example when T is a localization of a current on a compact manifold, by the following argument (see [6, Lemma 2.5] for details).

Let T be a directed harmonic current on $M\backslash E$, where M is a compact complex manifold and E is a finite set. The current T can be extended by zero through E in order to obtain the positive current $\tilde{T}$ on M. Next, we apply the following result.

THEOREM 2.9. (Alessandrini and Bassanelli [1, Theorem 5.6]) *Let Ω be an open subset of $\mathbb{C}^n$ and Y an analytic subset of Ω of dimension less than p. Suppose T is a negative current of bidimension (p, p) on $\Omega\backslash Y$ such that $dd^c T \geqslant 0$. Then the following assertions hold.*

(1) *The mass of T near Y is locally finite. In particular, T admits a trivial extension by 0 across Y, denoted by $\tilde{T}$.*
(2) *$dd^c\tilde{T} \geqslant 0$ on Ω.*

Here $-T$ is a negative current of bidimension $(1, 1)$ on $M\backslash E$ with $dd^c(-T) \geqslant 0$ and E has dimension 0. So for the trivial extension $\tilde{T}$ on M one has $dd^c(-\tilde{T}) \geqslant 0$. Moreover, $\tilde{T}$ is compactly supported since M is compact. Thus

$$\langle dd^c\tilde{T}, 1\rangle = \langle \tilde{T}, dd^c 1\rangle = 0.$$

Combining with $dd^c\tilde{T} \leqslant 0$ from the extension theorem, one concludes that $dd^c\tilde{T} = 0$ on M. Thus, locally near any singularity, the trivial extension $\tilde{T}$ is dd^c-closed.

Let $\beta := i dz \wedge d\bar{z} + i dw \wedge d\bar{w}$ be the standard Kähler form on $\mathbb{C}^2$. The *mass* of T on a domain $U \subset \mathbb{D}^2$ is denoted by $\|T\|_U := \int_U T \wedge \beta$. In this paper, all currents are assumed to have finite mass on $\mathbb{D}^2$.

Definition 2.10. (See [19, §2.4]) Let T be a directed harmonic current on $(\mathbb{D}^2, \mathscr{F}, \{0\})$. We define the *Lelong number* by the limit

$$\mathscr{L}(T, 0) = \limsup_{r\to 0+} \frac{1}{\pi r^2}\|T\|_{r\mathbb{D}^2} \in [0, +\infty].$$

The limit can be infinite when the trivial extension $\tilde{T}$ across the origin is not dd^c-closed [19, Example 2.11]. When $\tilde{T}$ is dd^c-closed, the following theorem ensures the finiteness.

THEOREM 2.11. (Skoda [22]) *Let T be a positive dd^c-closed $(1, 1)$-current in $\mathbb{D}^2$. Then the function $r \mapsto 1/\pi r^2\|T\|_{r\mathbb{D}^2}$ is increasing with $r \in (0, 1]$.*

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

In our case, the function

$$r \mapsto \frac{1}{\pi r^2}\|\tilde{T}\|_{r\mathbb{D}^2} = \frac{1}{\pi r^2}\|T\|_{r\mathbb{D}^2}$$

is increasing with $r \in (0, 1]$. In particular,

$$\mathscr{L}(T, 0) = \lim_{r\to 0+} \frac{1}{\pi r^2}\|T\|_{r\mathbb{D}^2} \in \left[0, \frac{1}{\pi}\|T\|_{\mathbb{D}^2}\right].$$

In this paper, the symbols $\lesssim$ and $\gtrsim$ stand for inequalities up to a multiplicative positive constant depending only on λ. We write $\approx$ when both inequalities are satisfied.

3. *Parametrization of leaves*

Recall the parametrization of an arbitrary leaf L_α:

$$\psi_\alpha(\zeta) = \Psi(\zeta, \alpha) = (e^{i\zeta}, \alpha e^{i\lambda\zeta}) \quad (\alpha \in \mathbb{C}^*, \zeta \in \mathbb{C}).$$

To calculate the mass $\|T\|_{\mathbb{D}^2}$ and the Lelong number $\mathscr{L}(T, 0)$, we shall study $\Psi^{-1}(r\mathbb{D}^2)$ for $r \in (0, 1]$. Define $P_\alpha := L_\alpha \cap \mathbb{D}^2$ and $P_\alpha^{(r)} := L_\alpha \cap r\mathbb{D}^2$. Define $\log^+(x) := \max\{0, \log(x)\}$ for $x > 0$.

LEMMA 3.1. *The range of (u, v) for a point $(z, w) \in P_\alpha$ and $P_\alpha^{(r)}$ is an upper half-plane when $\lambda > 0$, or a horizontal strip when $\lambda < 0$. More precisely:*

(1) *when $\lambda > 0$,*

$$(z, w) \in P_\alpha \iff v > \frac{\log^+ |\alpha|}{\lambda},$$

$$(z, w) \in P_\alpha^{(r)} \iff \begin{cases} v > \dfrac{\log|\alpha| - \log r}{\lambda} & (|\alpha| \geqslant r^{1-\lambda}), \\ v > -\log r & (|\alpha| < r^{1-\lambda}); \end{cases}$$

(2) *when $\lambda < 0$, $P_\alpha = \emptyset$ for $|\alpha| \geqslant 1$, $P_\alpha^{(r)} = \emptyset$ for $|\alpha| \geqslant r^{1-\lambda}$ and for the other α,*

$$(z, w) \in P_\alpha \iff 0 < v < \frac{\log|\alpha|}{\lambda},$$

$$(z, w) \in P_\alpha^{(r)} \iff -\log r < v < \frac{\log|\alpha| - \log r}{\lambda}.$$

Proof. Recall that $(z, w) = (e^{-v+iu}, \alpha e^{-\lambda v + i\lambda u})$ on L_α. So for any $r \in (0, 1]$, $(z, w) \in P_\alpha^{(r)}$ if and only if both $|z| = e^{-v} < r$ and $|w| = |\alpha| e^{-\lambda v} < r$.

When $\lambda > 0$ one has $v > -\log r$ and $v > (\log|\alpha| - \log r)/\lambda$. In particular, for $r = 1$, one has $v > 0$ and $v > \log|\alpha|/\lambda$.

When $\lambda < 0$ one has $-\log r < v < (\log|\alpha| - \log r)/\lambda$. In particular, for $r = 1$, one has $0 < v < \log|\alpha|/\lambda$. If there is no solution for v then $P_\alpha^{(r)} = \emptyset$. □

When $\lambda > 0$, the range of v is unbounded for each fixed $\alpha \in \mathbb{C}^*$. See Figures 1 and 2.

When $\lambda < 0$, the range of v is bounded for each fixed α. See Figures 3 and 4.

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

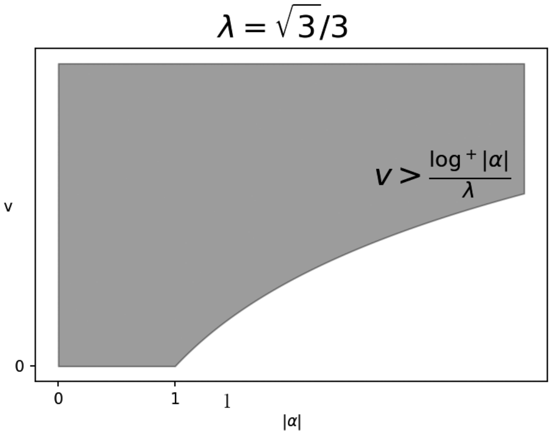

FIGURE 1. The region of $(|\alpha|, v)$ for P_α.

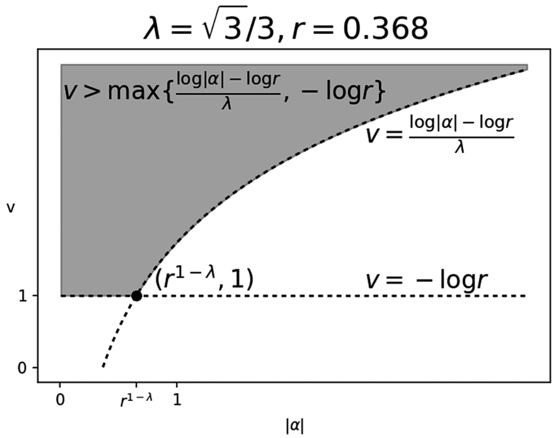

FIGURE 2. The region of $(|\alpha|, v)$ for $P_\alpha^{(r)}$.

3.1. *Positive case* $\lambda > 0$. For any $\alpha \in \mathbb{C}^*$ fixed, the leaf L_α is contained in a real three-dimensional Levi flat CR manifold† $|w| = |\alpha||z|^\lambda$, which can be viewed as a curve in $|z| = e^{-v}$, $|w| = |\alpha|e^{-\lambda v}$ coordinates. The norms $|z|$ and $|w|$ depend only on v. When $v \to +\infty$, the point on the leaf tends to the singularity $(0, 0)$ described by Figures 5 and 6.

If one fixes some $v = -\log r$, then $|z| = r$ and $|w| = |\alpha|r^\lambda$ is fixed. The set $\mathbb{T}_r^2 := \{(z, w) \in \mathbb{D}^2 : |z| = r, |w| = |\alpha|r^\lambda\}$ is a torus and the intersection of the leaf L_α with this torus is a smooth curve $L_{\alpha,r} := L_\alpha \cap \mathbb{T}_r^2$.

When $\lambda \in \mathbb{Q}$, this curve $L_{\alpha,r}$ is closed. See Figure 7.

When $\lambda \notin \mathbb{Q}$, this curve $L_{\alpha,r}$ is dense on the torus $\mathbb{T}_r^2$. See Figures 8 and 9.

† The name CR has its own history and interest in complex geometry, other than to say that CR stands both for Cauchy–Riemann and for Complex–Real.

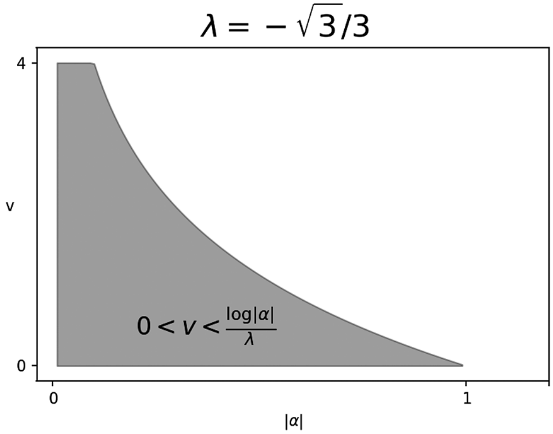

FIGURE 3. The region of $(|\alpha|, v)$ for P_α.

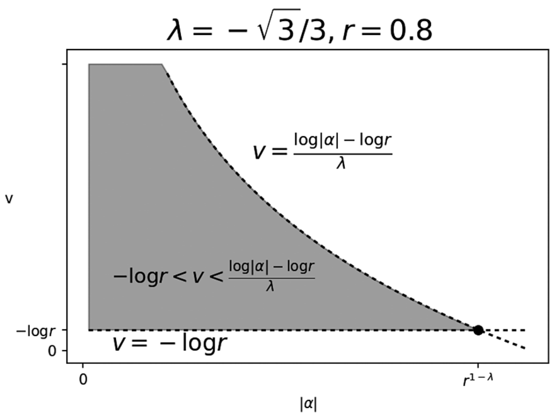

FIGURE 4. The region of $(|\alpha|, v)$ for $P_\alpha^{(r)}$.

