Note on the different proofs of Fourier's Series.

By Dr H. S. Carslaw.

The Use of Green's Functions in the Mathematical Theory of the Conduction of Heat.

By Dr H. S. Carslaw.

§ 1.

The use of Green's Functions in the Theory of Potential is well known. The function is most conveniently defined, for the closed surface S, as the potential which vanishes over S and is infinite as $\frac{1}{r}$, when r is zero, at the point $P(x_0, y_0, z_0)$, inside the surface. If this is represented by G(P), the solution with no infinity inside S and an arbitrary value V over the surface, is given by

$$v = \frac{1}{4\pi}\iint \frac{\partial}{\partial n} G(P) \, . \, V \, . \, dS,$$

$\frac{\partial}{\partial n}$, denoting differentiation along the outward drawn normal.

In the other Partial Differential Equations of Mathematical Physics similar functions may with advantage be employed, and,

https://doi.org/10.1017/S0013091500034507 Published online by Cambridge University Press

in particular they have been found of great value in the discussion of the equation

$$(\nabla^2 + \kappa^2)u = 0.*$$

It is the object of this paper to illustrate their use in the discussion of various questions in the Mathematical Theory of the Conduction of Heat. In this case *the Green's Function is taken as the temperature at* (x, y, z), *at the time* t, *due to an instantaneous point source generated at the point* $P(x_0, y_0, z_0)$, *at the time* τ, *the solid being initially at zero temperature, and the surface being kept at zero temperature.*

This solution may be written

$$u = \mathrm{F}(x, y, z, x_0, y_0, z_0, t - \tau),$$

and u satisfies the equation

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u, \qquad (t > \tau).$$

However since τ enters only in the form $t - \tau$, we have also the equation

$$\frac{\partial u}{\partial \tau} + \kappa \nabla^2 u = 0. \qquad (\tau < t).$$

Further

$$\underset{t=\tau}{\mathrm{Lt.}} \quad (u) = 0,$$

at all points inside S, except at the point (x_0, y_0, z_0), where the solution takes the form

$$\frac{1}{\left(2\sqrt{\pi\kappa(t-\tau)}\right)^3} e^{-\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4\kappa(t-\tau)}}$$

* *Cf.*

Pockels. *Über die Partielle Differential-gleichung*
$(\nabla^2 + \kappa^2)u = 0.$
Theil IV. § 4. Leipzig 1891.

Schwarzschild.
Die Beugung und Polarization des Lichts durch einen Spalt.
Math. Ann. Bd. 55. 1902.

Finally, at the surface of S,

$$u = 0, \qquad (\tau < t).$$

Now let v be the temperature at the time t in this solid due to the surface temperature $\phi(x, y, z, t)$, and the initial temperature $f(x, y, z)$. Then v satisfies the equations

$$\frac{\partial v}{\partial t} = \kappa \nabla^2 v, \qquad (t > 0)$$

$$v = f(x, y, z), \qquad \text{initially, inside S,}$$

$$v = \phi(x, y, z, t), \qquad (t > 0), \text{ at surface of S};$$

and, since the instant τ of our former equations lies within the interval for t, these equations may also be taken as

$$\frac{\partial v}{\partial \tau} = \kappa \nabla^2 v, \qquad (\tau < t)$$

$$\text{and} \qquad v = \phi(x, y, z, \tau) \qquad \text{at the surface.}$$

Therefore we have

$$\frac{\partial}{\partial \tau}(uv) = u\frac{\partial v}{\partial \tau} + v\frac{\partial u}{\partial \tau},$$

$$= \kappa\left[u\nabla^2 v - v\nabla^2 u\right],$$

and

$$\int_0^{t-\epsilon}\left[\iiint \frac{\partial}{\partial \tau}(uv).\,dxdydz\right]d\tau = \kappa\int_0^{t-\epsilon}\left[\iiint (u\nabla^2 v - v\nabla^2 u)\,dxdydz\right]d\tau,$$

the triple integration being taken throughout the solid, and ϵ being any positive quantity as small as we please.

Interchanging the order of integration on the left-hand side, and applying Green's Theorem to the right-hand side, we have

$$\iiint [uv]_{\tau = t-\epsilon}\,dxdydz - \iiint [uv]_{\tau = 0}\,dxdydz$$

$$= \kappa\int_0^{t-\epsilon}\left[\iint\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS\right]d\tau$$

$$= \kappa\int_0^{t-\epsilon}\left[\iint v\frac{\partial u}{\partial n_i}\,dS\right]d\tau,$$

https://doi.org/10.1017/S0013091500034507 Published online by Cambridge University Press

where $\dfrac{\partial}{\partial n_i}$ denotes differentiation along the inward drawn normal, and we have used the fact that u vanishes at S.

On the right-hand side we may put $\epsilon = 0$, as there is no singularity in the integrand, and the left-hand side as ϵ approaches zero takes the value

$$[v_{\mathrm{P}}]_t \left\{ \iiint [u]_{\tau = t - \epsilon} dxdydz \right\} - \iiint [u]_{\tau = 0} \cdot [v]_{\tau = 0} dxdydz,$$

the first integral being taken through an element of volume including the point $\mathrm{P}(x_0, y_0, z_0)$, where the infinity in u enters, and $[v_{\mathrm{P}}]_t$ standing for the value of v at the point $\mathrm{P}(x_0, y_0, z_0)$, at the instant t.

