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Formal and Tempered Solutions of Regular $\mathcal{D}$-Modules

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Abstract. A new family of sheaves has been recently studied by M. Kashiwara and P. Schapira generalizing to constructible sheaves the notion of moderate and formal cohomology. We prove comparison theorems when we regard these sheaves as solutions of a $\mathcal{D}$-module. These results are natural generalizations of those of Y. Laurent and the author.

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Introduction

Let X be a complex n-dimensional analytic manifold. Let F be an object of the derived category $D^b_{\mathbb{R}-c}(X)$, that is, a complex of $\mathbb{C}$-vector spaces with bounded and $\mathbb{R}$-constructible cohomology. We consider the object $t\mathcal{H}om(F, \mathcal{O}_X)$ of $D^b(\mathcal{D}_X)$ introduced by M. Kashiwara in [K3]. Here $\mathcal{O}_X$ denotes the sheaf of holomorphic functions and $\mathcal{D}_X$ the sheaf of linear holomorphic differential operators of finite order on X.

As a functor in $D^b_{\mathbb{R}-c}(X)$, $t\mathcal{H}om(\cdot, \mathcal{O}_X)$ generalizes the classical notions of moderate growth for holomorphic functions, or of tempered distributions in the real case.

Let us recall that there is a transformation of functors

$$t\mathcal{H}om(\cdot, \mathcal{O}_X) \longrightarrow \mathbf{R}\mathcal{H}om(\cdot, \mathcal{O}_X) \tag{1}$$

which contains as particular cases:

(i) when $F = \mathbb{C}_Y$, for Y an analytic subset of X, the morphism $\mathbf{R}\Gamma_{[Y]}(\mathcal{O}_X) \longrightarrow \mathbf{R}\Gamma_Y(\mathcal{O}_X)$ between algebraic and usual cohomology supported by Y;
(ii) when $F = \mathbb{C}_M$, for M a real analytic submanifold complexified by X, the inclusion $\mathcal{D}b_M \hookrightarrow B_M$ of the sheaf of distributions in the sheaf of Sato's hyperfunctions.

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When $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-Module, a well-known result of M. Kashiwara is the following general comparison theorem:

The natural morphism:

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, t\mathcal{H}om(F, \mathcal{O}_X)) \longrightarrow \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\mathcal{H}om(F, \mathcal{O}_X))$$

is an isomorphism.

More recently, M. Kashiwara and P. Schapira [K.S.1] introduced the so-called Whitney functor, denoted $\cdot \overset{W}{\otimes} \mathcal{O}_X$, dual of $t\mathcal{H}om(\cdot, \mathcal{O}_X)$, from $D^b_{\mathbb{R}-c}(X)$ to $D^b(\mathcal{D}_X)$ (cf. [K-S]).

There is a transformation of functors

$$\cdot \otimes \mathcal{O}_X \to \cdot \overset{W}{\otimes} \mathcal{O}_X.$$

In the case $F = \mathbb{C}_Y$ for Y an analytic submanifold, $\mathbb{C}_Y \overset{W}{\otimes} \mathcal{O}_X$ is identified with the formal completion $\mathcal{O}_{X\hat{|}Y}$; when $F = \mathbb{C}_M$ for a real analytic submanifold complexified by X, $F \overset{W}{\otimes} \mathcal{O}_X$ is identified to C^∞_M, the sheaf of C^∞-functions on M.

As a consequence of the duality between the two functors, these authors proved the following theorem:

Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_X$-Module. Let $F \in \mathcal{D}_{\mathbb{R}-c}(X)$. Then the natural morphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, F \otimes \mathcal{O}_X) \longrightarrow \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, F \overset{W}{\otimes} \mathcal{O}_X)$$

is an isomorphism.

Let now Y be a submanifold of X. Let us consider a Fuchsian (not necessarily holonomic) $\mathcal{D}_X$-Module $\mathcal{M}$ along Y. In a previous work with Y. Laurent [L-TMF], we have proved that the complexes of solutions of $\mathcal{M}$ in the sheaf of holomorphic hyperfunctions on Y, with either finite, or infinite order, are isomorphic. We have also shown that the complexes of either holomorphic solutions or formal solutions along Y are isomorphic.

Our aim here, in a first attempt, is to generalise both results of [L-TMF] replacing $\mathcal{O}_X$ by $t\mathcal{H}om(F, \mathcal{O}_X)$ is the first theorem and $\mathcal{O}_X$ by $F \overset{W}{\otimes} \mathcal{O}_X$ in the second. We treat two distinct situations: the case where Y is the intersection of two submanifolds each one containing either supp $\mathcal{M}$ or supp F; the case where $(\mathcal{M}; F)$ is an elliptic pair in the sense of Schapira–Schneiders, together with a Levi condition and F is $\mathbb{C}$-constructible. The proofs are completely different. Actually the proof in the second case relies on the reduction to the case treated by [L-TMF] thanks to the well-known result of M. Kashiwara [K3] which asserts that $t\mathcal{H}om(F, \mathcal{O}_X)$ is a complex of $\mathcal{D}_X$-Modules with regular holonomic cohomology when F is $\mathbb{C}$-constructible. Another essential tool is the duality theorem of [K-S,Th. 6.1].

The reader may wonder whether our two cases are actually contained in a general result. We are not able to give a definite answer to this question but we suspect it would be negative.

In a next attempt, we consider the following problem: let us suppose X is a complexification of a real analytic submanifold M in such a way that Y is a complexification of a real analytic submanifold N of M. It is then natural to consider an object F of $D^b_{\mathbb{R}-c}(M)$, $\mathcal{M}$ a coherent $\mathcal{D}_X$-module and investigate under which conditions the following morphisms are isomorphisms:

(3) $\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}; \mathbf{R}\Gamma_N(t\mathcal{H}om(F; Db_M)) \longleftarrow \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, t\mathcal{H}om(F_N, Db_M))$

(4) $\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}; \mathbb{C}_N \otimes (F \overset{W}{\otimes} C^\infty_M)) \longrightarrow \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, F_N \overset{W}{\otimes} C^\infty_M)$.

When codim $Y = 1$, $\mathcal{M}$ is defined by a strictly hyperbolic operator along N and $F = \mathbb{C}_M$, both results are consequence of a theorem of S. Alinhac [A] on the Cauchy Problem for flat functions on N (cf. also [H]).

We end with an application suggested by P. Schapira: to define new functors $I\mathcal{H}om(F; \mathcal{O}_X)$ and ITens $(F; \mathcal{O}_X)$ in $D^b_{\mathbb{R}-c}(X)$ using the technique of [K-S]: essentialy as third terms of distinguished triangles associated to

$$t\mathcal{H}om(F, \mathcal{O}_X) \longrightarrow \mathbf{R}\mathcal{H}om(F, \mathcal{O}_X)$$

$$F \otimes \mathcal{O}_X \longrightarrow F \overset{W}{\otimes} \mathcal{O}_X$$

and apply our comparison theorems to these sheaves.

1. Topics on Regularity for $\mathcal{D}$-Modules and the Functors $t\mathcal{H}om(\cdot, \mathcal{O}_X)$ and $\cdot \overset{W}{\otimes} \mathcal{O}_X$

We shall recall very briefly the concepts we need to state the results of this work.