In this case the two curves $L_{\alpha,r}$ and $L_{\alpha e^{2\pi i\lambda},r}$ are two different parametrizations of the same image. The dashed curve in Figure 8 is not only the image of $L_{\alpha,r}$ for $u \in [2\pi, 4\pi)$ but also the image of $L_{\alpha e^{2\pi i\lambda},r}$ for $u \in [0, 2\pi)$. This raises ambiguity while normalizing harmonic functions on a leaf L_α.

Such ambiguity can be resolved once one restricts everything to an open subset $U_\epsilon := \{(z, w) \in \mathbb{D}^2 \mid \arg(z) \in (0, 2\pi - \epsilon), z \neq 0, w \neq 0\}$ for some fixed $\epsilon \in [0, \pi)$. Any leaf L_α on U_ϵ decomposes into a disjoint union of infinitely many components:

$$L_\alpha \cap U_\epsilon = \bigcup_{k\in\mathbb{Z}} \left\{ (e^{-v+iu}, \alpha e^{2k\pi i\lambda} e^{-\lambda v + i\lambda u}) \mid u \in (0, 2\pi - \epsilon), v > \frac{\log^+ |\alpha|}{\lambda} \right\}.$$

For example, in Figure 10, the curve and the dashed curve are two distinct components of $L_{1,1} \cup U_\epsilon$.

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

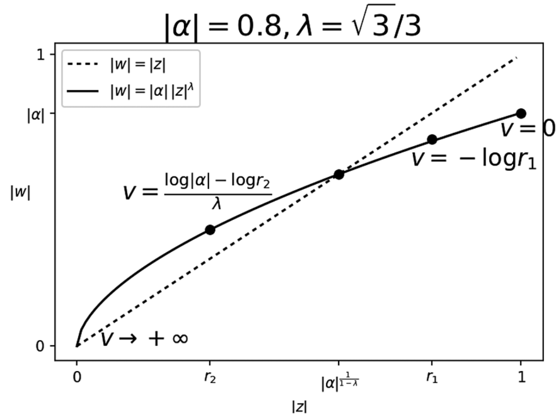

FIGURE 5. Case $|\alpha| < 1$.

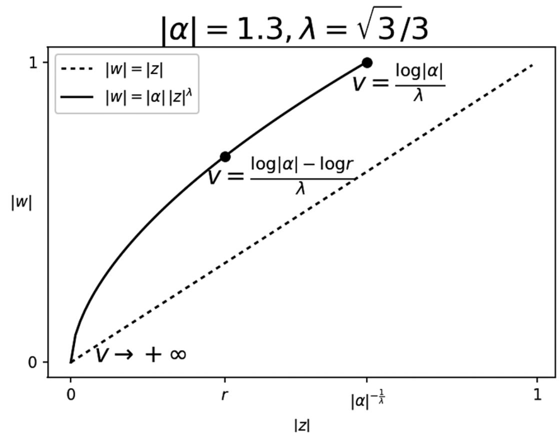

FIGURE 6. Case $|\alpha| \geqslant 1$.

Such a parametrization is yet not unique. For example, for any $k_0 \in \mathbb{Z}$ one can parametrize

$$L_\alpha \cap U_\epsilon = \bigcup_{k\in\mathbb{Z}} \bigg\{ (e^{-v+iu}, \alpha e^{2k\pi i\lambda} e^{-\lambda v + i\lambda u}) \,|\, u \in (2k_0\pi, 2k_0\pi + 2\pi - \epsilon), v > \frac{\log^+ |\alpha|}{\lambda} \bigg\}.$$

The parametrization is unique once one fixes k_0, for example, $k_0 = 0$. I remark for the time being that all other choices of k_0 will be used for analysing non-periodic currents in §5.2.

3.2. *Resolving ambiguity in the irrational case.* Let $\lambda \notin \mathbb{Q}$. Let T be a harmonic current directed by $\mathscr{F}$. Then $T|_{P_\alpha}$ has the form $h_\alpha(z, w)[P_\alpha]$. One may assume that h_α is nowhere

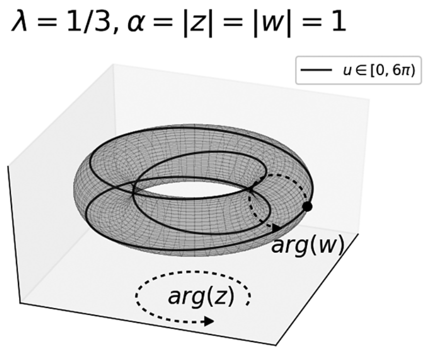

FIGURE 7. A closed curve on a torus.

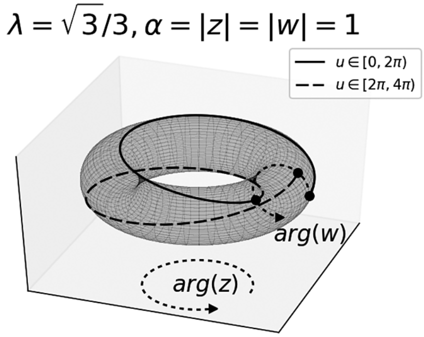

FIGURE 8. Two loops.

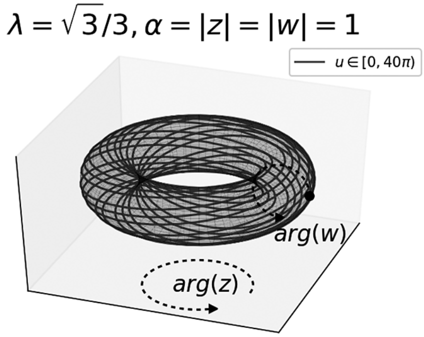

FIGURE 9. Twenty loops.

0 for every α. Let

$$H_\alpha(u+iv) := h_\alpha \circ \psi_\alpha\left(u+iv+i\frac{\log^+|\alpha|}{\lambda}\right).$$

This is a positive harmonic function for μ-almost all $\alpha \in \mathbb{C}^*$ defined in a neighbourhood of the upper half-plane $\mathbb{H} = \{(u+iv) \in \mathbb{C} \mid v > 0\}$, determined by the Poisson integral

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

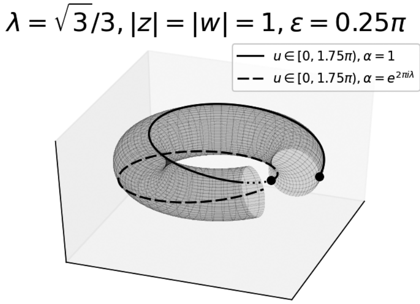

FIGURE 10. Two components of $L_{1,1} \cup U_\epsilon$.

formula

$$H_\alpha(u+iv) = \frac{1}{\pi}\int_{y\in\mathbb{R}} H_\alpha(y)\frac{v}{v^2+(y-u)^2}dy + C_\alpha v.$$

One can normalize H_α by setting $H_\alpha(0) = 1$. But by doing so one may normalize data over the same leaf for multiple times. Indeed, any pair of equivalent numbers $\alpha \sim \beta$ in $\mathbb{C}^*$, $\beta = \alpha e^{2k\pi i\lambda}$, may provide us with two different normalizations H_α and H_β on the same leaf $L_\alpha = L_\beta$. A major task is to find formulas for the mass and the Lelong number independent by the choice of normalization.

The ambiguity is described by the following proposition.

PROPOSITION 3.2. *If $\beta = \alpha e^{2k\pi i\lambda}$ for some $k \in \mathbb{Z}$, then the two normalized positive harmonic functions H_α and H_β satisfy*

$$H_\alpha(u+iv) = H_\alpha(2k\pi)H_\beta(u-2k\pi+iv).$$

In other words, they differ by a translation and a multiplication by a non-zero constant.

Proof. When $|\alpha| < 1$, by definition

$$H_\alpha(u+iv) = h_\alpha(e^{-v+iu}, \alpha e^{-\lambda v+i\lambda u}), \quad H_\alpha(0) = h_\alpha(1,\alpha).$$

Thus, the normalized harmonic function is

$$H_\alpha(u+iv) = \frac{h_\alpha(e^{-v+iu}, \alpha e^{-\lambda v+i\lambda u})}{h_\alpha(1,\alpha)},$$

and for the same reason

$$H_\beta(u+iv) = \frac{h_\beta(e^{-v+iu}, \beta e^{-\lambda v+i\lambda u})}{h_\beta(1,\beta)}.$$

The two functions h_α and h_β are the positive harmonic coefficient of T on the same leaf $L_\alpha = L_\beta$, hence they differ up to multiplication by a positive constant $C > 0$:

$$\begin{aligned} h_\alpha(e^{-v+iu}, \alpha e^{-\lambda v+i\lambda u}) &= C\cdot h_\beta(e^{-v+iu}, \alpha e^{-\lambda v+i\lambda u}) \\ &= C\cdot h_\beta(e^{-v+iu}, \beta e^{-2k\pi i\lambda}e^{-\lambda v+i\lambda u}) \\ &= C\cdot h_\beta(e^{-v+i(u-2k\pi)}, \beta e^{-\lambda v+i\lambda(u-2k\pi)}). \end{aligned}$$

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

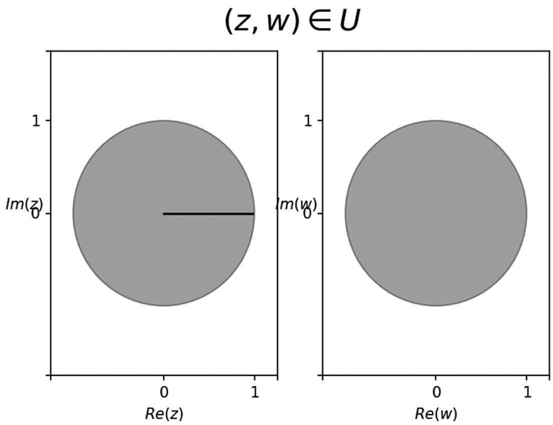

FIGURE 11. Domain U in coordinates (z, w).

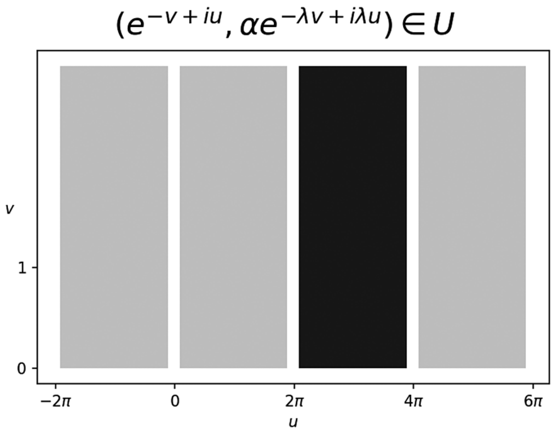

FIGURE 12. Domain U in coordinates (u, v).

Thus,

$$H_\alpha(u+iv) = \frac{h_\alpha(e^{-v+iu}, \alpha e^{-\lambda v+i\lambda u})}{h_\alpha(1,\alpha)} = \frac{C\cdot h_\beta(e^{-v+i(u-2k\pi)}, \beta e^{-\lambda v+i\lambda(u-2k\pi)})}{C\cdot h_\beta(1,\alpha)}$$
$$= \frac{h_\beta(e^{-v+i(u-2k\pi)}, \beta e^{-\lambda v+i\lambda(u-2k\pi)})}{h_\beta(1,\beta)}\cdot\frac{h_\beta(1,\beta)}{h_\beta(1,\alpha)}$$
$$= H_\beta(u-2k\pi+iv)\cdot\frac{h_\beta(1,\beta)}{h_\beta(1,\alpha)}.$$

When $u = 2k\pi$ and $v = 0$ one has $H_\alpha(2k\pi) = h_\beta(1,\beta)/h_\beta(1,\alpha)$. Thus, one gets the equality. The proof for the case $|\alpha| > 1$ is similar. □

Take the open subset $U := \{(z, w) \in \mathbb{D}^2 \mid z \notin \mathbb{R}_{\geqslant 0}, w \neq 0\}$. See Figures 11 and 12.