The choice of u, so that

$$\mathrm{Lt.}_{\tau = t}(u) = \mathrm{Lt.}_{\tau = t} \left(\frac{1}{[2\sqrt{\pi\kappa(t-\tau)}]^3} e^{-\frac{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}{4\kappa(t-\tau)}} \right)$$

makes the co-efficient of $[v_{\mathrm{P}}]_t$ unity, *and we obtain*

$$[v_{\mathrm{P}}]_t = \iiint [u]_{\tau = 0} f(x, y, z) dxdydz + \kappa \int_0^t \left[\iint \phi(x, y, z, \tau) \frac{\partial u}{\partial n_i} d\mathrm{S} \right] d\tau,$$

as the equation giving the temperature at $P(x_0, y_0, z_0)$, *at time* t, *due to the initial distribution* $f(x, y, z)$, *and the surface temperature* $\phi(x, y, z, t)$.

In the case of radiation at the surface, *the Green's Function* u *is taken as the temperature at* (x, y, z), *at time* t *due to an instantaneous source at* (x_0, y_0, z_0) *at time* τ, *the radiation taking place into a medium at zero temperature.*

The temperature at $\mathrm{P}(x_0, y_0, z_0)$, at the time t due to an initial distribution $f(x, y, z)$, and radiation at the surface into a medium

https://doi.org/10.1017/S0013091500034507 Published online by Cambridge University Press

at temperature $\phi(x, y, z, t)$, will then be found to be given by the equations,

$$[v_P]_t = \iiint u_{\tau=0} f(x, y, z)dxdydz + h\kappa \int_0^t \left[\iint u\phi(x, y, z, \tau)dS \right] d\tau,$$

$$= \iiint u_{\tau=0} f(x, y, z)dxdydz + \kappa \int_0^t \left[\iint \frac{\partial u}{\partial n_i} \phi(x, y, z, \tau)dS \right] d\tau,$$

the second of these equations being of the same form as that already obtained for the former case.

The use of Green's Function in the discussion of the equation of Conduction seems to have been noticed first by Minnigerode.* It is also developed in several papers by Betti, and is referred to in the other places noted below.†

The Green's Functions given in this paper in §§ 2, 5, may be written down by inspection, and the results obtained by the Synthetical Method * follow at once from our general theorems.

---

* Minnigerode.
Uber die Wärme-Leitung in Krystallen.
Diss. Göttingen. 1862.

† Betti.
(1) *Sopra la determinazione della temperatura variabile di un cylindro.*
Annali delle Università Toscane.
Tom. I. 1867.

(2) *Sopra la determinazione delle temperatura variabile di una lastra terminata.*
Annali di Matematica. Tom. I. 1867.

(3) *Sopra la determinazione delle temperatura nei corpi solidi ed omogenii.*
Mem. della Soc. Italiana delle Scienze.
Ser. III. Tom. I. 1868.

(4) *Sopra la propagazione del calore.*
Chelini Collezione 1881.

Sommerfeld.
Zur Analytische Theorie der Wärme-Leitung.
Math. Ann. Bd. 45. 1894.

Weber-Riemann.
Die Partiellen Differential-gleichungen der Physik.
Bd. II., § 51. 1901.

In the other cases, §§ 3, 4, 7, 8, these functions are obtained by the aid of Contour Integrals, following the method given by Dougall in his papers in these *Proceedings*.† This method is one of considerable power, and may be applied to many other problems of Mathematical Physics.

§ 2

Linear Flow of Heat.

Semi-Infinite Solid bounded by plane $x = 0$.

In this case our general result is simplified by the consideration of the plane source over $x = x_0$, instead of the point source at (x_0, y_0, z_0), and the Green's Function is to vanish at the surface, and become infinite for $x = x_0$, at $t = \tau$, in the form

$$\frac{1}{2\sqrt{\pi\kappa(t-\tau)}}e^{-\frac{(x-x_0)^2}{4\kappa(t-\tau)}}$$

When the solid is bounded by $x = 0$, but is unlimited in the direction $x > 0$, this function is clearly given by

$$u = \frac{1}{2\sqrt{\pi\kappa(t-\tau)}}\left[e^{-\frac{(x-x_0)^2}{4\kappa(t-\tau)}} - e^{-\frac{(x+x_0)^2}{4\kappa(t-\tau)}}\right],$$

and the solution of the problem, when the initial temperature is $f(x)$ and the boundary is kept at $\phi(t)$, is given by

---

* Hobson.

Synthetical Solutions in the Conduction of Heat.

Proc. Lond. Math. Soc. Vol. XIX. 1888.

† Dougall.

(i) *The Determination of Green's Function by means of Cylindrical or Spherical Harmonics.*

Proc. Edin. Math. Soc. Vol. XVIII. 1900.