1.1. REGULAR AND FUCHSIAN $\mathcal{D}_X$-MODULES

Let $Y \subset X$ be a complex $n-d$-dimensional submanifold of the n-dimensional complex analytic manifold X. The sheaf $\mathcal{D}_X$ of holomorphic differential operators may be endowed with two filtrations: let I_Y be the defining ideal of Y in X, denote $I_Y^j = \mathcal{O}_X$ if $j \leqslant 0$. The first filtration is

$$V_Y^j(\mathcal{D}_X) = \{P \in \mathcal{D}_X, \forall k \in \mathbb{Z}, PI_Y^j \subset I_Y^{j-k}\},$$

the so-called V-filtration. The second one is the usual filtration by the order and is denoted $\mathcal{D}_X(j)$, $j \geqslant 0$. We note θ the Euler vector field on $T_Y X$, the normal bundle to Y, and keep the same letter for any section of $V_Y^\circ(\mathcal{D}_X)$ whose class in $gr_V^\circ \mathcal{D}_X$

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is θ, as soon as there is no risk of confusion. Let $\mathcal{D}_X^\infty$ denote the sheaf of differential operator of infinite order. If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module $\mathcal{M}^\infty$ will denote $\mathcal{D}_X^\infty \underset{\mathcal{D}_X}{\otimes} \mathcal{M}$.

A coherent $\mathcal{D}_X$-Module $\mathcal{M}$ is regular along Y (cf. [K2]) if for any local section $u \in \mathcal{M}$ there exists a nontrivial polynomial $b \in \mathbb{C}[s]$ and a differential operator $Q \in V_Y^1(\mathcal{D}_X)$ such that

(1) $b(\theta)u + Qu = 0$
(2) order of Q is majorated by the degree of $b(s)$.

One also says that $b(s)$ is a regular b-function for u.

This condition is equivalent to the local existence of a sub-module $\mathcal{M}_0$ of $\mathcal{M}$, coherent over $\mathcal{O}_X$, and of a nontrivial polynomial $b(s) \in \mathbb{C}[s]$ such that:

$$b(\theta)\mathcal{M}_0 \subset (V_Y^1(\mathcal{D}_X) \cap \mathcal{D}_X(m))\mathcal{M}_0, \tag{1.0}$$

where m is the degree of $b(s)$, and $\mathcal{D}_X\mathcal{M}_0 = \mathcal{M}$.

We say that $b(s)$ is regular for $\mathcal{M}$.

A result due to Kashiwara and Kawai asserts that any regular holonomic $\mathcal{D}_X$-Module is regular along any submanifold Y (cf. [K-K]). A more general concept developped in [L-TMF] is that of *Fuchsian* $\mathcal{D}_X$-Module; a coherent $\mathcal{D}_X$-Module is *Fuchsian* along Y if, for any $x_0 \in Y$, one may choose local coordinates $(x_1, \dots, x_p, t_1, \dots, t_d)$ such that $Y = \{t_1 = \cdots = t_d = 0\}$, and for any local section u of M on U there exists an operator P such that $\mathrm{P}u = 0$ and

$$\mathrm{P}(x, t, D_x, D_t) = \sum_{\substack{|\beta|=|\alpha| \\ |\beta| \leqslant m}} p_{\alpha\beta}(x) t^\alpha D_t^\beta + Q(x, t, D_x, D_t)$$

such that

$$\begin{aligned} &\text{(i)} \quad Q \in V_Y^1(\mathcal{D}_X) \cap \mathcal{D}_X(m), \\ &\text{(ii)} \quad \forall \tau \in \mathbb{C}^d \setminus \{0\}, \sum_{\substack{|\beta|=|\alpha| \\ |\beta|=m}} p_{\alpha\beta}(x_0)\tau^\alpha \bar{\tau}^\beta \neq 0 \end{aligned} \tag{1.1}$$

(such operators were introduced by [B-G]).

Remark. When codim $Y = 1$, $\mathcal{M}$ is a Fuchsian $\mathcal{D}$-Module if and only if it is regular.

The main results of [L-TMF] are the following:

THEOREM 1.1.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-Module Fuchsian along Y. Then, the natural morphism in $D^b(X)$*

$$\mathbf{R}\mathcal{H}om_{D_X}(\mathcal{M}, \mathbf{R}\Gamma_{[Y]}(\mathcal{O}_X)) \longrightarrow \mathbf{R}\mathcal{H}om_{D_X}(\mathcal{M}, \mathbf{R}\Gamma_Y(\mathcal{O}_X))$$

is an isomorphism.

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THEOREM 1.1.2. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-Module Fuchsian along Y. Then the natural morphism in $D^b(X)$*

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}\left(\mathcal{M}, \mathcal{O}_X\Big|_Y\right) \longrightarrow \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X\hat{|}Y})$$

is an isomorphism.

1.2. THE FUNCTORS $t\mathcal{H}\mathrm{om}(\cdot, \mathcal{O}_X)$ AND $\cdot \overset{W}{\otimes} \mathcal{O}_X$

We refer to [K-S] for a detailed study of these functors.

To start with, let X be a real analytic manifold, C_X^∞ (respectively $\mathcal{D}b_X$) the sheaf of infinitely differentiable functions (respectively, the sheaf of distributions) on X. Let U be an open subanalytic subset of X. One sets

$$\mathbb{C}_U \overset{W}{\otimes} C_X^\infty := \mathcal{I}_{X\setminus U}^\infty$$

as the sheaf of flat functions on $X \setminus U$, i.e., $\mathbb{C}^\infty$-functions vanishing up to infinite order on $X \setminus U$.

On the other hand, one defines $t\mathcal{H}\mathrm{om}(\mathbb{C}_U, \mathcal{D}b_X)$ by the exact sequence

$$0 \to \Gamma_{X\setminus U}(\mathcal{D}b_X) \to \mathcal{D}b_X \to t\mathcal{H}\mathrm{om}(\mathbb{C}_U, \mathcal{D}b_X) \to 0,$$

that is, $t\mathcal{H}\mathrm{om}(\mathbb{C}_U, Db_X)$ is the sheaf of tempered distributions on U defined by M. Kashiwara in [K3]. In [K-S] these two constructions are proved to be particular cases of two local exact functors:

$$\cdot \overset{W}{\otimes} C_X^\infty : \mathbb{R} - \mathrm{Const}(X) \to \mathcal{M}\mathrm{od}(\mathcal{D}_X)$$
$$t\mathcal{H}\mathrm{om}(\cdot, Db_X) : (\mathbb{R} - \mathrm{Const}(X))^{\mathrm{op}} \to \mathcal{M}\mathrm{od}(\mathcal{D}_X)$$

where $\mathbb{R} - \mathrm{Const}(X)$ denotes the Abelian category of $\mathbb{R}$-constructible sheaves on X and $\mathcal{M}\mathrm{od}(\mathcal{D}_X)$ the category of $\mathcal{D}_X$-Modules. The first is the Whitney functor and the second the Schwartz functor.

Let us recall the classical Lemma (see [K-3] (Lemma 3.3)).

LEMMA 1.2.1. *Let U be an open subanalytic subset of X. Let Ω be an open relatively compact subset of X, u a distribution on Ω. Then, $u \in \Gamma(\Omega, t\mathcal{H}\mathrm{om}(\mathbb{C}_U, \mathcal{D}b_X))$ (that is, u is tempered on U) if and only if there exists $C \geqslant 0$ and $m \in \mathbb{N}$ such that for any $\varphi \in C_0^\infty(\Omega \cap U)$,*

$$|< u, \varphi >| \leqslant C \sum_{|\alpha| \leqslant m} \sup |D^\alpha \varphi|.$$

We now return to the complex analytic framework: let X be a complex analytic n-dimensional manifold, denote by $X_\mathbb{R}$ the underlying real analytic $2n$-dimensional manifold and by $\bar{X}$ the complex conjugate of X. A function on an open set $U \subset X_\mathbb{R}$ is holomorphic on $U \subset \bar{X}$ if and only if $\bar{f}$ is holomorphic on $U \subset X$.

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The functors of formal and moderate cohomology were introduced in [K-S] as functors on $D^b_{\mathbb{R}-c}(X)$ with values in $D^b(\mathcal{D}_X)$. Here $\mathcal{D}_X$ denotes the sheaf of holomorphic differential operators on X.