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

Any leaf L_α in U is a disjoint union of infinitely many components. Once α is fixed, there is a one-to-one correspondence between these components and strips in Figure 12.

$$L_\alpha \cap U = \bigcup_{k\in\mathbb{Z}} \tilde{L}_{\alpha e^{2k\pi i\lambda}} := \bigcup_{k\in\mathbb{Z}} \bigg\{(e^{-v+iu}, \alpha e^{2k\pi i\lambda} e^{-\lambda v+i\lambda u}) \,|\, u\in(0,2\pi),\, v>\frac{\log^+|\alpha|}{\lambda}\bigg\}.$$

Normalizing $H_{\alpha e^{2k\pi i\lambda}}$ on $\tilde{L}_{\alpha e^{2k\pi i\lambda}}$ avoids ambiguity. Thus, the mass

$$\begin{aligned}\|T\|_U &= \int_{(z,w)\in U} T\wedge i\partial\bar\partial(|z|^2+|w|^2)\\ &= \int_{\alpha\in\mathbb{C}^*}\int_{v>\log^+|\alpha|/\lambda}\int_{u=0}^{2\pi} H_\alpha(u+iv)2(e^{-2v}+\lambda^2|\alpha|^2e^{-2\lambda v})\,du\,dv\,d\mu(\alpha)\\ &= \int_{\alpha\in\mathbb{C}^*}\int_{v>0}\int_{u=0}^{2\pi} H_\alpha(u+iv)\|\psi'_\alpha\|^2\,du\,dv\,d\mu(\alpha)\end{aligned}$$

for some positive measure μ on $\mathbb{C}^*$. Here, $\|\psi'_\alpha\|^2$ is the jacobian coming from the $(1,1)$-form $i\partial\bar\partial(|z|^2+|w|^2)$ on L_α after a change of coordinates and a translation on v:

$$\|\psi'_\alpha\|^2 = \begin{cases} 2(e^{-2v}+\lambda^2|\alpha|^2e^{-2\lambda v}) & (|\alpha|<1),\\ 2(|\alpha|^{-2/\lambda}e^{-2v}+\lambda^2e^{-2\lambda v}) & (|\alpha|\geqslant 1).\end{cases} \tag{3}$$

Since H is harmonic in a neighbourhood of $\mathbb{H}$, it is continuous in $\mathbb{H}$. So

$$\begin{aligned}\|T\|_U &= \lim_{\epsilon\to0+}\int_{\alpha\in\mathbb{C}^*}\int_{v>0}\int_{u=0}^{2\pi+\epsilon} H_\alpha(u+iv)\|\psi'_\alpha\|^2\,du\,dv\,d\mu(\alpha)\\ &= \lim_{\epsilon\to0+}\|T\|_{\bigcup_{k\in\mathbb{Z}}\tilde{L}_{\alpha e^{2k\pi i\lambda}}}\\ &= \|T\|_{\mathbb{D}^2}.\end{aligned}$$

Thus, we can express the mass by a formula independent of the choice of normalization

$$\|T\|_{\mathbb{D}^2} = \int_{\alpha\in\mathbb{C}^*}\int_{v>0}\int_{u=0}^{2\pi} H_\alpha(u+iv)\|\psi'_\alpha\|^2\,du\,dv\,d\mu(\alpha).$$

LEMMA 3.3. *For each $k_0\in\mathbb{Z}$ fixed,*

$$\|T\|_{\mathbb{D}^2} = \int_{\alpha\in\mathbb{C}^*}\int_{v>0}\int_{u=2k_0\pi}^{2k_0\pi+2\pi} H_\alpha(u+iv)\|\psi'_\alpha\|^2\,du\,dv\,d\mu(\alpha). \tag{4}$$

Proof. The disjoint union $L_\alpha\cap U = \bigcup_{k\in\mathbb{Z}}\tilde{L}_{\alpha e^{2k\pi i\lambda}}$ can be parametrized in many other ways. For instance,

$$L_\alpha\cap U = \bigcup_{k\in\mathbb{Z}}\bigg\{(e^{-v+iu}, \alpha e^{2k\pi i\lambda}e^{-\lambda v+i\lambda u}) \mid u\in(2k_0\pi, 2k_0\pi+2\pi),\, v>\frac{\log^+|\alpha|}{\lambda}\bigg\}.$$

By the same argument as above one concludes. □

3.3. *Negative case $\lambda<0$.* As in the positive case, for any $\alpha\in\mathbb{C}^*$ fixed, the leaf L_α is contained in a real three-dimensional analytic Levi-flat CR manifold $|w|=|\alpha||z|^\lambda$, which can be viewed as a curve in $|z|$, $|w|$ coordinates. The norms $|z|$ and $|w|$ depend only on v.

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

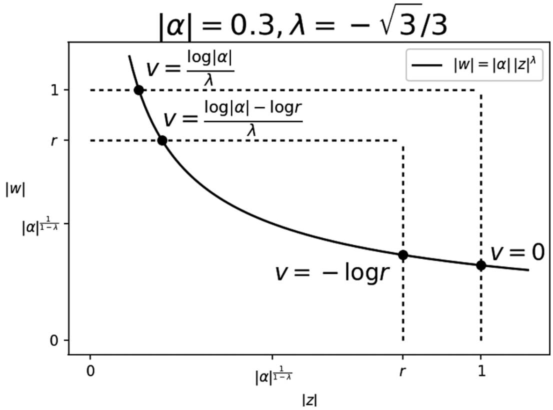

FIGURE 13. Case $\lambda < 0$.

The difference is that in the negative case, no leaf L_α tends to the singularity $(0, 0)$. For r sufficiently small, the leaf L_α is outside of $r\mathbb{D}^2$. See Figure 13.

Like the positive case $\lambda > 0$, when one fixes $|z| = r$ for some $r \in (0, 1)$, $|w| = |\alpha||z|^\lambda$ is uniquely determined and the real two-dimensional leaf L_α becomes a real 1-dimensional curve $L_{\alpha,r} := L_\alpha \cap \mathbb{T}_r^2$ on the torus $\mathbb{T}_r^2 := \{(z, w) \in \mathbb{D}^2 \mid |z| = r, |w| = |\alpha|r^\lambda\}$. It is a closed curve if $\lambda \in \mathbb{Q}$, and a dense curve on $\mathbb{T}_r^2$ if $\lambda \notin \mathbb{Q}$.

Let T be a harmonic current directed by $\mathscr{F}$. Then $T|_{P_\alpha}$ has the form $h_\alpha(z, w)[P_\alpha]$. Let $H_\alpha := h_\alpha \circ \psi_\alpha(u + iv)$. It is a positive harmonic function for μ-almost all $\alpha \in \mathbb{D}^*$ defined on a neighbourhood of a horizontal strip $\{(u, v) \in \mathbb{R}^2 \mid 0 < v < \log|\alpha|/\lambda\}$.

As in the case $\lambda > 0$, one only calculates the mass on an open subset $U := \{(z, w) \in \mathbb{D}^2 \mid z \notin \mathbb{R}_{\geqslant 0}, w \neq 0\}$. For each $\alpha \in \mathbb{D}^*$ one normalizes H_α by setting $H_\alpha(0) = 1$ to fix the expression $T := \int h_\alpha[P_\alpha]\, d\mu(\alpha)$. Similarly to Lemma 3.3, for each $k_0 \in \mathbb{Z}$ fixed,

$$\|T\|_{\mathbb{D}^2} = \int_{0<|\alpha|<1} \int_{v=0}^{\log|\alpha|/\lambda} \int_{u=2k_0\pi}^{2k_0\pi+2\pi} H_\alpha(u + iv)2(e^{-2v} + \lambda^2|\alpha|^2 e^{-2\lambda v})\, du\, dv\, d\mu(\alpha),$$

$$\begin{aligned}\mathscr{L}(T, 0) &= \lim_{r\to 0+} \frac{1}{r^2}\|T\|_{r\mathbb{D}^2} \\ &= \lim_{r\to 0+} \frac{1}{r^2} \int_{0<|\alpha|<r^{1-\lambda}} \int_{v=-\log r}^{(\log|\alpha|-\log r)/\lambda} \int_{u=2k_0\pi}^{2k_0\pi+2\pi} \\ &\qquad H_\alpha(u + iv)2(e^{-2v} + \lambda^2|\alpha|^2 e^{-2\lambda v})\, du\, dv\, d\mu(\alpha).\end{aligned}$$

These formulas will be calculated in later sections.

4. *Positive rational case:* $\lambda = (a/b) \in \mathbb{Q}$, $\lambda \in (0, 1]$

Write $\lambda = a/b$ where $a, b \in \mathbb{Z}_{\geqslant 1}$ are coprime. Then in $\mathbb{D}^2$, for any $\alpha \in \mathbb{C}^*$, the union $L_\alpha \cup \{0\}$ is the algebraic curve $\{w^b = \alpha^b z^a\} \cap \mathbb{D}^2$. In other words, every leaf is a separatrix. In this section it will be shown that any directed harmonic current T has non-zero Lelong number.

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

The parametrization map $\psi_\alpha(\zeta) := (e^{i\zeta}, \alpha e^{i\lambda\zeta})$ is now periodic: $\psi_\alpha(\zeta + 2\pi b) = \psi_\alpha(\zeta)$. Let T be a directed harmonic current. Then $T|_{P_\alpha}$ has the form $h_\alpha(z, w)[P_\alpha]$. Let

$$H_\alpha(u + iv) := h_\alpha \circ \psi_\alpha\bigg(u + iv + i\frac{\log^+ |\alpha|}{\lambda}\bigg).$$

This is a positive harmonic function for μ-almost all $\alpha \in \mathbb{C}^*$ defined in a neighbourhood of the upper half-plane $\mathbb{H} := \{(u + iv) \in \mathbb{C} \mid v > 0\}$. Moreover, it is periodic: $H_\alpha(u + iv) = H_\alpha(u + 2\pi b + iv)$. Periodic harmonic functions can be characterized by the following lemma.

LEMMA 4.1. *Let $F(u, v)$ be a harmonic function in a neighbourhood of $\mathbb{H}$. If $F(u, v) = F(u + 2\pi b, v)$ for all $(u, v) \in \mathbb{H}$, then*

$$F(u, v) = \sum_{k\in\mathbb{Z}, k\neq 0} \bigg(a_k e^{kv/b} \cos\bigg(\frac{ku}{b}\bigg) + b_k e^{kv/b} \sin\bigg(\frac{ku}{b}\bigg)\bigg) + a_0 + b_0 v,$$

for some $a_k, b_k \in \mathbb{R}$. Moreover, if $F|_{\mathbb{H}} \geqslant 0$, then $a_0, b_0 \geqslant 0$.