(ii) *Note on the Application of Complex Integration to the Equation of the Conduction of Heat.*

Proc. Edin. Math. Soc. Vol. XIX. 1901.

https://doi.org/10.1017/S0013091500034507 Published online by Cambridge University Press

$$v_P\Big]_t = \int_0^\infty u_{\tau=0} f(x)dx + \kappa\int_0^t \phi(\tau)\left(\frac{\partial u}{\partial x}\right)_{x=0} d\tau$$

$$= \frac{1}{2\sqrt{\pi\kappa t}}\int_0^\infty f(x)\left[e^{-\frac{(x-x_0)^2}{4\kappa t}} - e^{-\frac{(x+x_0)^2}{4\kappa t}}\right]dx$$

$$+ \frac{x_0}{2\sqrt{\pi\kappa}}\int_0^t \phi(\tau)\frac{e^{-\frac{x_0^2}{4\kappa(t-\tau)}}}{\sqrt{(t-\tau)^3}}d\tau.$$

This result is obtained by the Synthetical Method by the distribution of sources and sinks along the axis of x and of continuous doublets of strength $2\kappa\phi(t)$ at $x=0$.

The corresponding Green's Function for the case of radiation has been obtained by Bryan,* and also by the author,† by the method of Contour Integrals to be used later in this paper.

§ 3.

Linear Flow of Heat.

Finite Solid bounded by the planes $x=0$ and $x=a$.

To obtain the Green's Function for the solid bounded by the planes $x=0$, $x=a$, we proceed from the solution

$$v = \frac{1}{2\sqrt{\pi\kappa t}}\left[e^{-\frac{(x-x_0)^2}{4\kappa t}} - e^{-\frac{(x+x_0)^2}{4\kappa t}}\right]$$

which satisfies the conditions at $x=x_0$ and $x=0$. This may be written

$$v = \frac{1}{2\pi}\left[\int_{-\infty}^\infty e^{-\kappa a^2 t}\cos a(x-x_0)da - \int_{-\infty}^\infty e^{-\kappa a^2 t}\cos a(x+x_0)\right]da.$$

---

* Bryan.

An Application of the Method of Images to the Conduction of Heat. Proc. Lond. Math. Soc. Vol. XXII. 1891.

† *A Problem in Conduction of Heat.* Phil. Mag. July 1902.

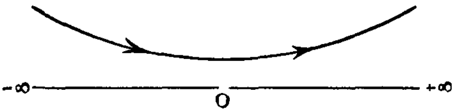
We replace these real integrals, by complex integrals in the a – plane, and obtain

$$v = \frac{1}{i\pi}\int e^{-\kappa a^2 t} \sin a x_0 \, e^{-iax} \, da, \qquad x > x_0$$

$$v = \frac{1}{i\pi}\int e^{-\kappa a^2 t} \sin a x \;\, e^{-iax_0} \, da, \qquad x < x_0,$$

the integrals being taken over the path (P) Fig. (1) in the a – plane, and the phase of a lying between 0 and $\frac{\pi}{4}$ to the right, and $\frac{3\pi}{4}$ and π to the left, at infinity.

FIGURE 1.



The path P in a-plane.

If we call this solution V, we must now choose a solution V_1, which will satisfy the conditions at $x = a$.

In this case we take

$$V_1 = -\frac{1}{i\pi}\int e^{-\kappa a^2 t} \frac{\sin a x_0 \sin a x}{\sin a a} e^{iaa} \, da,$$

over the same path (P), and we have now to examine the solution

$$\begin{aligned} v &= V + V_1 \\ &= \frac{1}{i\pi}\int e^{-\kappa a^2 t} \frac{\sin a x_0 \sin a(a - x)}{\sin a a} da \qquad x > x_0 \\ &= \frac{1}{i\pi}\int e^{-\kappa a^2 t} \frac{\sin a x \sin(a - x_0)}{\sin a a} da \qquad x < x_0 \end{aligned}$$

which we shall show satisfies all the conditions of the problem.

Initial Conditions.

We have seen that $\underset{t=0}{\text{Lt.}}\,(V)$ has the form required by the Green's Function; we have thus to show that $\underset{t=0}{\text{Lt.}}\,(V_1) = 0$.

When we put $t=0$ in the integrand in V_1 the expression vanishes: for

$$\int \frac{\sin\alpha x_0 \sin\alpha x}{\sin\alpha a} e^{i\alpha a}\, d\alpha$$

has no singularity in the α-plane *above* (P) and the integrand vanishes at infinity when the imaginary part of α is positive provided

$$x+x_0-2a<0.$$

Also the presence of the factor $e^{-\kappa\alpha^2 t}$ causes the integral over the path (P) to converge uniformly towards its value for $t=0$, and we are thus entitled to take

$$\underset{t=0}{\mathrm{Lt.}}\ [V_1]=0.$$

Boundary Conditions.