One sets, for $F \in D^b_{\mathbb{R}-c}(X)$,

$$F \overset{W}{\otimes} \mathcal{O}_X := \mathbf{R}\mathcal{H}om_{\mathcal{D}_{\bar{X}}}(\mathcal{O}_{\bar{X}}, F \overset{W}{\otimes} \mathcal{C}^\infty_X),$$
$$t\mathcal{H}om(F, \mathcal{O}_X) := \mathbf{R}\mathcal{H}om_{\mathcal{D}_{\bar{X}}}(\mathcal{O}_{\bar{X}}, t\mathcal{H}om(F, \mathcal{D}b_X)).$$

To end this section we shall recall a few facts about the topological duality in the derived categories. One notes FN (resp. DFN) a topological vector space with a Fréchet nuclear topology (resp. a dual of Fréchet nuclear topology). That is the case of $C^\infty(\Omega)$ and $\Gamma_c(\Omega, \check{Db}_X)$, for Ω an open subanalytic subset in X. Here $\mathcal{D}\check{b}_X$ denotes the sheaf Db_X tensorised by the sheaf of real analytic densities. Let $D^b(FN)$ (resp. $D^b(DFN)$) denote the localization of the additive category whose objects are bounded complexes of topological FN (resp. DFN) vector spaces and morphisms the linear continuous morphisms modulo homotopy, by the complexes which are algebraically exact. We shall use one of the main results of [K-S]:

Let Ω_X denote the sheaf of holomorphic differential forms of degree n.

PROPOSITION 1.2.2 ([K-S], Theorem 6.1). *Let $\mathcal{M}$ be a bounded complex with coherent $\mathcal{D}_X$-module cohomology, $F \in D^b_{\mathbb{R}-c}(X)$. Then we can define $\mathbf{R}\Gamma(X, \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, F \overset{W}{\otimes} \mathcal{O}_X))$ and $\mathbf{R}\Gamma_c(X; t\mathcal{H}om(F, \Omega_X)[n] \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M})$ as objects of $D^b(FN)$ and $D^b(DFN)$ functorially with respect to $\mathcal{M}$ and F and they are dual to each other.*

Remark that $t\mathcal{H}om(F, \Omega_X)[n] \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M}$ is naturally isomorphic to

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*, t\mathcal{H}om(F, \Omega_X))[n],$$

where $\mathcal{M}^* = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$.

2. Comparison Theorems

2.1. THE COMPLEX CASE

Let X be a n-dimensional complex analytic manifold and Y a d codimensional submanifold. Let us note $D^b_{\mathbb{C}-c}(X)$ the derived category whose objects are complexes of $\mathbb{C}$-vector spaces with bounded and $\mathbb{C}$-constructible cohomology. Let $SS(F)$ denote the microsupport of $F \in D^b_{\mathbb{R}-c}(X)$ and $\mathrm{Car}(\mathcal{M})$ denote the characteristic variety of an object $\mathcal{M}$ of $D^b_c(\mathcal{D}_X)$, that is, of a bounded complex of $\mathcal{D}_X$-modules with coherent cohomology.

Remark that when F belong to $D^b_{\mathbb{C}-c}(X)$, $t\mathcal{H}om(F, \mathcal{O}_X)$ belongs to $D^b_c(\mathcal{D}_X)$ and has regular holonomic cohomology (also denoted by $\mathcal{M} \in D^b_{rh}(\mathcal{D}_X)$).

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, V a smooth involutive submanifold of the bundle $\dot{T}^*X$ (the cotangent bundle T^*X minus the zero-section). The 1-microcharacteristic variety $C^1_V(\mathcal{M}) \subset T_V(T^*X)$ (the normal bundle to V) was defined in [TMF] as well as in [L] with another notation. Now, if $\mathcal{M} \in D^b_c(\mathcal{D}_X)$ one defines $C^1_V(\mathcal{M}) = \cup_j C^1_V(\mathcal{H}^j(\mathcal{M}))$. Recall that if $\eta \in T_V(T^*X)$, η is non 1-microcharacteristic for $\mathcal{M}$ along V if $\eta \notin C^1_V(\mathcal{M})$.

Let $F \in D^b_{\mathbb{C}-c}(X)$, $\mathcal{N} \in D^b_{rh}(\mathcal{D}_X)$ such that $\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_X) \simeq F$, let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, $Y \subset X$ a smooth submanifold. We say that Y is non-1-microcharacteristic for $(\mathcal{M}, F)$ if the diagonal $\Delta \subset X \times X$ is non-1-microcharacteristic for $\mathcal{M} \underline{\boxtimes}\, \underline{D}\mathcal{N}$ along $\dot{T}^*_{Y\times X} X \times X$.

When S is a subset of X, I_S will denote the ideal of germs of holomorphic functions vanishing on S.

Our first purpose is to investigate when the following morphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, t\mathcal{H}om(F_Y, \mathcal{O}_X)\Big) \to \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, \mathbf{R}\Gamma_Y(t\mathcal{H}om(F, \mathcal{O}_X))\Big). \qquad (2.1)$$

is an isomorphism in $D^b(X)$.

We shall prove the two following theorems:

THEOREM 2.1.1. *Let $\mathcal{M}$ belong to $D^b_c(\mathcal{D}_X)$, Y be a smooth submanifold of X and F belong to $D^b_{\mathbb{R}-c}(X)$. Let us assume that Y is of the form $Y_1 \cap Y_2$, for Y_1, Y_2 two submanifolds such that* $\operatorname{supp} \mathcal{M} \subset Y_1$, $\operatorname{supp} F \subset Y_2$. *Then (2.1) is an isomorphism.*

Remark. In particular (2.1) is an isomorphism for arbitrary F if $\operatorname{supp} \mathcal{M} \subset Y$ and for arbitrary $\mathcal{M}$ if $\operatorname{supp} F \subset Y$, this last case being obvious.

THEOREM 2.1.2. *Let us assume $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, $F \in D^b_{\mathbb{C}-c}(X)$, $\mathcal{M}$ is regular along Y and Y is non-1-microcharacteristic for $(\mathcal{M}, F)$. Then (2.1) is an isomorphism.*

Proof of Theorem 2.1.1. We get

$$\begin{aligned}
&\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_Y t\mathcal{H}om(F, \mathcal{O}_X)) \simeq \\
&\qquad \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_{Y_1} R\Gamma_{Y_2} t\mathcal{H}om(F, \mathcal{O}_X)) \\
&\qquad \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_{Y_1} t\mathcal{H}om(F, \mathcal{O}_X)) \\
&\qquad \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_{Y_1} t\mathcal{H}om(F_{Y_2}, \mathcal{O}_X))
\end{aligned}$$

Hence it is enough to consider the case $Y = Y_1$. We shall argue by induction on the codimension d of Y. Suppose $d = 1$ and choose t a local coordinate on X such that $Y = \{t = 0\}$. We shall take in account the object $t\mathcal{H}om(F_{X\setminus Y}, \mathcal{O}_X)$.