Proof. By periodicity

$$F(u, v) = \sum_{k=1}^{\infty} \bigg(A_k(v) \cos\bigg(\frac{ku}{b}\bigg) + B_k(v) \sin\bigg(\frac{ku}{b}\bigg)\bigg) + A_0(v),$$

for some functions $A_k(v)$, $B_k(v)$. They are smooth since F is harmonic. Moreover,

$$\begin{aligned} 0 &= \Delta F(u, v) \\ &= \sum_{k=1}^{\infty}\bigg(\bigg(A_k''(v) - \bigg(\frac{k}{b}\bigg)^2 A_k(v)\bigg)\cos\bigg(\frac{ku}{b}\bigg) + \bigg(B_k''(v) - \bigg(\frac{k}{b}\bigg)^2 B_k(v)\bigg)\sin\bigg(\frac{ku}{b}\bigg)\bigg) + A_0''(v). \end{aligned}$$

Thus,

$$A_k''(v) = \bigg(\frac{k}{b}\bigg)^2 A_k(v), \quad B_k''(v) = \bigg(\frac{k}{b}\bigg)^2 B_k(v), \quad A_0''(v) = 0.$$

Hence,

$$A_k(v) = a_k e^{kv/b} + a_{-k} e^{-kv/b}, \quad B_k(v) = b_k e^{kv/b} - b_{-k} e^{-kv/b}, \quad A_0(v) = a_0 + b_0 v,$$

for some $a_k, a_{-k}, b_k, b_{-k} \in \mathbb{R}$. One obtains the equality.

If $F|_{\mathbb{H}} \geqslant 0$, then for any $v \geqslant 0$,

$$\int_{u=0}^{2\pi b} F(u, v)\, du = 2\pi b(a_0 + b_0 v) \geqslant 0.$$

Thus, $a_0, b_0 \geqslant 0$. □

For $\alpha, \beta \in \mathbb{C}^*$, the two maps ψ_α and ψ_β parametrize the same leaf $L_\alpha = L_\beta$ if and only if $\beta = \alpha e^{2\pi i(k/b)}$ for some $k \in \mathbb{Z}$, that is α and β differ from multiplying a bth root of unity. Thus, a transversal can be chosen as the sector $\mathbb{S} := \{\alpha \in \mathbb{C}^* \mid \arg(\alpha) \in [0, 2\pi/b)\}$. One fixes a normalization by setting $H_\alpha(0) = h_\alpha \circ \psi_\alpha(i(\log^+ |\alpha|/\lambda)) = 1$.

The mass of the current T is

$$\|T\|_{\mathbb{D}^2} = \int_{(z,w)\in\mathbb{D}^2} T \wedge i\partial\bar{\partial}(|z|^2 + |w|^2).$$

In particular, one calculates the $(1,1)$-form $i\partial\bar{\partial}(|z|^2 + |w|^2)$ on L_α, where $z = e^{-v+iu}$, $w = \alpha e^{-\lambda v + i\lambda u}$, using

$$dz = ie^{-v+iu}\,du - e^{-v+iu}\,dv, \qquad d\bar{z} = -ie^{-v-iu}\,du - e^{-v-iu}\,dv,$$
$$dw = i\alpha\lambda e^{-\lambda v+i\lambda u}\,du - \alpha\lambda e^{-\lambda v+i\lambda u}\,dv, \quad d\bar{w} = -i\bar{\alpha}\lambda e^{-\lambda v-i\lambda u}\,du - \bar{\alpha}\lambda e^{-\lambda v-i\lambda u}\,dv,$$

whence

$$\begin{aligned} i\partial\bar{\partial}(|z|^2 + |w|^2) &= i\,dz \wedge d\bar{z} + i\,dw \wedge d\bar{w} \\ &= 2(e^{-2v} + \lambda^2|\alpha|^2 e^{-2\lambda v})\,du \wedge dv. \end{aligned}$$

Thus,

$$\begin{aligned} \|T\|_{\mathbb{D}^2} &= \int_{\alpha\in\mathbb{S}} h_\alpha(z,w) \int_{P_\alpha} i\partial\bar{\partial}(|z|^2+|w|^2)\,d\mu(\alpha) \\ &= \int_{\alpha\in\mathbb{S}} \int_{u=0}^{2\pi b} \int_{v>0} H_\alpha(u+iv) 2(e^{-2(v+\log^+|\alpha|/\lambda)} \\ &\qquad + \lambda^2|\alpha|^2 e^{-2\lambda(v+\log^+|\alpha|/\lambda)})\,du\wedge dv\,d\mu(\alpha) \\ &= \int_{\alpha\in\mathbb{S},|\alpha|<1} \int_{u=0}^{2\pi b} \int_{v>0} H_\alpha(u+iv) 2(e^{-2v} + \lambda^2|\alpha|^2 e^{-2\lambda v})\,du\wedge dv\,d\mu(\alpha) \\ &\quad + \int_{\alpha\in\mathbb{S},|\alpha|\geqslant 1} \int_{u=0}^{2\pi b} \int_{v>0} H_\alpha(u+iv) 2(|\alpha|^{-2/\lambda} e^{-2v} + \lambda^2 e^{-2\lambda v})\,du\wedge dv\,d\mu(\alpha). \end{aligned}$$

By Lemma 4.1,

$$H_\alpha(u+iv) = \sum_{k\in\mathbb{Z},k\neq 0} \left(a_k(\alpha) e^{kv/b} \cos\left(\frac{ku}{b}\right) + b_k(\alpha) e^{kv/b} \sin\left(\frac{ku}{b}\right)\right) + a_0(\alpha) + b_0(\alpha)v, \tag{5}$$

where $a_0(\alpha)$, $b_0(\alpha)$ are positive for μ-almost all α. Thus,

$$\begin{aligned} &\|T\|_{\mathbb{D}^2} \\ &\quad = 2\pi b\bigg\{ \int_{\alpha\in\mathbb{S},|\alpha|<1} \int_{v>0} (a_0(\alpha) + b_0(\alpha)v) 2(e^{-2v} + \lambda^2|\alpha|^2 e^{-2\lambda v})\,dv\,d\mu(\alpha) \\ &\qquad + \int_{\alpha\in\mathbb{S},|\alpha|\geqslant 1} \int_{v>0} (a_0(\alpha) + b_0(\alpha)v) 2(|\alpha|^{-2/\lambda} e^{-2v} + \lambda^2 e^{-2\lambda v})\,dv\,d\mu(\alpha)\bigg\} \\ &\quad = 2\pi b\bigg\{\int_{\alpha\in\mathbb{S},|\alpha|<1} a_0(\alpha)(1+|\alpha|^2\lambda)\,d\mu(\alpha) + \int_{\alpha\in\mathbb{S},|\alpha|\geqslant 1} a_0(\alpha)(|\alpha|^{-2/\lambda}+\lambda)\,d\mu(\alpha) \\ &\qquad + \int_{\alpha\in\mathbb{S},|\alpha|<1} b_0(\alpha)\left(\frac{1}{2} + \frac{1}{2}|\alpha|^2\right)d\mu(\alpha) + \int_{\alpha\in\mathbb{S},|\alpha|\geqslant 1} b_0(\alpha)\left(\frac{1}{2}+\frac{1}{2}|\alpha|^{-2/\lambda}\right)d\mu(\alpha)\bigg\} \\ &\quad \approx \int_{\alpha\in\mathbb{S}} a_0(\alpha)\,d\mu(\alpha) + \int_{\alpha\in\mathbb{S}} b_0(\alpha)\,d\mu(\alpha). \end{aligned}$$

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

The Lelong number can now be calculated as follows:

$$
\begin{aligned}
&\mathscr{L}(T,0)\\
&= \lim_{r\to 0+} \frac{1}{r^2}\|T\|_{r\mathbb{D}^2}\\
&= \lim_{r\to 0+} \frac{1}{r^2} 2\pi b\bigg\{ \int_{\alpha\in\mathbb{S},|\alpha|<r^{1-\lambda}} \int_{v>-\log r} (a_0(\alpha)+b_0(\alpha)v)2(e^{-2v}\\
&\qquad\qquad + \lambda^2|\alpha|^2 e^{-2\lambda v})\, dv\, d\mu(\alpha)\\
&\quad + \int_{\alpha\in\mathbb{S},r^{1-\lambda}\leqslant|\alpha|<1} \int_{v>(\log|\alpha|-\log r)/\lambda} (a_0(\alpha)+b_0(\alpha)v)2(e^{-2v}\\
&\qquad\qquad + \lambda^2|\alpha|^2 e^{-2\lambda v})\, dv\, d\mu(\alpha)\\
&\quad + \int_{\alpha\in\mathbb{S},|\alpha|\geqslant 1} \int_{v>-\log r/\lambda} (a_0(\alpha)+b_0(\alpha)v)2(|\alpha|^{-2/\lambda}e^{-2v}+\lambda^2 e^{-2\lambda v})\, dv\, d\mu(\alpha)\bigg\}\\
&= \lim_{r\to 0+} 2\pi b\bigg\{ \int_{\alpha\in\mathbb{S},|\alpha|<r^{1-\lambda}} a_0(\alpha)(1+\lambda|\alpha|^2 r^{2\lambda-2})\, d\mu(\alpha)\\
&\quad + \int_{\alpha\in\mathbb{S},|\alpha|\geqslant r^{1-\lambda}} a_0(\alpha)(|\alpha|^{-2/\lambda}r^{2/\lambda-2}+\lambda)\, d\mu(\alpha)\\
&\quad + \int_{\alpha\in\mathbb{S},|\alpha|<r^{1-\lambda}} b_0(\alpha)\bigg(\frac{1}{2}+\frac{1}{2}|\alpha|^2 r^{2\lambda-2} - \log r - \lambda|\alpha|^2 r^{2\lambda-2}\log r\bigg)\, d\mu(\alpha)\\
&\quad + \int_{\alpha\in\mathbb{S},r^{1-\lambda}\leqslant|\alpha|<1} b_0(\alpha)\bigg(\frac{1}{2}+\frac{1}{2}|\alpha|^{-2/\lambda}r^{2/\lambda-2} - \log r - |\alpha|^{-2/\lambda}\lambda^{-1}r^{2\lambda-2}\log r\\
&\qquad\qquad + \log|\alpha| + \lambda^{-1}|\alpha|^{-2/\lambda}\log|\alpha| r^{2\lambda-2}\bigg)\, d\mu(\alpha)\\
&\quad + \int_{\alpha\in\mathbb{S},|\alpha|\geqslant 1} b_0(\alpha)\bigg(\frac{1}{2}+\frac{1}{2}|\alpha|^{-2/\lambda}r^{2/\lambda-2} - \log r - \lambda^{-1}|\alpha|^{-2/\lambda}r^{2\lambda-2}\log r\bigg)\, d\mu(\alpha)\bigg\}.
\end{aligned}
$$

First one analyses the $a_0(\alpha)$ part. When $|\alpha| < r^{1-\lambda}$,

$$1 < 1+\lambda|\alpha|^2 r^{2\lambda-2} < 1+\lambda r^{2-2\lambda} r^{2\lambda-2} = 1+\lambda, \tag{6}$$

is uniformly bounded with respect to α and r. When $|\alpha| \geqslant r^{1-\lambda}$

$$\lambda < |\alpha|^{-2/\lambda} r^{2/\lambda-2} + \lambda < 1+\lambda, \tag{7}$$

is also uniformly bounded with respect to α and r. Thus,

$$\mathscr{L}(T,0) \approx \underbrace{\int_{\alpha\in\mathbb{S}} a_0(\alpha)\, d\mu(\alpha)}_{\text{linear part}} + \underbrace{\lim_{r\to 0+} (b_0(\alpha)\text{part})}_{\text{with } v \text{ part}}.$$

Next one analyses the $b_0(\alpha)$ part.