These have been already satisfied by our choice of V_1 and it is clear that the two expressions

$$v=\frac{1}{i\pi}\int e^{-\kappa\alpha^2 t}\frac{\sin\alpha x_0 \sin\alpha(a-x)}{\sin\alpha a}\,d\alpha \qquad x>x_0$$

$$=\frac{1}{i\pi}\int e^{-\kappa\alpha^2 t}\frac{\sin\alpha x \sin\alpha(a-x_0)}{\sin\alpha a}\,d\alpha \qquad x<x_0$$

vanish at the boundaries $x=0$ and $x=a$.

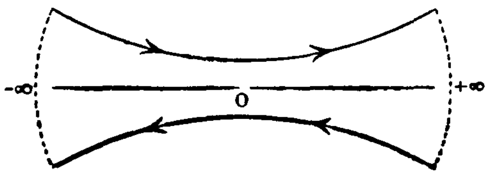
Hence the temperature at x, at time t, due to a source at x_0, at $t=0$, is given by these integrals over the path (P), and since the integrand is an odd function of α we may replace them by the forms

$$\frac{1}{2i\pi}\int e^{-\kappa\alpha^2 t}\frac{\sin\alpha x_0 \sin\alpha(a-x)}{\sin\alpha a}\,d\alpha \qquad x_0<x<a$$

$$\frac{1}{2i\pi}\int e^{-\kappa\alpha^2 t}\frac{\sin\alpha x \ \sin\alpha(a-x_0)}{\sin\alpha a}\,d\alpha \qquad 0<x<x_0$$

over the path (Q) of figure (2).

FIGURE 2.



The path Q in α-plane.

Expansion in Series.

From this result we deduce the expression for the temperature due to a source at $x = x_0$, in the form of an infinite series. Using Cauchy's Residue Theorem, since the singularities occur along the real axis at $\alpha = \frac{n\pi}{a}$, our expressions become

$$\frac{2}{a}\sum_1^\infty \sin\frac{n\pi}{a}x \sin\frac{n\pi}{a}x_0\, e^{-\frac{\kappa n^2\pi^2}{a^2}t}$$

Hence the Green's Function for this case is

$$u = \frac{2}{a}\sum_1^\infty \sin\frac{n\pi}{a}x \sin\frac{n\pi}{a}x_0\, e^{-\frac{\kappa n^2\pi^2}{a^2}(t-\tau)};$$

and the temperature at x_0 at time t when the initial temperature is $f(x)$ and the boundaries are kept at $\phi_1(t)$ and $\phi_2(t)$, is given by

$$v = \frac{2}{a}\sum_1^\infty \sin\frac{n\pi}{a}x_0 \int_0^a \sin\frac{n\pi}{a}x f(x) e^{-\kappa\frac{n^2\pi^2}{a^2}t}\, dx$$

$$+\frac{2\kappa n\pi}{a}\sum_1^\infty \sin\frac{n\pi}{a}x_0 \int_0^t [\phi_1(\tau) - (-1)^n\phi_2(\tau)]\, e^{-\kappa\frac{n^2\pi^2}{a^2}(t-\tau)}\, d\tau.$$

In the Synthetical Method this result is obtained by the distribution of sources and doublets along the axis.

§ 4.

Linear Flow.

Finite Solid: Radiation at Boundaries $x=0$, and $x=a$ into a medium at zero.

The Green's Function in this case is obtained in a similar fashion. Starting with the solution

$$V = \frac{1}{2\sqrt{\pi\kappa t}} e^{-\frac{(x-x_0{}^2)}{4\kappa t}}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\kappa\alpha^2 t}\cos\alpha(x - x_0)\, d\alpha.$$

https://doi.org/10.1017/S0013091500034507 Published online by Cambridge University Press

we transform it into the integrals over the path (P) in the a – plane

$$\frac{1}{2\pi}\int e^{-\kappa a^2 t}\, e^{ia(x-x_0)}\, da, \qquad x > x_0,$$

or

$$\frac{1}{2\pi}\int e^{-\kappa a^2 t}\, e^{-ia(x-x_0)}\, da, \qquad x < x_0.$$

Associate with this solution, another given by

$$V_1 = \frac{1}{2\pi}\int e^{-\kappa a^2 t}\left[A e^{iax} + B e^{-iax}\right] da$$

over the path (P) and determine A and B as functions of (a) by the boundary conditions

$$\mp\frac{\partial v}{\partial u} + hv = 0, \qquad \text{at } x = 0 \text{ and } x = a.$$

In this way we obtain

$$A = -(h + ia)\frac{h\sin a(a - x_0) + a\cos a(a - x_0)}{(h^2 - a^2)\sin aa + 2ah\cos aa}$$

$$B = -(h + ia)\frac{h\sin ax_0 + a\cos ax_0}{(h^2 - a^2)\sin aa + 2ah\cos aa}\, e^{iaa}$$

and

$$V + V_1 = -\frac{i}{\pi}\int e^{-\kappa a^2 t}\frac{(h\sin ax_0 + a\cos ax_0)(h\sin a(a - x) + a\cos a(a - x))}{(h^2 - a^2)\sin aa + 2ah\cos aa}\, da,$$

while, when $x < x_0$, we interchange x and x_0 in this expression.