By classical arguments we may reduce $\mathcal{M}$ to the case of a single local generator u such that $t^m u = 0$ for some $m \in \mathbb{N}$. On the other hand t is invertible on $t\mathcal{H}om(F_{X\setminus Y}, \mathcal{O}_X)$, hence

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, t\mathcal{H}om(F_{X\setminus Y}, \mathcal{O}_X)) = 0.$$

Using the distinguished triangle

$$t\mathcal{H}\mathrm{om}(F_Y, \mathcal{O}_X) \longrightarrow t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X) \longrightarrow t\mathcal{H}\mathrm{om}(F_{X\setminus Y}, \mathcal{O}_X) \underset{+1}{\longrightarrow}$$

we deduce the isomorphism:

$$R\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, t\mathcal{H}\mathrm{om}(F_Y, \mathcal{O}_X)) \simeq R\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X)) \tag{2.2}$$

Let us now assume $d > 1$, and that (2.2) holds for any submanifold Y' of codimension $d' < d$. Let $(t_1, \dots, t_d)$ be coordinate functions on X such that $Y = \{t_1 = \cdots = t_d = 0\}$, $t' = (t_1, \dots, t_j)$, $1 < j < d$, $t'' = (t_{j+1}, \dots, t_d)$. We have:

$$\begin{aligned}
&\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\Gamma_Y(t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X))) \simeq \\
&\qquad \simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\Gamma_{\{t'=0\}}\mathbf{R}\Gamma_{\{t''=0\}}(t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X))) \\
&\qquad \simeq \mathbf{R}\Gamma_{\{t'=0\}}\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\Gamma_{\{t''=0\}}(t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X)) \\
&\qquad \simeq \mathbf{R}\Gamma_{\{t'=0\}}\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, t\mathcal{H}\mathrm{om}(F_{\{t''=0\}}, \mathcal{O}_X)) \\
&\qquad \simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, t\mathcal{H}\mathrm{om}(F_Y, \mathcal{O}_X)) \\
&\qquad \text{(by the induction hypotesis).}
\end{aligned}$$

□

Proof of Theorem 2.1.2. Let $\mathcal{D}_{X\leftarrow Y}$ and $\mathcal{D}_{Y\to X}$ denote the 'transfer bimodules' associated to $Y \hookrightarrow X$. By Theorem 5.7 of [K-S] we get

$$t\mathcal{H}\mathrm{om}(F_Y, \mathcal{O}_X) \xleftarrow{\sim} \mathcal{D}_{X\leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} t\mathcal{H}\mathrm{om}(i^{-1}F, \mathcal{O}_Y)[-d]$$

On the other hand, by Theorem 5.8. of [K-S],

$$t\mathcal{H}\mathrm{om}(i^{-1}F, \mathcal{O}_Y) \simeq \mathcal{D}_{Y\to X} \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X)[-d]$$

Therefore, by the results of [K1] one gets

$$\begin{aligned}
t\mathcal{H}\mathrm{om}(F_Y, \mathcal{O}_X) &\simeq \mathcal{D}_{X\leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} \left(\mathcal{D}_{Y\to X} \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X)[-d]\right) \\
&\simeq \mathbf{R}\Gamma_{[Y]}(t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X)).
\end{aligned} \tag{2.3}$$

By the 'way-out' argument we may assume that the complex $t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X)$ is concentrated in degree zero and we shall denote $\mathcal{N} = \mathcal{H}^\circ(t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X))$. Let us identify X to the diagonal Δ of $X \times X$ and let ℓ be the inclusion of Δ in $X \times X$. Let us also denote $\underline{D}\mathcal{N}$ the left $\mathcal{D}_X$-module adjoint of $\mathcal{N}$:

$$\underline{D}\mathcal{N} := \mathcal{N}^* \underset{\mathcal{O}_X}{\otimes} \Omega_X^{\otimes -1}$$

The assumption entails that Δ is noncharacteristic for $\mathcal{M} \boxtimes \underline{D}\mathcal{N}$. Therefore the induced $\mathcal{D}_\Delta$-module $\underline{\ell}^*(\mathcal{M} \boxtimes \underline{D}\mathcal{N}) := \mathcal{D}_{\Delta\to X\times X} \underset{\mathcal{D}_{X\times X}}{\otimes} (\mathcal{M} \boxtimes \underline{D}\mathcal{N})$ is coherent and so is $\underline{\ell}^!(\mathcal{M} \boxtimes \underline{D}\mathcal{N})$. Here $\underline{\ell}^!$ stands for the composition of functors $\underline{D} \circ \underline{\ell}^* \circ \underline{D}$.

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On the other side, $\mathcal{M} \boxtimes \underline{D}\mathcal{N}$ is regular along $W := Y \times X$. We claim that $\underline{\ell}^*(\mathcal{M} \boxtimes \underline{D}\mathcal{N})$ and, hence, $\underline{\ell}^!(\mathcal{M} \boxtimes \underline{D}\mathcal{N})$ is regular along Y.

This will be a consequence of the following sublemma, which is itself the differential version of ([TMF], Th. 1.3.8):

SUBLEMMA 2.1.3. *Let X be a complex n-dimensional manifold, Y and Z be transversal submanifolds of X and $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module such that Z is non 1-microcharacteristic for $\mathcal{M}$ along $\dot{T}^*_Y X$ and $\mathcal{M}$ is regular along Y. Then the induced system $\underline{\ell}^*\mathcal{M}$ is regular along $Z \cap Y$, where ℓ: $Z \hookrightarrow X$ is the inclusion.*

Proof. Let us consider a system of local coordinates on X, (t, x, s), $t = (t_1, \ldots, t_d)$, $x = (x_1, \ldots, x_r)$, $s = (s_1, \ldots, s_h)$ with $d + r + h = n$, such that $Y = \{t = 0\}$, $Z = \{x = 0\}$. Therefore, $W = Y \cap Z$ is the submanifold of Z defined by $t = 0$. We have

$$\underline{\ell}^*\mathcal{M} = \frac{\mathcal{M}}{x_1\mathcal{M} + \cdots + x_r\mathcal{M}}.$$

Let $\mathcal{M}_0$ be a coherent $\mathcal{O}_X$-submodule of $\mathcal{M}$ generating $\mathcal{M}$ and $b(s)$ a b-function satisfying (1.0) with respect to $\mathcal{M}_0$. That is,

$$b\left(\sum_{i=1}^{d} t_i D_{t_i}\right)\mathcal{M}_0 \subset \left(V^1_Y(\mathcal{D}_X) \cap \mathcal{D}_X(m)\right)\mathcal{M}_0,$$

where m is the degree of b.

Remark that $x_i, s_\ell, D_{x_i}, D_{s_\ell}$ are in $V^\circ_Y(\mathcal{D}_X)$. Let $\{u_1, \ldots, u_\ell\}$ be a system of local generators of $\mathcal{M}_0$, hence of $\mathcal{M}$ as a $\mathcal{D}_X$-module.

Because Z is non-1-microcharacteristic for $\mathcal{M}$ along $\dot{T}^*_Y X$, for each $j \in \{1, \ldots, \ell\}$, $i \in \{1, \ldots, r\}$ there is an order p_{ij} such that

$$D^{p_{ij}}_{x_i} u_j = \sum_{\substack{|\alpha+\beta+\gamma| \leqslant p_{ij} \\ \gamma_i \leqslant p_{ij}-1}} A_{\alpha\beta\gamma}(t, x, s)(t_p D_{t_k})^\alpha D^\beta_s D^\gamma_x, \tag{2.4}$$

where $A_{\alpha\beta\gamma}$ are holomorphic functions.

Let $M = \sum p_{ij}$. In particular, setting $w_{\gamma i}$ the class of $D^\gamma_x u_j$ in $\underline{\ell}^*\mathcal{M}$, $|\gamma| \leqslant M$, the set $\{w_{\gamma i}\}$ is a system of local generators of $\underline{\ell}^*\mathcal{M}$ over $\mathcal{D}_Z$.