LEMMA 4.2. *The Lelong number of T at 0 is finite only if $b_0(\alpha) = 0$ for μ-almost all $\alpha \in \mathbb{S}$.*

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

Proof. Suppose not, that is, $\int_{\alpha\in\mathbb{S}} b_0(\alpha)\, d\mu(\alpha) = B_0 > 0$. Then

$$\begin{aligned}\mathscr{L}(T,0) &\geqslant \lim_{r\to 0+} 2\pi b\bigg\{\int_{\alpha\in\mathbb{S},|\alpha|<r^{1-\lambda}} b_0(\alpha)(-\log r)\, d\mu(\alpha) \\ &\qquad + \int_{\alpha\in\mathbb{S},|\alpha|\geqslant r^{1-\lambda}} b_0(\alpha)(-\log r)\, d\mu(\alpha)\bigg\} \\ &= 2\pi b B_0 \lim_{r\to 0+}(-\log r) = +\infty,\end{aligned}$$

contradicting the finiteness of the Lelong number stated in Theorem 2.11. □

Thus, one may assume $b_0(\alpha) = 0$ for μ-almost all $\alpha \in \mathbb{S}$. Then the Lelong number

$$\mathscr{L}(T,0) \approx \int_{\alpha\in\mathbb{S}} a_0(\alpha)\, d\mu(\alpha) \approx \|T\|_{\mathbb{D}^2}$$

is strictly positive.

5. *Positive irrational case* $\lambda \notin \mathbb{Q}$, $\lambda \in (0, 1)$

Now $\{z = 0\}$ and $\{w = 0\}$ are the only two separatrices in $\mathbb{D}^2$. For each fixed $\alpha \in \mathbb{C}^*$, the map $\psi_\alpha(\zeta) = (e^{i\zeta}, \alpha e^{i\lambda\zeta})$ is injective since $\lambda \notin \mathbb{Q}$.

5.1. *Periodic currents, still a Fourier series.* Periodic currents behave similarly to currents in the rational case $\lambda \in \mathbb{Q}$. Suppose H_α is periodic, that is, there is some $b \in \mathbb{Z}_{\geqslant 1}$ such that $H_\alpha(u + iv) = H_\alpha(u + 2\pi b + iv)$ for any $u + iv \in \mathbb{H}$. Periodic harmonic functions are characterized as in (5) of Lemma 4.1.

According to Lemma 3.3, the mass is

$$\|T\|_{\mathbb{D}^2} = \int_{\alpha\in\mathbb{C}^*}\int_{v>0}\int_{u=2k_0\pi}^{2k_0\pi+2\pi} H_\alpha(u+iv)\|\psi'_\alpha\|^2\, du\wedge dv\, d\mu(\alpha),$$

for any $k_0 \in \mathbb{Z}$, in particular for $k_0 = 0, 1, \ldots, b-1$. Thus, we may calculate

$$\begin{aligned} b\|T\|_{\mathbb{D}^2} &= \int_{\alpha\in\mathbb{C}^*}\int_{v>0}\int_{u=0}^{2\pi b} H_\alpha(u+iv)\|\psi'_\alpha\|^2\, du\wedge dv\, d\mu(\alpha) \\ \|T\|_{\mathbb{D}^2} &= \frac{1}{b}\int_{\alpha\in\mathbb{C}^*}\int_{v>0}\int_{u=0}^{2\pi b} H_\alpha(u+iv)\|\psi'_\alpha\|^2\, du\wedge dv\, d\mu(\alpha), \\ &= \frac{1}{b}\bigg\{\int_{|\alpha|<1}\int_{v>0}\int_{u=0}^{2\pi b} H_\alpha(u+iv)2(e^{-2v}+\lambda^2|\alpha|^2e^{-2\lambda v})\, du\wedge dv\, d\mu(\alpha) \\ &\quad + \int_{|\alpha|\geqslant 1}\int_{v>0}\int_{u=0}^{2\pi b} H_\alpha(u+iv)2(|\alpha|^{-2/\lambda}e^{-2v}+\lambda^2e^{-2\lambda v})\, du\wedge dv\, d\mu(\alpha)\bigg\}, \\ &= \frac{2\pi b}{b}\bigg\{\int_{|\alpha|<1}\int_{v>0}(a_0(\alpha)+b_0(\alpha)v)2(e^{-2v}+\lambda^2|\alpha|^2e^{-2\lambda v})\, dv\, d\mu(\alpha) \\ &\quad + \int_{|\alpha|\geqslant 1}\int_{v>0}(a_0(\alpha)+b_0(\alpha)v)2(|\alpha|^{-2/\lambda}e^{-2v}+\lambda^2e^{-2\lambda v})\, dv\, d\mu(\alpha)\bigg\}, \\ &= 2\pi\bigg\{\int_{|\alpha|<1} a_0(\alpha)(1+|\alpha|^2\lambda)\, d\mu(\alpha) + \int_{|\alpha|\geqslant 1} a_0(\alpha)(|\alpha|^{-2/\lambda}+\lambda)\, d\mu(\alpha)\end{aligned}$$

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

$$+\int_{|\alpha|<1} b_0(\alpha)\left(\frac{1}{2}+\frac{1}{2}|\alpha|^2\right) d\mu(\alpha)+\int_{|\alpha|\geqslant 1} b_0(\alpha)\left(\frac{1}{2}+\frac{1}{2}|\alpha|^{-2/\lambda}\right) d\mu(\alpha)\bigg\}$$
$$\approx \int_{\alpha\in\mathbb{C}^*} a_0(\alpha)\, d\mu(\alpha)+\int_{\alpha\in\mathbb{C}^*} b_0(\alpha)\, d\mu(\alpha),$$

which is the same expression as in the case $\lambda\in\mathbb{Q}_{>0}$.

Next, the Lelong number is calculated as

$$\begin{aligned}
&\mathscr{L}(T,0)\\
&=\lim_{r\to 0+}\frac{1}{r^2}\|T\|_{r\mathbb{D}^2}\\
&=\lim_{r\to 0+}\frac{1}{r^2}2\pi\bigg\{\int_{|\alpha|<r^{1-\lambda}}\int_{v>-\log r}(a_0(\alpha)+b_0(\alpha)v)2(e^{-2v}\\
&\qquad\qquad +\lambda^2|\alpha|^2e^{-2\lambda v})\, dv\, d\mu(\alpha)\\
&\quad+\int_{r^{1-\lambda}\leqslant|\alpha|<1}\int_{v>(\log|\alpha|-\log r/\lambda)}(a_0(\alpha)+b_0(\alpha)v)2(e^{-2v}\\
&\qquad\qquad +\lambda^2|\alpha|^2e^{-2\lambda v})\, dv\, d\mu(\alpha)\bigg\}\\
&\quad+\int_{|\alpha|\geqslant 1}\int_{v>-\log r/\lambda}(a_0(\alpha)+b_0(\alpha)v)2(|\alpha|^{-2/\lambda}e^{-2v}+\lambda^2e^{-2\lambda v})\, dv\, d\mu(\alpha)\bigg\}\\
&=\lim_{r\to 0+}2\pi\bigg\{\int_{|\alpha|<r^{1-\lambda}}a_0(\alpha)(1+\lambda|\alpha|^2r^{2\lambda-2})\, d\mu(\alpha)\\
&\quad+\int_{|\alpha|\geqslant r^{1-\lambda}}a_0(\alpha)(|\alpha|^{-2/\lambda}r^{2/\lambda-2}+\lambda)\, d\mu(\alpha)\\
&\quad+\int_{|\alpha|<r^{1-\lambda}}b_0(\alpha)\left(\frac{1}{2}+\frac{1}{2}|\alpha|^2r^{2\lambda-2}-\log r-\lambda|\alpha|^2r^{2\lambda-2}\log r\right) d\mu(\alpha)\\
&\quad+\int_{r^{1-\lambda}\leqslant|\alpha|<1}b_0(\alpha)\bigg(\frac{1}{2}+\frac{1}{2}|\alpha|^{-2/\lambda}r^{2/\lambda-2}-\log r-\lambda^{-1}|\alpha|^{-2/\lambda}r^{2\lambda-2}\log r\\
&\quad+\log|\alpha|+\lambda^{-1}|\alpha|^{-2/\lambda}\log|\alpha|r^{2\lambda-2}\bigg)\, d\mu(\alpha)\\
&\quad+\int_{|\alpha|\geqslant 1}b_0(\alpha)\left(\frac{1}{2}+\frac{1}{2}|\alpha|^{-2/\lambda}r^{2/\lambda-2}-\log r-\lambda^{-1}|\alpha|^{-2/\lambda}r^{2\lambda-2}\log r\right) d\mu(\alpha)\bigg\},
\end{aligned}$$

exactly the same expression as in the positive rational case with $b=1$. Using the same argument as in Lemma 4.2, one may assume that $b_0(\alpha)=0$ for μ-almost all $\alpha\in\mathbb{C}^*$. One concludes that

$$\mathscr{L}(T,0)\approx\int_{\alpha\in\mathbb{C}^*}a_0(\alpha)\, d\mu(\alpha)\approx\|T\|_{\mathbb{D}^2}.$$

The Lelong number is strictly positive, the same as in the case $\lambda\in\mathbb{Q}\cup(0,1)$.

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

5.2. *Non-periodic current.* For periodic currents, one takes an average among b expressions (4) in the previous section. For non-periodic currents, there is no canonical way of normalization. The key technique is to calculate expressions (4) for all $k_0 \in \mathbb{Z}$.

The Lelong number is expressed as

$$\begin{aligned}\mathscr{L}(T,0) = \lim_{r\to 0+} \frac{1}{r^2}\bigg\{ &\int_{|\alpha|<r^{1-\lambda}} \int_{v>-\log r} \int_{u=0}^{2\pi} H_\alpha(u+iv)\|\psi'_\alpha\|^2 du\, dv\, d\mu(\alpha)\\ &+ \int_{r^{1-\lambda}\leqslant|\alpha|<1} \int_{v>(\log|\alpha|-\log r)/\lambda} \int_{u=0}^{2\pi} H_\alpha(u+iv)\|\psi'_\alpha\|^2 du\, dv\, d\mu(\alpha)\\ &+ \int_{|\alpha|\geqslant 1} \int_{v>-\log r/\lambda} \int_{u=0}^{2\pi} H_\alpha(u+iv)\|\psi'_\alpha\|^2 du\, dv\, d\mu(\alpha)\bigg\}\end{aligned}$$

Recall the Poisson integral formula after multiplying by a non-zero constant:

$$H_\alpha(u+iv) = \frac{1}{\pi}\int_{y\in\mathbb{R}} H_\alpha(y)\frac{v}{v^2+(y-u)^2}dy + C_\alpha v.$$

Using the same argument as in Lemma 4.2, one may assume $C_\alpha = 0$ for all $\alpha \in \mathbb{C}^*$.