Initial Conditions.

We have chosen V to satisfy the condition at $t = 0$ of the source at $x = x_0$: hence we have only to prove that

$$\underset{t=0}{\text{Lt.}}\,(V_1) = 0.$$

From the form of the expression for V_1 it will be seen that the singularities enter only at the roots of the equation

$$(h^2 - a^2)\sin aa + 2ah\cos aa = 0.$$

These are real and simple and there are thus no poles above the path (P). Also by examining the expression for V_1 it will be seen that this vanishes at infinity in the upper part of the plane, provided that

$$x + x_0 > 0$$

$$x + x_0 - 2a < 0,$$

conditions which are both satisfied.

Hence when we put $t=0$ in V_1, its value is zero, and the convergency factor $e^{-\kappa\alpha^2 t}$, and the choice of the path P, cause the integral to converge uniformly to its value for $t=0$.

Hence
$$\underset{t=0}{\mathrm{Lt.}}\,(V_1) = 0.$$

Boundary Conditions.

The choice of A and B causes the conditions at $x=0$ and $x=a$ to be satisfied.

Expansion in Series.

Taking the form for $x > x_0$,

$$v = -\frac{i}{\pi}\int e^{-\kappa\alpha^2 t}\frac{(h\sin\alpha x + \alpha\cos\alpha x_0)(h\sin\alpha(a-x) + \alpha\cos\alpha(a-x))}{(h^2-\alpha^2)\sin\alpha a + 2\alpha h\cos\alpha a}d\alpha,$$

over the path (P), we obtain

$$v = -\frac{i}{2\pi}\int e^{-\kappa\alpha^2 t}\frac{(h\sin\alpha x_0 + \alpha\cos\alpha x_0)(h\sin\alpha(a-x) + \alpha\cos\alpha(a-x))}{(h^2-\alpha^2)\sin\alpha a + 2\alpha h\cos\alpha a}d\alpha,$$

over the path (Q),

$$= 2\Sigma\, e^{-\kappa\alpha^2 t}\frac{(h\sin\alpha x_0 + \alpha\cos\alpha x_0)(h\sin\alpha x + \alpha\cos\alpha x)}{a(h^2+\alpha^2) + 2h},$$

the summation being taken over the positive roots of the equation

$$(h^2-\alpha^2)\sin\alpha a + 2\alpha h\cos\alpha a = 0.$$

The symmetry of this result shows that the expression also holds for $0 < x < x_0$, since in that case we had only to interchange x and x_0 in our former work.

https://doi.org/10.1017/S0013091500034507 Published online by Cambridge University Press

Hence the Green's Function for this solid is given by the equation

$$u = 2\sum \frac{(h\sin\alpha x_0 + \alpha\cos\alpha x_0)(h\sin\alpha x + \alpha\cos\alpha x)}{a(h^2+\alpha^2)+2h} e^{-\kappa\alpha^2(t-\tau)}.$$

The solution for an arbitrary initial distribution $v = f(x)$ follows at once, and we obtain for the case of the medium at zero temperature,

$$v = 2\int_0^a f(x')\left[\sum e^{-\kappa\alpha^2 t}\frac{(h\sin\alpha x' + \alpha\cos\alpha x')(h\sin\alpha x + \alpha\cos\alpha x)}{a(h^2+\alpha^2)+2h}\right]dx'$$

as the temperature at x at the time t. This admits of integration term by term and may be written

$$v = 2\sum e^{-\kappa\alpha^2 t}\frac{(h\sin\alpha x + \alpha\cos\alpha x)}{a(h^2+\alpha^2)+2h}\int_0^a f(x')[h\sin\alpha x' + \alpha\cos\alpha x']\,dx'.$$

It follows that

$$f(x) = 2\ \underset{t=0}{\text{Lt.}}\ \sum e^{-\kappa\alpha^2 t}\frac{(h\sin\alpha x + \alpha\cos\alpha x)}{a(h^2+\alpha^2)+2h}\int_0^a f(x')[h\sin\alpha x' + \alpha\cos\alpha x']dx',$$

when $0 < x < a$.

This expansion differs from that obtained by the Fourier Method * by the presence of the Convergency Factor $e^{-\kappa\alpha^2 t}$, and in the above proof we are not at liberty to proceed to the value $t = 0$, the expansion occurring only as the limit when $t = 0$. For the discussion of the convergency of the series *when* $t = 0$, reference may be made to the two dissertations noted below.†

---

* Fourier's Heat. Chapter V., Section I.
Kirchhoff. *Vorlesungen über Mathematische Physik*, Bd. IV., pp. 30-33.

† Knake. *Über die Wärme-bewegung in einem von zwei parallelen Wänden begrenzten Korper dessen Begrenzungen mit einem Gase in Berührung stehen.*
Diss. Halle. 1871.

Fudzisawa. *Über eine in der Wärme-Leitungs-Theorie auftretende, nach den Wurzeln einer transcendenten Gleichung fortschreitende, unendliche Reihe.*
Diss. Strassburg. 1886.

§ 5.

Two Dimensional Problems.