Let $\mathcal{M}'_0$ be the coherent $\mathcal{O}_X$-submodule of $\mathcal{M}$ generated by $\{D^\gamma_x u_j\}_{|\gamma| \leqslant M}$, $j = 1, \ldots, \ell$ and set

$$\widetilde{\mathcal{M}}_0 = \frac{\mathcal{M}'_0}{M'_0 \cap (x_1\mathcal{M} + \cdots + x_r\mathcal{M})}.$$

$\widetilde{\mathcal{M}}_0$ is a coherent $\mathcal{O}_Z$-module generating $\underline{\ell}^*\mathcal{M}$. We will check that

$$b\left(\sum t_i D_{t_i}\right)\widetilde{\mathcal{M}}_0 \subset \left(V^1_W(\mathcal{D}_Z) \cap \mathcal{D}_Z(m)\right)\widetilde{\mathcal{M}}_0.$$

Set $\theta = \sum_{i=1}^{d} t_i D_{t_i}$. It is enough to prove that $b(\theta)w_{\gamma i}$ may be written as a finite sum of terms of the form $t_p Q w_{\beta t}$ where $Q \in V^{\circ}_{W}(\mathcal{D}_Z) \cap \mathcal{D}_Z(m)$. By assumption $b(\theta)D^{\gamma}_{x} u_j$ is a finite sum of terms of the form $t_p \widetilde{Q} D^{\beta}_{x} u_t$ with $\widetilde{Q} \in V^{\circ}_{Y}(\mathcal{D}_X) \cap \mathcal{D}_X(m)$, $|\beta| \leqslant M$.

By division, each $\widetilde{Q}$ may be written in the form

$$\widetilde{Q} = \sum_{i=1}^{r} x_i \widetilde{R}_i + \sum_{i=1}^{r} \widetilde{T}_i D_{x_i} + Q',$$

where $Q' \in V^{\circ}_{W}(\mathcal{D}_Z) \cap \mathcal{D}_Z(m)$, $\widetilde{R}_i \in \mathcal{D}_X(m)$, $\widetilde{T}_i \in \mathcal{D}_X(m-1) \cap V^{\circ}_{Y}(\mathcal{D}_X)$; at this point (2.4) implies that the class of $t_p \widetilde{T}_i D_{x_i} D^{\beta}_{x} u_t$ is a sum of terms of the form $t_p T w_{\beta' t'}$ with $T \in V^{\circ}_{W}(\mathcal{D}_Z) \cap \mathcal{D}_Z(m)$, hence the result. □

Continuation of the proof of Theorem 2.1.2.

Hence $\underline{\ell}^* \widetilde{M}$ is regular along Y.

Let us now identify $\mathcal{D}_X$ to $B_{\Delta|X\times X} \underset{\mathcal{O}_X}{\otimes} \Omega_X$. Using again [K1] we have

$$\begin{aligned}
&\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\Gamma_{[Y]}(\mathcal{N})) \\
&\quad \simeq \mathbf{R}\Gamma_{[Y]}(\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)) \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{N} \\
&\quad \simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\Gamma_{[Y]}(\mathcal{D}_X)) \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{N} \\
&\quad \simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M} \underline{\boxtimes} \underline{D}\mathcal{N}, \mathbf{R}\Gamma_{[Y]}(B_{\Delta|X\times X})).
\end{aligned}$$

Therefore by (2.3) the left-hand side of (2.1) becomes

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X\times X}}(\widetilde{M}, \mathbf{R}\Gamma_{[Y]}(B_{\Delta|X\times X}))$$

and the right-hand side becomes $\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X\times X}}(\widetilde{M}, \mathbf{R}\Gamma_{Y}(B_{\Delta|X\times X}))$. We shall now consider the following lemma:

LEMMA 2.1.4. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, $Z \underset{\ell}{\hookrightarrow} X$ a submanifold of X (of codimension s) and $Y \underset{i}{\hookrightarrow} Z$ a submanifold of Z (of codimension t). Assume that Z is noncharacteristic for $\mathcal{M}$ and $\underline{\ell}^! \mathcal{M}$ Fuchsian along Y. Then the morphism in $D^b(X)$*

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\Gamma_{[Y]}(B_{Z|X})) \to \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\Gamma_{Y}(B_{Z|X}))$$

is an isomorphism.

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Proof. We have $\mathbf{R}\Gamma_{[Y]}(B_{Z|X}) \simeq B_{Y|X}[-t]$ and the isomorphisms:

$$\begin{aligned}\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, B_{Y|X}[-t])\Big|_Y &\simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Y}((\underline{\ell \circ i})^! \mathcal{M}, \mathcal{O}_Y)[-t] \\ &\simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Z}(\underline{\ell}^! \mathcal{M}, B_{Y|Z})[-t]\Big|_Y \\ &\simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Z}(\underline{\ell}^! \mathcal{M}, B^{\infty}_{Y|Z})[-t]\Big|_Y\end{aligned}$$

because $\underline{\ell}^! \mathcal{M}$ is Fuchsian along Y.

The last complex is isomorphic to

$$\begin{aligned}\mathbf{R}\Gamma_Y(\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Z}(\underline{\ell}^! \mathcal{M}, \mathcal{O}_Z))\Big|_Y &\simeq \mathbf{R}\Gamma_Y(\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, B_{Z|X}))\Big|_Y \\ &\simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\Gamma_Y(B_{Z|X}))\Big|_Y.\end{aligned}$$ □

To end the proof of Theorem 2.1.2. we only have to apply the preceding lemma for $\widetilde{M}, \Delta$ and Y. □

Remark 1. Theorem 1.1.1. is a particular case of Theorem 2.1.2.

Remark 2. Lemmas 2.1.3. and 2.1.4. were the essential tools for the proof of Theorem 2.1.2. The following example shows that the assumption $\underline{\ell}^! \mathcal{M}$ Fuchsian along Y is crucial. However, one may possibly weaken the assumption Z noncharacteristic for $\mathcal{M}$ and replace it by the condition of ellipticity defined by [L-S].

Let $X = \mathbb{C}^2$ with the coordinates (x, t), $Z = \{x = 0\}$, $Y = \{t = 0\}$, and $\mathcal{M}$ be defined by the equation $(D_t + D_x)u = 0$. Hence Z is non characteristic for $\mathcal{M}$ and $\mathcal{M}$ is regular along Y. However, $\underline{\ell}^! \mathcal{M} \simeq \mathcal{D}_Z$ is not Fuchsian along $Y \cap Z = \{0\}$. One gets

$$\begin{aligned}\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\Gamma_Y(t\mathcal{H}\mathrm{om}(\mathbb{C}_Z, \mathcal{O}_X))_0 &\simeq \mathbf{R}\Gamma_Y \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, B_{Z|X}[-1])_0 \\ &\simeq \mathbf{R}\Gamma_{Y\cap Z} \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Z}(\underline{\ell}^! \mathcal{M}, \mathcal{O}_Z[-1])_0 \\ &\simeq B^{\infty}_{\{0\}|Z}[-2]_0 \\ &\not\simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, B_{\{0\}|\mathbb{C}^2})_0[-2].\end{aligned}$$

We also conjecture that we may replace the assumption $\mathcal{M}$ regular along Y by $\mathcal{M}$ Fuchsian along Y in Theorem 2.1.2.

Remark 3. If $Y \subset Z$ as in Lemma 2.1.4., and $\mathcal{M}$ is regular along Y and along Z, the result will obviously be true.

EXAMPLE. Let $X = \mathbb{C}^3$ with the coordinates (t_1, t_2, x), $Y = \{t_1 = t_2 = 0\}$, let $\mathcal{M}$ be defined by $D_{t_1} u = 0$, $(t_2 D_{t_2} - \frac{1}{2})^4 u = t_2 D_x^4 u$, $F = \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_X)$ where $\mathcal{N}$ is defined

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by

$$(t_1 D_{t_1} + 1)^2 w = 0, \quad (D_{t_2} + t_2 D_x) w = 0 \text{ and } \left(D_x^2 - \frac{1}{2}\right)^3 w = 0.$$

Then $\mathcal{M}$, Y and F satisfy the assumptions of Theorem 2.1.2.

Our next purpose is to investigate the natural morphism

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbb{C}_Y \otimes (F \overset{W}{\otimes} \mathcal{O}_X)) \longrightarrow \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, F_Y \overset{W}{\otimes} \mathcal{O}_X). \tag{2.5}$$

We shall prove the following theorems:

THEOREM 2.1.5. *Under the assumptions of Theorem 2.1.1 the morphism (2.5) is an isomorphism in* $D^b(X)$.