LEMMA 5.1. *For any $v \geqslant 1/\lambda > 1$ and for any $u \in \mathbb{R}$,*

$$\frac{\partial/\partial v(-\frac{1}{2}(v/(v^2+(u-y)^2)e^{-2v}))}{v/(v^2+(u-y)^2)e^{-2v}} \in \left(\frac{1}{2},2\right),$$

$$\frac{\partial/\partial v(-(1/2\lambda)(v/(v^2+(u-y)^2))e^{-2\lambda v})}{v/(v^2+(u-y)^2)e^{-2\lambda v}} \in \left(\frac{1}{2},2\right).$$

Proof. This can be calculated directly:

$$\begin{aligned}\frac{\partial}{\partial v}\left(-\frac{1}{2}\frac{v}{v^2+(u-y)^2}e^{-2v}\right) &= \left(\frac{v}{v^2+(u-y)^2} + \left(-\frac{1}{2}\right)\frac{1}{v^2+(u-y)^2}\right.\\ &\qquad \left. + \left(-\frac{1}{2}\right)\frac{v(-2v)}{(v^2+(u-y)^2)^2}\right)e^{-2v}\\ \frac{\partial/\partial v(-\frac{1}{2}(v/(v^2+(u-y)^2))e^{-2v})}{v/(v^2+(u-y)^2)e^{-2v}} &= 1 + \left(-\frac{1}{2}\frac{1}{v}\right) + \frac{v}{v^2+(u-y)^2}\\ &\in \left(1-\frac{1}{2v}, 1+\frac{1}{v}\right) \subseteq \left(\frac{1}{2},2\right) \quad (v>1),\\ \frac{\partial}{\partial v}\left(-\frac{1}{2\lambda}\frac{v}{v^2+(u-y)^2}e^{-2\lambda v}\right) &= \left(\frac{v}{v^2+(u-y)^2} + \left(-\frac{1}{2\lambda}\right)\frac{1}{v^2+(u-y)^2}\right.\\ &\qquad \left. + \left(-\frac{1}{2\lambda}\right)\frac{v(-2v)}{(v^2+(u-y)^2)^2}\right)e^{-2\lambda v}\\ \frac{\partial/\partial v(-(1/2\lambda)(v/(v^2+(u-y)^2))e^{-2\lambda v})}{v/(v^2+(u-y)^2)e^{-2\lambda v}} &= 1 + \left(-\frac{1}{2\lambda}\frac{1}{v}\right) + \frac{1}{\lambda}\frac{v}{v^2+(u-y)^2}\\ &\in \left(1-\frac{1}{2\lambda v}, 1+\frac{1}{\lambda v}\right) \subseteq \left(\frac{1}{2},2\right) \ \left(v \geqslant \frac{1}{\lambda}\right).\end{aligned}$$

□

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

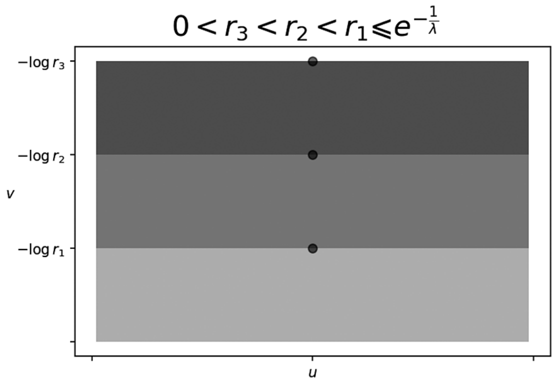

FIGURE 14. $1/r^2$ (The integration on $v > -\log r$) $\approx$ (The value at $v = -\log r$).

COROLLARY 5.2. *For any r such that* $0 < r \leqslant e^{-1/\lambda}$,

$$\frac{1}{r^2}\int_{v>-\log r} H_\alpha(u+iv)\|\psi_\alpha'\|^2 dv \approx H_\alpha(u+(-\log r)i) \quad (0<|\alpha|<r^{1-\lambda}),$$

$$\frac{1}{r^2}\int_{v>(\log|\alpha|-\log r)/\lambda} H_\alpha(u+iv)\|\psi_\alpha'\|^2 dv \approx H_\alpha\bigg(u+\bigg(\frac{\log|\alpha|-\log r}{\lambda}\bigg)i\bigg) \quad (r^{1-\lambda}\leqslant|\alpha|<1),$$

$$\frac{1}{r^2}\int_{v>(\log|\alpha|-\log r)/\lambda} H_\alpha(u+iv)\|\psi_\alpha'\|^2 dv \approx H_\alpha\bigg(u+\bigg(\frac{-\log r}{\lambda}\bigg)i\bigg) \quad (|\alpha|\geqslant 1).$$

Figure 14 explains Corollary 5.2. We remark that Corollary 5.2 is true for $r \in (0,1)$ after a dilation $(z,w) \mapsto (e^{1/2\lambda}z, e^{1/2\lambda}w)$.

Proof. The assumption $0 < r \leqslant e^{-1/\lambda}$ implies $-\log r \geqslant 1/\lambda$. Hence, for $v \geqslant -\log r \geqslant 1/\lambda$, Lemma 5.1 holds.

First, when $0 < |\alpha| \leqslant r^{1-\lambda}$,

$$\begin{aligned}
&\int_{v>-\log r} H_\alpha(u+iv)\|\psi_\alpha'\|^2 dv\\
&= \frac{1}{\pi}\int_{v>-\log r}\int_{y\in\mathbb{R}} H_\alpha(y)\frac{v}{v^2+(u-y)^2}2(e^{-2v}+\lambda^2|\alpha|^2e^{-2\lambda v})\,dy\,dv\\
&\approx \frac{1}{\pi}\int_{y\in\mathbb{R}} H_\alpha(y)\bigg\{\int_{v>-\log r}\frac{\partial}{\partial v}\bigg(\frac{v}{v^2+(u-y)^2}(-e^{-2v}-\lambda|\alpha|^2e^{-2\lambda v})\bigg)\,dv\bigg\}\,dy\\
&= \frac{1}{\pi}\int_{y\in\mathbb{R}} H_\alpha(y)\frac{-\log r}{(-\log r)^2+(u-y)^2}(r^2+\lambda|\alpha|^2r^{2\lambda})\,dy\\
&= H_\alpha(u+(-\log r)i)(r^2+\lambda|\alpha|^2r^{2\lambda})\\
&\approx r^2H_\alpha(u+(-\log r)i).
\end{aligned}$$

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

For the same reason, when $r^{1-\lambda} \leqslant |\alpha| < 1$, which implies $(\log|\alpha| - \log r)/\lambda \geqslant -\log r \geqslant 1/\lambda$,

$$\begin{aligned}\int_{v>(\log|\alpha|-\log r)/\lambda} & H_\alpha(u+iv)\|\psi'_\alpha\|^2\, dv \\ &\approx H_\alpha\bigg(u+\bigg(\frac{\log|\alpha|-\log r}{\lambda}\bigg)i\bigg)(|\alpha|^{-2/\lambda}r^{2/\lambda}+\lambda r^2) \\ &\approx r^2 H_\alpha\bigg(u+\bigg(\frac{\log|\alpha|-\log r}{\lambda}\bigg)i\bigg).\end{aligned}$$

Finally, when $|\alpha| \geqslant 1$ one has $-\log r/\lambda \geqslant -\log r \geqslant 1/\lambda$ and

$$\begin{aligned}\int_{v>-\log r/\lambda} H_\alpha(u+iv)\|\psi'_\alpha\|^2 dv &\approx H_\alpha\bigg(u+\bigg(\frac{-\log r}{\lambda}\bigg)i\bigg)(|\alpha|^{-2/\lambda}r^{2/\lambda}+\lambda r^2) \\ &\approx r^2 H_\alpha\bigg(u+\bigg(\frac{-\log r}{\lambda}\bigg)i\bigg).\end{aligned}$$

□

Thus,

$$\begin{aligned}\mathscr{L}(T,0) \approx \lim_{r\to 0+}\bigg\{ &\int_{|\alpha|<r^{1-\lambda}}\int_{u=0}^{2\pi} H_\alpha(u+(-\log r)i)\, du\, d\mu(\alpha) \\ &+\int_{r^{1-\lambda}\leqslant|\alpha|<1}\int_{u=0}^{2\pi} H_\alpha\bigg(u+\bigg(\frac{\log|\alpha|-\log r}{\lambda}\bigg)i\bigg)\, du\, d\mu(\alpha) \\ &+\int_{|\alpha|\geqslant 1}\int_{u=0}^{2\pi} H_\alpha\bigg(u+\bigg(\frac{-\log r}{\lambda}\bigg)i\bigg)\, du\, d\mu(\alpha)\bigg\},\end{aligned}$$

by inequalities (6) and (7) in the previous subsection. All terms are positive, so the order of taking the limit and integration can change:

$$\begin{aligned}\mathscr{L}(T,0) &\approx \lim_{v\to+\infty}\int_{\alpha\in\mathbb{C}^*}\int_{u=0}^{2\pi} H_\alpha(u+iv)\, du\, d\mu(\alpha) \\ &= \lim_{k\to+\infty}\int_{\alpha\in\mathbb{C}^*}\int_{u=0}^{2\pi}\int_{y\in\mathbb{R}} H_\alpha(y)\frac{2k\pi}{(2k\pi)^2+(u-y)^2}\, dy\, du\, d\mu(\alpha).\end{aligned}$$

Fix some $k \in \mathbb{Z}$, $k \geqslant 2$. Define intervals I_N for all $N \in \mathbb{Z}$ as follows:

$$\begin{aligned} I_0 &= [-2k\pi+2\pi, 2k\pi), \\ I_N &= \begin{cases}[2kN\pi, 2k(N+1)\pi) & (N>0), \\ [2k(N-1)\pi+2\pi, 2kN\pi+2\pi) & (N<0).\end{cases}\end{aligned}$$

Thus, $\mathbb{R} = \bigcup_{N\in\mathbb{Z}} I_N$ is a disjoint union.

LEMMA 5.3. *For any $u \in (0, 2\pi)$, one has*

$$\frac{2k\pi}{(2k\pi)^2+(u-y)^2} \geqslant \frac{1}{1+(N+1)^2}\frac{1}{2k\pi} \quad (y \in I_N).$$

Proof. Elementary. □

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

Thus,

$$\begin{aligned}\mathscr{L}(T,0) &\approx \lim_{k\to+\infty}\sum_{N\in\mathbb{Z}}\int_{\alpha\in\mathbb{C}^*}\int_{u=0}^{2\pi}\int_{y\in I_N} H_\alpha(y)\frac{2k\pi}{(2k\pi)^2+(u-y)^2}\,dy\,du\,d\mu(\alpha)\\ &\geqslant \lim_{k\to+\infty}\sum_{N\in\mathbb{Z}}\int_{\alpha\in\mathbb{C}^*}\int_{y\in I_N}\int_{u=0}^{2\pi} H_\alpha(y)\frac{1}{1+(N+1)^2}\frac{1}{2k\pi}\,du\,dy\,d\mu(\alpha)\\ &= \lim_{k\to+\infty}\sum_{N\in\mathbb{Z}}\int_{\alpha\in\mathbb{C}^*}\int_{y\in I_N} H_\alpha(y)\frac{1}{1+(N+1)^2}\frac{1}{k}\,dy\,d\mu(\alpha).\end{aligned}$$

By Lemma 3.3 and Corollary 5.2 after a dilation,

$$\begin{aligned}\|T\|_{\mathbb{D}^2} &= \int_{\alpha\in\mathbb{C}^*}\int_{v>0}\int_{u=2k_0\pi}^{2k_0\pi+2\pi} H_\alpha(u+iv)\|\psi_\alpha'\|^2\,du\wedge dv\,d\mu(\alpha)\quad (k_0\in\mathbb{Z})\\ &\approx \int_{\alpha\in\mathbb{C}^*}\int_{\alpha\in\mathbb{C}^*}\int_{y=2k_0\pi}^{2k_0\pi+2\pi} H_\alpha(y)\,dy\,d\mu(\alpha)\end{aligned}$$

is the integral of y on any interval of length 2π. Since I_0 has length $(2k-1)2\pi$ and I_N has length $2k\pi$ for $N\neq 0$,

$$\begin{aligned}\int_{\alpha\in\mathbb{C}^*}\int_{y\in I_0} H_\alpha(y)\,dy\,d\mu(\alpha) &\approx (2k-1)\|T\|_{\mathbb{D}^2}\\ &\geqslant k\|T\|_{\mathbb{D}^2},\\ \int_{\alpha\in\mathbb{C}^*}\int_{y\in I_N} H_\alpha(y)\,dy\,d\mu(\alpha) &\approx k\|T\|_{\mathbb{D}^2}\quad (N\neq 0).\end{aligned}$$

Thus,

$$\mathscr{L}(T,0) \gtrsim \lim_{k\to+\infty}\sum_{N\in\mathbb{Z}}\frac{1}{1+(N+1)^2}\|T\|_{\mathbb{D}^2}\approx\|T\|_{\mathbb{D}^2}$$

is non-zero.