In the cases where the equation of conduction reduces to

$$\frac{\partial v}{\partial t}=\kappa\left[\frac{\partial^2 v}{\partial^2 x}+\frac{\partial^2 v}{\partial y^2}\right].$$

we use as Green's Function u the temperature at (x, y) at time t due to a Line Source generated at the instant τ along $x=x_0$, $y=y_0$, the surface being in the one case kept at zero, and in the other radiation taking place into a medium at zero.

With these values for the Green's Function, the temperature at $\mathrm{P}(x_0, y_0)$ at time t, when the initial temperature is $f(x, y)$ and the boundary is either kept at $\phi(x, y, t)$ or radiation takes place into a medium at that temperature, is given by the equation

$$[v_{\mathrm{P}}]_t=\iint u_{\tau=0}f(x, y)dxdy+\kappa\int_0^t\left[\int\frac{\partial u}{\partial n_i}\phi(x, y, \tau)ds\right]d\tau,$$

integration taking place along the bounding arcs.

By means of this result we are able to write down the solutions of the two problems in which the solid is bounded by the plane $y=0$, and extends to infinity in the direction $y>0$: the initial temperature is $f(x, y)$: and, in the first case, the boundary $y=0$ is kept at temperature $f(x, t)$, while in the second, radiation takes place into a medium at that temperature.

When the boundary is kept at temperature $\mathrm{F}(x, t)$, the Green's Function is obviously given by

$$u=\frac{1}{4\pi\kappa(t-\tau)}\left[e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\kappa(t-\tau)}}-e^{-\frac{(x-x_0)^2+(y+y_0)^2}{4\kappa(t-\tau)}}\right],$$

and

$$\left[\frac{\partial u}{\partial n_i}\right]=\left[\frac{\partial u}{\partial y}\right]_{y=0}=\frac{y_0}{4\pi\kappa^2(t-\tau)^2}e^{-\frac{(x-x_0)^2+y_0^2}{4\kappa(t-\tau)}}$$

Hence $[v_P]_t$

$$= \frac{1}{4\pi\kappa t}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(x, y)\left[e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\kappa(t-\tau)}} - e^{-\frac{(x-x_0)^2+(y+y_0)^2}{4\kappa(t-\tau)}}\right]dxdy$$

$$+ \frac{y_0}{4\pi\kappa}\int_0^t\int_{-\infty}^{\infty}\frac{F(x, \tau)}{(t-\tau)^2}e^{-\frac{(x-x_0)^2+y_0^2}{4\kappa(t-\tau)}}dx\, d\tau$$

gives the temperature at $P(x_0, y_0)$ at the time t.

In the case of radiation the Green's Function is given by Bryan,*

$$u = \frac{1}{4\pi\kappa(t-\tau)}\left[e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4\kappa(t-\tau)}} - e^{-\frac{(x-x_0)^2+(y+y_0)^2}{4\kappa(t-\tau)}} - 2h\int_0^{\infty} e^{-h\eta}\, e^{-\frac{(x-x_0)^2+(y+y_0+\eta)^2}{4\kappa(t-\tau)}}d\eta\right],$$

and may be obtained also by the method followed by the author in the similar case of Linear Flow.

Hence

$$\left[\frac{\partial u}{\partial n_i}\right] = \left[\frac{\partial u}{\partial y}\right]_{y=0} = \frac{h}{4\pi\kappa^2(t-\tau)^2}\int_0^{\infty} e^{-h\eta}\, e^{-\frac{(x-x_0)^2+(y_0+\eta)^2}{4\kappa(t-\tau)}}(y_0+\eta)d\eta$$

and the solution of the general problem, when the initial temperature is zero, is given by the equation

$$[v_P]_t = \frac{h}{4\pi\kappa}\int_0^t\int_{-\infty}^{\infty}\int_0^{\infty}\frac{F(x, \tau)}{(t-\tau)^2}\cdot e^{-h\eta}\cdot e^{-\frac{(x-x_0)^2+(y_0+\eta)^2}{4\kappa(t-\tau)}}d\tau dx d\eta$$

This agrees with the solution obtained by Hobson by the Synthetical Method.

§ 6.

The Circular Cylinder.

Before discussing the corresponding problems for the cylinder, it will be necessary to define the solutions of Bessel's Equation which we employ.

* loc. cit. p. 427.

The *Bessel's Function of the First Kind* is, as usual, defined by the equation

$$J_n(z) = \sum_0^\infty (-1)^s \frac{z^{n+2s}}{2^{n+2s}\,\Pi(s)\,\Pi(n+2s)},$$

where to make the function uniform we have to restrict the complex variable to a complete revolution about the origin, and we assume that the argument of z varies from $-\frac{\pi}{2}$ to $\frac{3\pi}{2}$.