THEOREM 2.1.6. *Under the assumptions of Theorem 2.1.2 the morphism (2.5) is an isomorphism in* $D^b(X)$.

Proof of Theorem 2.1.5. We shall adapt the proof of Theorem 2.1.1. When $\operatorname{supp} F \subset Y$ the result is obvious for arbitrary $\mathcal{M}$. Now, suppose $\operatorname{codim} Y = 1$ and $\operatorname{supp} \mathcal{M} \subset Y$.

We have the natural isomorphism

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbb{C}_Y \otimes (F_{X\setminus Y} \overset{W}{\otimes} \mathcal{O}_X)) \simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, F_{X\setminus Y} \overset{W}{\otimes} \mathcal{O}_X)$$

and the result is true for F_Y; by the exact sequence

$$0 \longrightarrow F_{X\setminus Y} \longrightarrow F \longrightarrow F_Y \longrightarrow 0$$

it is enough to check that $\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, F_{X\setminus Y} \overset{W}{\otimes} \mathcal{O}_X) = 0$. But this is a consequence of the duality theorem 1.2.2 because of the equality

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, t\mathcal{H}\mathrm{om}(F_{X\setminus Y}, \mathcal{O}_X)) = 0$$

obtained in Theorem 2.1.1.

Next, let us reason by induction on d and consider local coordinates on X, (x, t', t''), such that $Y = \{t_1 = \cdots = t_j = t_{j+1} = \cdots = t_d = 0\}, j < d$ and $\operatorname{supp} \mathcal{M} \subset Y$ as in the proof of Theorem 2.1.1. We have

$$\begin{aligned}
&\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbb{C}_Y \otimes (F \overset{W}{\otimes} \mathcal{O}_X)) \\
&\quad \simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbb{C}_{Y''} \otimes (F_{Y'} \overset{W}{\otimes} \mathcal{O}_X)) \\
&\quad \simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, F_Y \overset{W}{\otimes} \mathcal{O}_X),
\end{aligned}$$

where $Y' = \{t_1 = \cdots = t_j = 0\}$, $Y'' = \{t_{j+1} = \cdots = t_d\} = 0$. □

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Before embarking in the proof of Theorem 2.1.6 we shall prepare two useful formulas. Let us identify again X with the diagonal of $X \times X$.

Denote p_2 the second projection of $X \times X$ over X.

PROPOSITION 2.1.7. *Let F be an object of $D^b_{\mathbb{C}-c}(X)$. Then one has a natural isomorphism:*

$$\mathbf{R}\mathcal{H}\mathrm{om}_{p_2^{-1}\mathcal{D}_X}(p_2^{-1}t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X), \mathcal{O}_{X\times X\hat{|}X}) \xleftarrow{\sim} F \overset{W}{\otimes} \mathcal{O}_X. \tag{2.6}$$

Proof. Once again we may assume that $t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X)$ is a coherent (regular holonomic) $\mathcal{D}_X$-Module. For any open subanalytic set U of X, U' any open subanalytic neighborhood of U in $X \times X$, such that $U' \cap X = U$, by Proposition 1.2.2 we have that $\mathbf{R}\Gamma(U, F \overset{W}{\otimes} \mathcal{O}_X)$ is dual of $\mathbf{R}\Gamma_c(U, t\mathcal{H}\mathrm{om}(F, \Omega_X)[n])$. This last complex is quasi-isomorphic to

$$\mathbf{R}\Gamma_c(U', \mathbf{R}\mathcal{H}\mathrm{om}_{p_2^{-1}\mathcal{D}_X}(p_2^{-1}\underline{D}t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X), B_{X|X\times X}))[-n]$$

by a continuous linear quasi-isomorphism.

On the other hand, by Proposition 1.2.2 this last complex is dual to

$$\begin{aligned}&\mathbf{R}\Gamma(U', \mathbf{R}\mathcal{H}\mathrm{om}_{p_2^{-1}\mathcal{D}_X}(p_2^{-1}t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X), \mathbb{C}_\Delta \overset{W}{\otimes} \mathcal{O}_{X\times X})\\ &\quad\simeq \mathbf{R}\Gamma(U', \mathbf{R}\mathcal{H}\mathrm{om}_{p_2^{-1}\mathcal{D}_X}(p_2^{-1}t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X), \mathcal{O}_{X\times X\hat{|}X}).\end{aligned}$$

Now we use the argument of [R-R]. If two complexes in $D^b(FN)$ (resp. in $D^b(DFN)$) are algebraically quasi-isomorphic by a continous linear quasi-isomorphism the dual complexes are isomorphic in $D^b(DFN)$ (resp. in $D^b(FN)$) (Lemma 2 of [R-R]) $\square$

PROPOSITION 2.1.8. *Let F be an object of $D^b_{\mathbb{C}-c}(X)$. Then one has a natural isomorphism:*

$$\mathbf{R}\mathcal{H}\mathrm{om}_{p_2^{-1}\mathcal{D}_X}(p_2^{-1}t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X), \mathcal{O}_{X\times X\hat{|}Y}) \simeq F_Y \overset{W}{\otimes} \mathcal{O}_X.$$

Proof. We use the same argument as in the preceding proof. For any open subanalytic open set $U \subset X$ and any open subanalytic neighborhood U' of U in $X \times X$, such that $U' \cap X = U$, since $\mathcal{O}_{X\times X\hat{|}Y} \simeq \mathbb{C}_Y \overset{W}{\otimes} \mathcal{O}_{X\times X}$ and denoting by $\mathcal{N}$ the coherent $\mathcal{D}_X$-Module $t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X)$ by Proposition 1.2.2 we know that

$$\mathbf{R}\Gamma(U', \mathbf{R}\mathcal{H}\mathrm{om}_{p_2^{-1}\mathcal{D}_X}(p_2^{-1}\mathcal{N}, \mathbb{C}_Y \overset{W}{\otimes} \mathcal{O}_{X\times X}))$$

is dual to $\mathbf{R}\Gamma_c(U', \mathbf{R}\mathcal{H}\mathrm{om}_{p_2^{-1}\mathcal{D}_X}(p_2^{-1}\mathcal{N}^*, \mathbf{R}\Gamma_{[Y]}(B_{X|X\times X}) \underset{\mathcal{O}_X}{\otimes} \Omega_{X\times X}[n])$ and this last

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complex is continuously quasi-isomorphic to $\mathbf{R}\Gamma_c(U, \mathbf{R}\Gamma_{[Y]}(\mathcal{N}) \underset{\mathcal{O}_X}{\otimes} \Omega_X)[n]$, that is, to

$$\mathbf{R}\Gamma_c(U, t\mathcal{H}\mathrm{om}(F_Y, \Omega_X))[n].$$

Now we apply Proposition 1.2.2 and Lemma 2 of [R-R] to get the conclusion. □

Proof of Theorem 2.1.6. Once again we may suppose $t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X)$ concentrated in degree zero. Let us keep the notation $\mathcal{N}$ for the regular holonomic $\mathcal{D}_X$-Module $t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X)$. Since the morphisms in the statement of the theorem are well defined we are reduced to prove the result germwise. Let $y_0 \in Y$ and U belong to a fundamental system of open subanalytic neighborhood of y_0. Let U' be as in Proposition 2.1.7.