6. *Periodic currents in the negative case* $\lambda < 0$

Now we treat the case $\lambda < 0$. We assume the currents are periodic. Recall that when $\lambda\in\mathbb{Q}$ all directed currents are periodic. So such currents include all currents for $\lambda\in\mathbb{Q}_{<0}$.

Recall the formulas of the mass and of the Lelong number obtained in §3.3, for each $k_0\in\mathbb{Z}$ fixed:

$$\|T\|_{\mathbb{D}^2} = \int_{0<|\alpha|<1}\int_{v=0}^{\log|\alpha|/\lambda}\int_{u=2k_0\pi}^{2k_0\pi+2\pi} H_\alpha(u+iv)2(e^{-2v}+\lambda^2|\alpha|^2e^{-2\lambda v})\,du\,dv\,d\mu(\alpha),$$

$$\begin{aligned}\mathscr{L}(T,0) &= \lim_{r\to 0+}\frac{1}{r^2}\|T\|_{r\mathbb{D}^2}\\ &= \lim_{r\to 0+}\frac{1}{r^2}\int_{0<|\alpha|<r^{1-\lambda}}\int_{v=-\log r}^{(\log|\alpha|-\log r)/\lambda}\int_{u=2k_0\pi}^{2k_0\pi+2\pi} H_\alpha(u+iv)2(e^{-2v}+\lambda^2|\alpha|^2e^{-2\lambda v})\,du\,dv\,d\mu(\alpha).\end{aligned}$$

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

We now prove Theorem 1.5. Suppose that there exists some $b \in \mathbb{Z}_{\leqslant 1}$ such that $H_\alpha(u + iv) = H_\alpha(u + 2\pi b + iv)$ for all $\alpha \in \mathbb{D}^*$ and all (u, v) in a neighbourhood of the strip $\{(u + iv) \in \mathbb{C} \mid u \in \mathbb{R}, v \in [0, \log|\alpha|/\lambda]\}$. One proves the following result.

LEMMA 6.1. *Let $F(u, v)$ be a positive harmonic function on a neighbourhood of the horizontal strip $\{(u + iv) \in \mathbb{C} \mid u \in \mathbb{R}, v \in [0, C]\}$ for some $C > 0$. Suppose $F(u, v) = F(u + 2\pi b, v)$ on this strip. Then*

$$F(u, v) = \sum_{k\in\mathbb{Z},k\neq 0} \left(a_k e^{kv/b} \cos\left(\frac{ku}{b}\right) + b_k e^{kv/b} \sin\left(\frac{ku}{b}\right)\right) + a_0(1 - C^{-1}v) + b_0 v,$$

for some $a_k, b_k \in \mathbb{R}$ with $a_0 \geqslant 0$ and $b_0 \geqslant 0$.

Proof. The proof is almost the same as that of Lemma 4.1. Using Fourier series and calculating the Laplacian, one concludes that

$$F(u, v) = \sum_{k\in\mathbb{Z},k\neq 0} \left(a_k e^{kv/b} \cos\left(\frac{ku}{b}\right) + b_k e^{kv/b} \sin\left(\frac{ku}{b}\right)\right) + p + qv,$$

for some $a_k, b_k, p, q \in \mathbb{R}$. For any $v \in [0, C]$, $F(u, v) \geqslant 0$ implies

$$\int_{u=0}^{2\pi b} F(u, v)du = 2\pi b(p + qv) \geqslant 0.$$

Thus, $p \geqslant 0$ and $q \geqslant -C^{-1}p$. One may write $p + qv = p(1 - C^{-1}v) + (q + C^{-1}p)v$ with $p =: a_0 \geqslant 0$ and $q + C^{-1}p =: b_0 \geqslant 0$. □

For periodic currents one may assume

$$H_\alpha(u + iv) = \sum_{k\in\mathbb{Z},k\neq 0} \left(a_k(\alpha) e^{kv/b} \cos\left(\frac{ku}{b}\right) + b_k(\alpha) e^{kv/b} \sin\left(\frac{ku}{b}\right)\right) + a_0(\alpha)\left(1 - \frac{\lambda}{\log|\alpha|}v\right) + b_0(\alpha)v, \tag{8}$$

for some $a_k(\alpha), b_k(\alpha) \in \mathbb{R}$ with $a_0(\alpha) \geqslant 0$ and $b_0(\alpha) \geqslant 0$. According to Lemma 3.3, for any $k_0 \in \mathbb{Z}$, use the Jacobian (3):

$$\|T\|_{\mathbb{D}^2} = \int_{0<|\alpha|<1} \int_{v=0}^{\log|\alpha|/\lambda} \int_{u=2k_0\pi}^{2k_0\pi+2\pi} H_\alpha(u + iv) 2(e^{-2v} + \lambda^2|\alpha|^2 e^{-2\lambda v})\, du\, dv\, d\mu(\alpha).$$

Next, using $0 = \int_0^{2\pi b} \cos(ku/b)du$ for $k \neq 0$ and the same for $\sin(ku/b)$, let us calculate the average among $k_0 = 0, 1, \ldots, b - 1$ for the mass

$$\begin{aligned}\|T\|_{\mathbb{D}^2} &= \frac{1}{b} \int_{0<|\alpha|<1} \int_{v=0}^{\log|\alpha|/\lambda} \int_{u=0}^{2\pi b} H_\alpha(u + iv) 2(e^{-2v} + \lambda^2|\alpha|^2 e^{-2\lambda v})\, du\, dv\, d\mu(\alpha) \\ &= \frac{2\pi b}{b} \int_{0<|\alpha|<1} \int_{v=0}^{\log|\alpha|/\lambda} \\ &\qquad \left(a_0(\alpha)\left(1 - \frac{\lambda}{\log|\alpha|}v\right) + b_0(\alpha)v\right) 2(e^{-2v} + \lambda^2|\alpha|^2 e^{-2\lambda v})\, dv\, d\mu(\alpha),\end{aligned}$$

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

and for the Lelong number

$$\begin{aligned}
&\mathscr{L}(T, 0)\\
&= \lim_{r\to 0+} \frac{1}{r^2}\|T\|_{r\mathbb{D}^2}\\
&= \lim_{r\to 0+} \frac{1}{br^2}\int_{0<|\alpha|<r^{1-\lambda}}\int_{v=-\log r}^{(\log|\alpha|-\log r)/\lambda}\int_{u=0}^{2\pi b}\\
&\qquad H_\alpha(u+iv)2(e^{-2v}+\lambda^2|\alpha|^2e^{-2\lambda v})\,du\,dv\,d\mu(\alpha)\\
&= \lim_{r\to 0+} \frac{2\pi b}{br^2}\int_{0<|\alpha|<r^{1-\lambda}}\int_{v=-\log r}^{(\log|\alpha|-\log r)/\lambda}\\
&\qquad \bigg(a_0(\alpha)\bigg(1-\frac{\lambda}{\log|\alpha|}v\bigg)+b_0(\alpha)v\bigg)2(e^{-2v}+\lambda^2|\alpha|^2e^{-2\lambda v})\,dv\,d\mu(\alpha).
\end{aligned}$$

We introduce the two functions of $r \in (0, 1]$ given by elementary integrals,

$$\begin{aligned}
I_a(r) &:= \frac{1}{r^2}\int_{v=-\log r}^{(\log|\alpha|-\log r)/\lambda} 2\bigg(1-\frac{\lambda}{\log|\alpha|}v\bigg)(e^{-2v}+\lambda^2|\alpha|^2e^{-2\lambda v})\,dv\\
&= 1+\lambda|\alpha|^2r^{2\lambda-2}+\frac{1}{2\log|\alpha|}(-2|\alpha|^{-2/\lambda}r^{2/\lambda-2}\log(r)+\lambda|\alpha|^{-2/\lambda}r^{2/\lambda-2}\\
&\qquad +2\lambda^2|\alpha|^2r^{2\lambda-2}\log(r)-\lambda|\alpha|^2r^{2\lambda-2}),\\
I_b(r) &:= \frac{1}{r^2}\int_{v=-\log r}^{(\log|\alpha|-\log r)/\lambda} 2v(e^{-2v}+\lambda^2|\alpha|^2e^{-2\lambda v})\,dv\\
&= \frac{1}{2}\bigg(-\frac{|\alpha|^{-2/\lambda}r^{2/\lambda-2}(\lambda+2\log|\alpha|-2\log(r))}{\lambda}\\
&\qquad +|\alpha|^2r^{2\lambda-2}(1-2\lambda\log(r))-2\log|\alpha|\bigg),
\end{aligned}$$

to describe the contributions from the $a_0(\alpha)$ part and from the $b_0(\alpha)$ part. Here we recall that every positive linear function of v on $[0, (\log|\alpha|)/\lambda]$ is a sum of $a_0(\alpha)(1-\lambda/(\log|\alpha|)v)$ and $b_0(\alpha)\,v$ with $a_0(\alpha), b_0(\alpha) \geqslant 0$. The two summands correspond to the dotted line and the dashed line in Figure 15.

Then we can express

$$\|T\|_{\mathbb{D}^2} = 2\pi\int_{0<|\alpha|<1}(a_0(\alpha)I_a(1)+b_0(\alpha)I_b(1))\,d\mu(\alpha),$$

$$\mathscr{L}(T, 0) = 2\pi\lim_{r\to 0+}\int_{0<|\alpha|<r^{1-\lambda}}(a_0(\alpha)I_a(r)+b_0(\alpha)I_b(r))\,d\mu(\alpha).$$

Observe that

$$I_a(1) = 1+\lambda|\alpha|^2+\frac{\lambda(|\alpha|^{-2/\lambda}-|\alpha|^2)}{2\log|\alpha|},$$

$$I_b(1) = \frac{1}{2}\bigg(-\frac{|\alpha|^{-2/\lambda}(\lambda+2\log|\alpha|)}{\lambda}+|\alpha|^2-2\log|\alpha|\bigg).$$

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

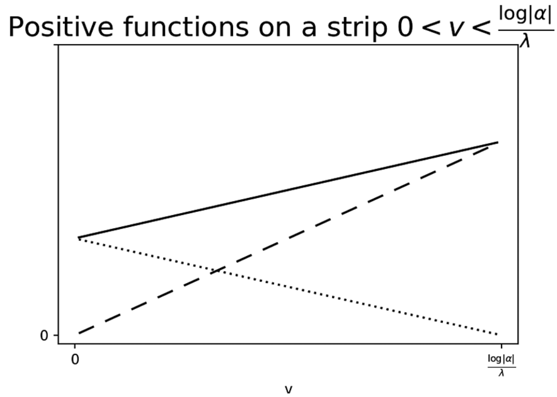

FIGURE 15. A positive function = a dotted one (gives $I_a(r)$) + a dashed one ($I_b(r)$).