For the *Bessel's Function of the Second Kind*, Hankel * uses

$$Y_n(z) = \frac{2\pi e^{ni\pi}}{\sin 2n\pi}(\cos n\pi\, J_n(z) - J_{-n}(z)),$$

and he obtains the following expressions for the limiting values of these two solutions when z becomes infinite, the real part of z being positive :—

$$J_n(z) = \sqrt{\frac{2}{\pi z}}.\cos\left\{z - (n+\tfrac{1}{2})\frac{\pi}{2}\right\}$$

$$Y_n(z) = \sqrt{\frac{2\pi}{z}}\,\frac{e^{ni\pi}}{\cos n\pi}\sin\left\{z - (n+\tfrac{1}{2})\frac{\pi}{2}\right\} \dagger$$

In this paper it is necessary to use as Second Solution a function which vanishes at the positive imaginary infinity. Hankel shows that the function

$$U_n(z) = \frac{\pi}{2\sin n\pi}(J_{-n}(z) - e^{-in\pi}J_n(z))$$

has the limiting value

$$\sqrt{\frac{\pi}{2z}}\,e^{-\frac{in\pi}{2}}\,e^{i\left(z+\frac{\pi}{4}\right)},$$

whether the real part of z be positive or negative, and it is obvious that this solution vanishes at the positive imaginary infinity.

We shall use this as our Bessel's Function of the Second Kind.

---

* Hankel. *Die Cylinder-Functionen erster und zweiter Art.* Math. Ann. Bd. VI., p. 494 (3) and (4).

† loc. cit. pp. 496-7.

It will be seen that

$$U_n(z) = \frac{i\pi}{2} J_n(z) - \frac{\cos n\pi}{2e^{ni\pi}} Y_n(z)$$

and it is to be noticed that what we here write as Y_n is not the solution given under that symbol in Gray and Mathew's *Treatise on Bessel Functions.* These writers follow Neumann's Notation, and denote Hankel's Y_n by $\overline{Y}_n$. The relation which connects the two is given (p. 66) by :—

$$\overline{Y}_n(z) = Y_n(z) - (\log 2 - \gamma) J_n(z),$$

γ, being Euler's Constant.

It also follows from the definition of U_n that when the real part of z is positive

$$i\pi J_n(z) = U_n(z) - e^{in\pi} U_n(ze^{i\pi}).*$$

§ 7.

Infinite Circular Cylinder: $r = a$; boundary at zero temperature.

To obtain the Green's Function for this case we proceed from the solution

$$v = \frac{1}{4\pi\kappa t} e^{-\frac{r^2 + r'^2 - 2rr' \cos(\theta - \theta')}{4\kappa t}},$$

corresponding to a Line Source at $(r'\theta')$ in the infinite solid.

We transform this into

$$\frac{1}{2\pi}\int_0^\infty e^{-\kappa\lambda^2 t} J_0(\lambda R)\lambda d\lambda,†$$

where $R^2 = r^2 + r'^2 - 2rr'\cos(\theta - \theta')$.

---

* Reference might also be made to the discussion in Graf and Gubler's *Einleitung in die Theorie der Bessel'schen Functionen.* Erster Heft. Cf. pp. 34, 35, 82-86. Bern, 1898.

† Cf. Gray and Mathew's Treatise, p. 77 (158).

Since, by Neumann's Formula,

$$J_0(\lambda R) = J_0(\lambda r)J_0(\lambda r') + 2\sum_1^\infty J_m(\lambda r)J_m(\lambda r')\cos m(\theta - \theta')$$

our expansion becomes

$$\frac{1}{2\pi}\int_0^\infty \lambda e^{-\kappa\lambda^2 t}\left[J_0(\lambda r)\,J_0(\lambda r') + 2\sum_1^\infty J_m(\lambda r)\,J_m(\lambda r')\cos m(\theta - \theta')\right]d\lambda.$$

If we assume that this series is uniformly convergent and can be integrated term by term,* this expression may be written

$$\frac{1}{\pi}\sum_{m=0}^\infty a_m \cos m(\theta - \theta')\int_0^\infty \lambda e^{-\kappa\lambda^2 t} J_m(\lambda r)\,J_m(\lambda r')d\lambda,$$

where $a_0 = \frac{1}{2}$ and $a_m = 1$, $m \geqq 1$.

Now

$$\frac{1}{i\pi}\int_{-\infty}^\infty \lambda e^{-\kappa\lambda^2 t} J_m(\lambda r')\,U_m(\lambda r)d\lambda$$

$$= \frac{1}{i\pi}\int_0^\infty \lambda e^{-\kappa\lambda^2 t}\left(U_m(\lambda r) - e^{im\pi}\,U_m(-\lambda r)\right)J_m(\lambda r')d\lambda$$

$$= \int_0 \lambda e^{-\kappa\lambda^2 t} J_m(\lambda r')\,J_m(\lambda r)d\lambda$$

since

$$i\pi\,J_m(\lambda r) = U_m(\lambda r) - e^{im\pi}\,U_m(-\lambda r)$$

in this case.