We have by Proposition 2.1.7.

$$\begin{aligned}
&\mathbf{R}\Gamma(U, \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, (F \overset{W}{\otimes} \mathcal{O}_X) \otimes \mathbb{C}_Y)) \\
&\quad \simeq \mathbf{R}\Gamma(U', \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\mathcal{H}\mathrm{om}_{p_2^{-1}\mathcal{D}_X}(p_2^{-1}\mathcal{N}, \mathcal{O}_{X\times X\hat{|}X}|_Y)) \\
&\quad \simeq R\Gamma(U', \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X\times X}}(\mathcal{M}\underline{\boxtimes}\mathcal{N}, \mathcal{O}_{X\times X\hat{|}X}|_Y)).
\end{aligned}$$

On the other hand, the right-hand side of (2.5) becomes $\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X\times X}}(\mathcal{M}\underline{\boxtimes}\mathcal{N}, \mathcal{O}_{X\times X\hat{|}Y})$ by Proposition 2.1.8. we shall now use the following lemma:

LEMMA 2.1.9. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, Z and Y submanifolds of X satisfying the assumptions of Lemma 2.1.4. Then the natural morphism*

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X\hat{|}Z})|_Y \longrightarrow \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X\hat{|}Y})|_Y$$

is an isomorphism.

Proof. We have the chain of isomorphism:

$$\begin{aligned}
\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X\hat{|}Z})|_Y &\simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Z}(\underline{\ell}^*\mathcal{M}, \mathcal{O}_Z)|_Y \\
&\simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Z}(\underline{\ell}^*\mathcal{M}, \mathcal{O}_{Z\hat{|}Y})|_Y \\
&\simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Y}(\underline{\ell \circ i})^*\mathcal{M}, \mathcal{O}_Y) \\
&\simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X\hat{|}Y})|_Y,
\end{aligned}$$

where the second isomorphism is a consequence of Theorem 1.1.2. □

The proof of Theorem 2.1.6 is an immediate application of the preceding lemma for $\mathcal{M}\underline{\boxtimes}\mathcal{N}$, $X \times X$, Δ and Y. □

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Remark. A natural question is whether a duality theorem analogous to Theorem 1.2.2 would still be true if we consider

$$\mathbf{R}\Gamma(X, t\mathcal{H}om(F, \Omega_X)[n]) \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M}) \text{ and } \mathbf{R}\Gamma_c(X, \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, F \overset{W}{\otimes} \mathcal{O}_X)).$$

Of course, this situation is outside the *FN* and *DFN* framework, and it would simplify our proofs. However, the answer is negative. For that purpose we may consider the elegant example of P. Schapira: let $X = \mathbb{C}^2$, Y the diagonal of X and $F = \mathbb{C}_Y$. It is easy to check that $\mathbf{R}\Gamma_c(X, \mathbb{C}_Y \overset{W}{\otimes} \mathcal{O}_X)$ is concentrated in a single degree; on the other side

$$R\Gamma(X, t\mathcal{H}om(\mathbb{C}_Y, \Omega_X))[2] \simeq R\Gamma(\mathbb{C}, \mathcal{D}_{\mathbb{C}})[1].$$

But $R\Gamma(\mathbb{C}, \mathcal{D}_{\mathbb{C}})$ is concentrated in two degrees, 0 and 1, therefore the duality fails: we may identify $\mathcal{D}_{\mathbb{C}}$ to $\mathcal{O}_{\mathbb{C}}[t]$ the sheaf of polynomials with holomorphic coefficients and then consider the Dolbeault (soft) resolution of $\mathcal{O}_{\mathbb{C}}[t]$:

$$\circ \longrightarrow \mathcal{O}_{\mathbb{C}}[t] \longrightarrow C^\infty_{\mathbb{C}}[t] \xrightarrow{\bar{\partial}} C^\infty_{\mathbb{C}}[t] \longrightarrow \circ\,.$$

One may easily choose a family $(f_i)_{i\in\mathbb{N}}$ of C^∞-functions with compact support K_i with (K_i) locally finite so that $u = \sum_{i\geqslant 0} f_i(x)t^i$ is a global section of $C^\infty_{\mathbb{C}}[t]$ and $u \notin \operatorname{Im} \bar{\partial}$.

2.2. THE REAL CASE IN CODIMENSION ONE

We shall henceforth consider the following situation: Y is an hypersurface of X, X is the complexified of a real analytic n-dimensional manifold M, such that Y complexifies a smooth hypersurface $N \subset M$. We shall say that for a coherent $\mathcal{D}_X$-module $\mathcal{M}$, or, more generally, for an object $\mathcal{M}$ of $D^b_c(\mathcal{D}_X)$ the Cauchy problem for flat functions on N is well posed if

$$\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{J}^\infty_N) = 0. \tag{2.7}$$

Examples of such $\mathcal{D}$-modules are provided by the Fuchsian strictly hyperbolic operators studied by Alinhac [A]. Roughly speaking, these are Fuchsian operators along N such that, in some local coordinate system on X, (x, t), real on M, satisfying $Y = \{t = 0\}$, the total symbol $\sigma(P)$ is of the form $t^m p(x, t, \xi, \tau)$ where p is a hyperbolic polynomial on τ.

In particular, the strictly hyperbolic operators in the sense of Hörmander [H] are Fuchsian strictly hyperbolic.

PROPOSITION 2.2.1 (Chauchy problem for C^∞_M). *Let $\mathcal{M}$ be an object of $D^b_c(\mathcal{D}_X)$. Then the Cauchy problem for flat functions on N is well posed for $\mathcal{M}$ if and only*

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if the following natural morphism is an isomorphism:

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}\left(\mathcal{M}, C^\infty_M\Big|_N\right) \longrightarrow \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbb{C}_N \overset{W}{\otimes} C^\infty_M)$$

is an isomorphism.

Proof. One considers the exact sequence of sheaves

$$0 \to \mathbb{C}_{M\setminus N} \to \mathbb{C}_M \to \mathbb{C}_N \to 0.$$

Since by definition $\mathbb{C}_{M\setminus N} \overset{W}{\otimes} C^\infty_M = \mathcal{J}^\infty_N$, we have the distinguished triangle

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}\left(\frac{\mathcal{D}_X}{\mathcal{D}_X P}, C^\infty_M\Big|_N\right) \to \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}\left(\frac{\mathcal{D}_X}{\mathcal{D}_X P}, \mathbb{C}_N \overset{W}{\otimes} C^\infty_M\right)$$
$$\overset{+1}{\to} \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}\left(\frac{\mathcal{D}_X}{\mathcal{D}_X P}, \mathcal{J}^\infty_N\right)$$

hence the result. □

Remark. $\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbb{C}_N \overset{W}{\otimes} C^\infty_M) \simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Y}(\mathcal{M}_Y, C^\infty_N)$.

PROPOSITION 2.2.2. *Let $\mathcal{M}$ belong to $D^b_c(\mathcal{D}_X)$ and assume that the Cauchy problem for flat functions along N is well posed for $\mathcal{M}$. Then*

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, t\mathcal{H}\mathrm{om}(\mathbb{C}_{M\setminus N}, \mathcal{O}_X) = 0$$

Proof. One has $\mathbf{R}\Gamma_c(X, \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, t\mathcal{H}\mathrm{om}(\mathbb{C}_{M\setminus N}, \mathcal{O}_X)) = 0$ as well as replacing X by $X \setminus \{x\}$, for arbitrary $x \in X$. Hence, denoting $\mathcal{F}$, to simplify, the complex: $\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, t\mathcal{H}\mathrm{om}(\mathbb{C}_{M\setminus N}, \mathcal{O}_X))$ and considering the distinguished triangle:

$$\mathbf{R}\Gamma_c(X \setminus \{x\}, \mathcal{F}) \longrightarrow \mathbf{R}\Gamma_c(X, \mathcal{F}) \underset{+1}{\longrightarrow} \mathbf{R}\Gamma(\{x\}, \mathcal{F})[1]$$

we get that $\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, t\mathcal{H}\mathrm{om}(\mathbb{C}_{M\setminus N}, \mathcal{O}_X))_x = 0$, hence the result.