Fix any $\alpha \in \mathbb{D}^*$; by definition $r^2 I_a(r)$ and $r^2 I_b(r)$ are increasing for $r \in (0, 1]$, since the interval of integration $(-\log r, (\log|\alpha| - \log r)/\lambda)$ is expanding and the function integrated is positive. In particular, for any $r \in (0, 1]$,

$$I_a(r) \leqslant r^{-2} I_a(1), \quad I_b(r) \leqslant r^{-2} I_b(1).$$

It is more subtle to talk about monotonicity of $I_a(r)$ and $I_b(r)$. We expect upper bounds of $I_a(r)/I_a(1)$ and $I_b(r)/I_b(1)$ for $r \in (0, 1]$ which are independent of α, that is, depend only on λ.

LEMMA 6.2. *For any $r \in (0, 1)$ and any $\alpha \in \mathbb{C}$ with $0 < |\alpha| < r^{1-\lambda} < 1$, one has*

$$0 < I_a(r) < I_a(1).$$

Proof. Differentiation gives

$$\frac{d}{dr} I_a(r) = \underbrace{\frac{|\alpha|^{-2/\lambda}}{\lambda r^3 \log|\alpha|}}_{>0} \Big(\lambda^2 (|\alpha|^{2+2/\lambda} r^{2\lambda} - r^{2/\lambda}) - 2(1-\lambda)(\lambda^3 |\alpha|^{2+2/\lambda} r^{2\lambda} + r^{2/\lambda}) \log(r) - 2(1-\lambda)\lambda^2 |\alpha|^{2+2/\lambda} r^{2\lambda} \log|\alpha|\Big).$$

It suffices to show that $(d/dr) I_a(r) > 0$ when $r \in (0, 1)$ and $0 < |\alpha| < r^{1-\lambda}$.

Introduce the new variable $t := |\alpha|/r^{1-\lambda} \in (0, 1)$. In the big parentheses, replace $|\alpha|$ by $tr^{1-\lambda}$ and $\log|\alpha|$ by $\log(t) + (1-\lambda)\log(r)$:

$$\begin{aligned} \frac{d}{dr} I_a(r) &= \underbrace{\frac{|\alpha|^{-2/\lambda} r^{2/\lambda}}{\lambda r^3 \log|\alpha|}}_{>0} (\lambda^2 (t^{2+2/\lambda} - 1) - 2(1-\lambda)(t^{2+2/\lambda} + 1)\log(r) \\ &\qquad \underbrace{-2(1-\lambda)\lambda^2 t^{2+2/\lambda} \log(t))}_{>0} \\ &> \frac{|\alpha|^{-2/\lambda} r^{2/\lambda}}{\lambda r^3 \log|\alpha|} (\lambda^2 \underbrace{(t^{2+2/\lambda} - 1)}_{\geqslant 0} \underbrace{-2(1-\lambda)(t^{2+2/\lambda} + 1)\log(r))}_{>0} > 0, \end{aligned}$$

since $\lambda \in [-1, 0)$ implies $t^{2+2/\lambda} \geqslant 1$. □

https://doi.org/10.1017/etds.2022.46 Published online by Cambridge University Press

It is not true that $I_b(r)$ is increasing on $(0, 1]$, but on a smaller half-neighbourhood of 0, independent of α, it is increasing. This suffices to give an upper bound for $I_b(r)/I_b(1)$.

LEMMA 6.3. *For any* $r \in (0, e^{1/2\lambda(1-\lambda)})$ *and any* $\alpha \in \mathbb{C}$ *with* $0 < |\alpha| < r^{1-\lambda} < 1$, *one has*

$$0 < I_b(r) < I_b(e^{1/2\lambda(1-\lambda)}) \leqslant e^{1/(-\lambda(1-\lambda))} I_b(1).$$

Proof. Differentiation gives

$$\frac{d}{dr} I_b(r) = \underbrace{\frac{|\alpha|^{-2/\lambda}}{\lambda^2 r^3}}_{>0} (-\lambda^2(|\alpha|^{2+2/\lambda} r^{2\lambda} - r^{2/\lambda}) + 2(1-\lambda)(\lambda^3 |\alpha|^{2+2/\lambda} r^{2\lambda} + r^{2/\lambda}) \log(r) \\ - 2(1-\lambda) r^{2/\lambda} \log|\alpha|).$$

It suffices to show that $d/dr\, I_b(r) > 0$ when $0 < r < e^{1/2\lambda(1-\lambda)}$ and $0 < |\alpha| < r^{1-\lambda}$.

Again, introduce the variable $t := |\alpha|/r^{1-\lambda} \in (0, 1)$ and replace α and $\log|\alpha|$ in the parentheses:

$$\begin{aligned} \frac{d}{dr} I_b(r) &= \underbrace{\frac{|\alpha|^{-2/\lambda} r^{2/\lambda}}{\lambda^2 r^3}}_{>0} (-\lambda^2(t^{2+2/\lambda} - 1) + 2\lambda(1-\lambda)(\lambda^2 t^{2+2/\lambda} + 1) \log(r) \\ &\qquad \underbrace{-2(1-\lambda)\log(t)}_{>0}) \\ &> \frac{|\alpha|^{-2/\lambda} r^{2/\lambda}}{\lambda^2 r^3} (-\lambda^2(t^{2+2/\lambda} - 1) + \underbrace{2\lambda(1-\lambda)(\lambda^2 t^{2+2/\lambda} + 1)}_{<0} \underbrace{\log(r)}_{<1/(2\lambda(1-\lambda))<0}) \\ &> \frac{|\alpha|^{-2/\lambda} r^{2/\lambda}}{\lambda^2 r^3} (-\lambda^2(t^{2+2/\lambda} - 1) + \lambda^2 t^{2+2/\lambda} + 1) = \frac{|\alpha|^{-2/\lambda} r^{2/\lambda}}{\lambda^2 r^3} (\lambda^2 + 1) > 0. \end{aligned}$$

□

End of proof of Theorem 1.5. From the foregoing, the Lelong number is zero:

$$\begin{aligned} \mathscr{L}(T, 0) &= 2\pi \lim_{r<e^{1/2\lambda(1-\lambda)}, r\to 0+} \int_{0<|\alpha|<r^{1-\lambda}} (a_0(\alpha) I_a(r) + b_0(\alpha) I_b(r))\, d\mu(\alpha) \\ &\leqslant 2\pi \lim_{r\to 0+} \int_{0<|\alpha|<r^{1-\lambda}} (a_0(\alpha) I_a(1) + b_0(\alpha) e^{1/(-2\lambda(1-\lambda))} I_b(1))\, d\mu(\alpha) \\ &\approx 2\pi \lim_{r\to 0+} \int_{0<|\alpha|<r^{1-\lambda}} (a_0(\alpha) I_a(1) + b_0(\alpha) I_b(1))\, d\mu(\alpha) = 0, \end{aligned}$$

since $\|T\|_{\mathbb{D}^2} = 2\pi \int_{0<|\alpha|<1} (a_0(\alpha) I_a(1) + b_0(\alpha) I_b(1))\, d\mu(\alpha)$ is finite.

Acknowledgements. The author thanks Joël Merker and an anonymous referee for valuable suggestions which help to improve the presentation.

REFERENCES

[1] L. Alessandrini and G. Bassanelli. Plurisubharmonic currents and their extension across analytic subsets. *Forum Math.* **5**(6) (1993), 577–602.

[2] V. I. Arnold and Y. S. Ilyashenko. Ordinary differential equations. *Dynamical Systems, I (Encylopaedia of Mathematical Sciences, 1)*. Eds D. V. Anosov and V. I. Arnold. Springer, Berlin, 1988, pp. 1–148.

[3] B. Berndtsson and N. Sibony. The $\bar\partial$-equation on a positive current. *Invent. Math.* **147** (2002), 371–428.

[4] A. D. Brjuno. *A Local Method of Nonlinear Analysis for Differential Equations*. Nauka, Moscow, 1979.

[5] M. Brunella. *Birational Geometry of Foliations (IMPA Monographs, 1)*. Springer, Cham, 2015.

[6] T. C. Dinh, V. A. Nguyên and N. Sibony. Heat equation and ergodic theorems for Riemann surface laminations. *Math. Ann.* **354**(1) (2012), 331–376.

[7] T. C. Dinh, V. A. Nguyên and N. Sibony. Unique ergodicity for foliations on compact Kähler surfaces. *Duke Math. J.* to appear.

[8] T. C. Dinh and N. Sibony. Unique ergodicity for foliations in $\mathbb{P}^2$ with an invariant curve. *Invent. Math.* **211**(1) (2018), 1–38.

[9] T. C. Dinh and N. Sibony. Some open problems on holomorphic foliation theory. *Acta Math. Vietnam.* **45**(1) (2020), 103–112.

[10] T. C. Dinh and H. Wu. Harmonic currents directed by foliations by Riemann surfaces. *Proc. Amer. Math. Soc.* **149**(8) (2021), 3453–3461.

[11] J. E. Fornæss and N. Sibony. Riemann surface laminations with singularities. *J. Geom. Anal.* **18**(2) (2008), 400–442.

[12] J. E. Fornæss and N. Sibony. Unique ergodicity of harmonic currents on singular foliations of $\mathbb{P}^2$. *Geom. Funct. Anal.* **19**(5) (2010), 1334–1377.

[13] J. E. Fornæss, N. Sibony and E. F. Wold. Examples of minimal laminations and associated currents. *Math. Z.* **269**(1–2) (2011), 495–520.

[14] E. F. Lee. The structure and topology of the Brjuno numbers. *Proceedings of the 1999 Topology and Dynamics Conference (Salt Lake City, UT) (Topology Proceedings, 24)*. Ed. G. Gruenhage. Auburn University, Auburn, AL, 1999, pp. 189–201.

[15] V. A. Nguyên. Oseledec multiplicative ergodic theorem for laminations. *Mem. Amer. Math. Soc.* **246**(1164) (2017).

[16] V. A. Nguyên. Directed harmonic currents near hyperbolic singularities. *Ergod. Th. & Dynam. Sys.* **38**(8) (2018), 3170–3187.

[17] V. A. Nguyên. Ergodic theory for Riemann surface laminations: a survey. *Geometric Complex Analysis (Springer Proceedings in Mathematics & Statistics, 246)*. Eds J. Byun, H. Cho, S. Kim, K. H. Lee and J. D. Park. Springer, Singapore, 2018, pp. 291–327.

[18] V. A. Nguyên. Ergodic theorems for laminations and foliations: recent results and perspectives. *Acta Math. Vietnam.* **46**(1) (2021), 9–101.

[19] V. A. Nguyên. Singular holomorphic foliations by curves. III: Zero Lelong numbers. *Preprint*, 2022, arXiv:2009.06566v2.

[20] V. A. Nguyên. Singular holomorphic foliations by curves I: Integrability of holonomy cocycle in dimension 2. *Invent. Math.* **212**(2) (2018), 531–618.

[21] A. Seidenberg. *Reduction of singularities of the differential equation* $A\,dy = B\,dx$. *Amer. J. Math.* **90** (1968), 248–269.

[22] H. Skoda. Prolongement des courants, positifs, fermés de masse finie [Extension of closed, positive currents of finite mass]. *Invent. Math.* **66**(3) (1982), 361–376 (in French)