In particular, if $\mathcal{M}$ satisfies (2.7) we have

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\Gamma_N(t\mathcal{H}\mathrm{om}(\mathbb{C}_{M\setminus N}, \mathcal{O}_X)) = 0 \tag{2.8}$$

therefore we conclude:

PROPOSITION 2.2.3. *Let $\mathcal{M}$ belong to $D^b_c(\mathcal{D}_X)$ such that the Cauchy problem for flat functions on N is well posed for $\mathcal{M}$. Then the natural morphism*

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, t\mathcal{H}\mathrm{om}(\mathbb{C}_N, Db_M))\Big|_N \longrightarrow \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\Gamma_N(Db_M))\Big|_N \tag{2.9}$$

is an isomorphism.

The following examples show that we cannot avoid strict hyperbolicity if we want (2.8) and, hence, (2.9) to be an isomorphism:

EXAMPLES

(1) Let $X=\mathbb{C}^2$ with the coordinates (x,t) and $P(x,t,D_x,D_t)=D_t^2+D_x^2$, $\mathcal{M}=\mathcal{D}_X/\mathcal{D}_XP$, $Y=\{t=0\}$, $M=\mathbb{R}^2$. P is not hyperbolic, although it is Fuchsian along N (or Y) and (2.8) does not hold. One checks that $u(x,y)=\mathrm{e}^{1/t+ix}$ is a solution of $Pu=0$ defined for $t>0$ (or $t<0$) and it is not tempered at the points of N.

(2) (Example given by Boutet de Monvel): Let $X=\mathbb{C}^2$ with coordinates (x,t), $P(x,t,D_x,D_t)=D_t^2-D_x$ the heat operator, $\mathcal{M}=(\mathcal{D}_X/\mathcal{D}_XP)$ P is Fuchsian along Y, weakly hyperbolic (the Cauchy Problem with respect to N is well posed for hyperfunctions) but it is not strictly hyperbolic. The function

$$u(x,t)=\frac{1}{\sqrt{4\pi x}}\mathrm{e}^{-t^2/4x},$$

solves $Pu=0$ on $t>0$ (or $t<0$). Let $\psi(D_x)=\sum_{n\geqslant 0}(iD_x)^{2n}/6n!$. Then $\psi(D_x)u(x,t)$ is still a solution of $Pu=0$ for $t>0$ but it is not a tempered distribution at the origin.

2.3. APPLICATION

Following an idea of P. Schapira, one may define two new functors in $D^b_{\mathbb{R}-c}(X)$ which we shall denote by $I\mathcal{H}om(\cdot,\mathcal{O}_X)$ and $I\mathrm{Tens}(F,\mathcal{O}_X)$ respectively and natural transformation of functors

$$\begin{aligned}\mathbf{R}\mathcal{H}om(\cdot,\mathcal{O}_X)&\longrightarrow I\mathcal{H}om(\cdot,\mathcal{O}_X)\longrightarrow t\mathcal{H}om(\cdot,\mathcal{O}_X)[1]\cdot\overset{W}{\otimes}\mathcal{O}_X\\&\longrightarrow I\,\mathrm{Tens}(\cdot,\mathcal{O}_X)\longrightarrow\cdot\overset{\mathbb{L}}{\otimes}\mathcal{O}_X[1]\end{aligned}\tag{2.10}$$

such that the associated triangles when (2.10) applied to F are distinguished.

These two functors are constructed by application of Theorem 1.1 and 1.4 of [K-S]: for any open subanalytic set of X, denote $I\mathcal{H}om(\mathbb{C}_U,Db_X)$ the $\mathcal{D}_X$-Module defined by the exact sequence:

$$0\longrightarrow t\mathcal{H}om(\mathbb{C}_U,Db_X)\longrightarrow\Gamma_U(Db_X)\longrightarrow I\mathcal{H}om(\mathbb{C}_U,Db_X)\longrightarrow 0$$

Similarly, we define $I\mathrm{Tens}(\mathbb{C}_U,\mathcal{O}_X)$ by the exact sequence

$$0\longrightarrow\mathbb{C}_U\otimes C_X^\infty\longrightarrow\mathbb{C}_U\overset{W}{\otimes}C_X^\infty\longrightarrow I\mathrm{Tens}(\mathbb{C}_U,C_X^\infty)\longrightarrow 0$$

$I\mathrm{Tens}(\cdot,C_X^\infty)$ and $I\mathcal{H}om(\cdot,Db_X)$ extend to exact local functors in the category $\mathbb{R}$-Const(X) (resp. in $\mathbb{R}$-Const$(X)^{op}$).

Finally one defines

$$\begin{aligned}
&I\mathcal{H}\mathrm{om}(F, \mathcal{O}_X) := \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{\bar{X}}}(\mathcal{O}_{\bar{X}}, I\mathcal{H}\mathrm{om}(F, Db_X)),\\
&I\mathrm{Tens}(F, \mathcal{O}_X) := \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{\bar{X}}}(\mathcal{O}_{\bar{X}}, I\mathrm{Tens}(F, C_X^{\infty})),
\end{aligned}$$

and by Proposition 1.4 of [K-S] these new functors are left derivable.

COROLLARY 2.3.1. *Let $\mathcal{M}$ be a regular $\mathcal{D}_X$-Module along a submanifold Y. Let F be an object of $D^b_{\mathbb{C}-c}(X)$ such that $\mathcal{M}$ and F satisfy the assumption of Theorem 2.1.2. Then*

(1) $\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, I\mathcal{H}\mathrm{om}(F_Y, \mathcal{O}_X)) = 0$,
(2) $\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, I\mathrm{Tens}(F_Y, \mathcal{O}_X)) = 0$.

Proof. The condition $(\mathcal{M}, F)$ elliptic pair entails that the natural morphism

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X)) \longrightarrow \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X)^{\infty})$$

is an isomorphism hence

$$\begin{aligned}
\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}&(\mathcal{M}, \mathbf{R}\Gamma_Y(t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X)))\\
&\simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\Gamma_Y(t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X)^{\infty}))\\
&\simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, (\mathbf{R}\Gamma_{[Y]}(t\mathcal{H}\mathrm{om}(F, \mathcal{O}_X))^{\infty})\\
&\simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\mathcal{H}\mathrm{om}(F_Y, \mathcal{O}_X)).
\end{aligned}$$

Similarly $\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, F \overset{W}{\otimes} \mathcal{O}_X) \simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, F \otimes \mathcal{O}_X)$ thanks to Proposition 2.1.7. Hence

$$\begin{aligned}
\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbb{C}_Y \otimes F \overset{W}{\otimes} \mathcal{O}_X) &\simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, F_Y \otimes \mathcal{O}_X)\\
&\simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, F_Y \overset{W}{\otimes} \mathcal{O}_X). \qquad \square
\end{aligned}$$

Remark. Under the assumptions of Theorem 2.1.1. Corollary 2.3.1. does not hold in general, as the following example shows in the $\mathbb{C}$-constructible case:

$$X = \mathbb{C}^2, \quad F = \mathbb{C}_{\{x=0\}}, \quad Y = \{t = 0\}, \quad \mathcal{M}$$

defined by $tu = 0$.

Then

$$\begin{aligned}
&t\mathcal{H}\mathrm{om}(F_Y, \mathcal{O}_X) \simeq B_{\{0\}|\mathbb{C}^2}[-2],\\
&R\mathcal{H}\mathrm{om}(F_Y, \mathcal{O}_X) \simeq B^{\infty}_{\{0\}|\mathbb{C}^2}[-2]
\end{aligned}$$

and obviously t is not an isomorphism on

$$\frac{B^{\infty}_{\{0\}|\mathbb{C}^2}}{B_{\{0\}|\mathbb{C}^2}}.$$

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