Therefore

$$\int_0^\infty \lambda e^{-\kappa\lambda^2 t} J_m(\lambda r)\,J_m(\lambda r')d\lambda$$

$$= \frac{1}{i\pi}\int_{-\infty}^\infty \lambda e^{-\kappa\lambda^2 t} J_m(\lambda r')\,U_m(\lambda r)d\lambda$$

$$= \frac{1}{i\pi}\int \lambda e^{-\kappa\lambda^2 t} J_m(\lambda r')\,U_m(\lambda r)d\lambda, \qquad r > r'$$

the path of integration being now the path (P) of Fig. (1) in the plane of the complex variable λ: and we must interchange r and r' in this result when $r < r'$.

---

* Cf. Sommerfeld. *Die Willkurlichen Functionen in der Mathematischen Physik*, §§ 7, 12.
Diss. Königsberg, 1891.

This follows by Cauchy's Theorem since there are no poles of the integrand inside the contour formed by the real axis, the dotted lines, and the path (P) fig. (1). Further the integrand vanishes over the dotted lines when the part of the path is taken at an infinite distance: * and the argument of λ on the path P at infinity, must, on the right, lie between 0 and $\frac{\pi}{4}$, and on the left between $\frac{3\pi}{4}$ and π, since otherwise the factor $e^{-\kappa\lambda^2 t}$ would become infinite.

We have thus transformed the expression for the source into an infinite series each of whose terms is an integral over the path (P) in the λ plane.

We denote this solution, as before by V, and have the equation.

$$V=\frac{1}{i\pi^2}\Sigma a_m \cos m(\theta-\theta')\int \lambda e^{-\kappa\lambda^2 t} J_m(\lambda r')\, U_m(\lambda r)d\lambda \qquad (r>r')$$

the integrals being over the path (P) and r, r' being interchanged, when $r<r'$.

To obtain the conditions at the boundary $r=a$, we associate with this solution, another, denoted by V_1, where

$$V_1=\frac{1}{i\pi^2}\Sigma a_m \cos m(\theta-\theta')\int A\lambda e^{-\kappa\lambda^2 t} J_m(\lambda r')\, U_m(\lambda r)d\lambda \qquad (r>r'),$$

and choose the term A so that the Boundary Conditions are satisfied.

We find, at once,

$$A=-\frac{U_m(\lambda a)}{J_m(\lambda a)}$$

and putting

$$v=V+V_1$$

we obtain the solution of our problem in the form

$$v=\frac{1}{i\pi^2}\Sigma a_m \cos m(\theta-\theta')$$

$$\int \lambda e^{-\kappa\lambda^2 t}\frac{J_m(\lambda r')}{J_m(\lambda a)}\left(U_m(\lambda r)J_m(\lambda a)-U_m(\lambda a)J_m(\lambda r)\right)d\lambda$$

when $r>r'$,

the integrals being taken over the path (P).

We shall now show that this expression satisfies all the conditions of the problem and then obtain an infinite series to which it is equivalent.

* Cf. The approximate value given below for the Bessel's Functions.

Boundary and Initial Conditions.

The Boundary Conditions are satisfied by our choice of the supplementary function V_1: and we have only to show that

$$\underset{t=0}{\mathrm{Lt.}}\ (V_1) = 0,$$

since $\underset{t=0}{\mathrm{Lt.}}\ (V)$ satisfies the conditions for a source at (r', θ').

Hence we have to show that

$$\underset{t=0}{\mathrm{Lt.}} \int \lambda e^{-\kappa\lambda^2 t} \frac{J_m(\lambda r')\, U_m(\lambda a)\, J_m(\lambda r)}{J_m(\lambda a)}\, d\lambda,$$

over the path (P), vanishes.

The limiting forms, when λ is very large and lies in the upper part of the plane, of the Bessel's Functions occurring in this expression are given as follows :—

$$\left.\begin{aligned}
J_m(\lambda r) &= \frac{1}{2\sqrt{\pi\lambda r}}\, e^{-i\xi r + i\left(m+\frac{1}{2}\right)\frac{\pi}{2}} \,.\, e^{r\eta} \\
J_m(\lambda r') &= \frac{1}{2\sqrt{\pi\lambda r'}}\, e^{-i\xi r' + i\left(m+\frac{1}{2}\right)\frac{\pi}{2}} \,.\, e^{r'\eta} \\
J_m(\lambda a) &= \frac{1}{2\sqrt{\pi\lambda a}}\, e^{-i\xi a + i\left(m+\frac{1}{2}\right)\frac{\pi}{2}} \,.\, e^{a\eta} \\
U_m(\lambda a) &= \sqrt{\frac{\pi}{2\lambda a}}\,.\, e^{i\xi a - i\left(m-\frac{1}{2}\right)\frac{\pi}{2}} \,.\, e^{-a\eta}
\end{aligned}\right\} \lambda = \xi + i\eta.$$

Thus

$$\frac{\lambda\, J_m(\lambda r')\, J_m(\lambda r)\, U_m(\lambda a)}{J_m(\lambda a)}$$

vanishes at the positive imaginary infinity, when $r + r' - 2a < 0$.

Also since the zeroes of

$$J_m(\lambda a) = 0$$

are real and simple, there are no poles of the integrand, above (P), and

$$\int \frac{\lambda J_m(\lambda r')\, J_m(\lambda r)\, U_m(\lambda r)}{J_m(\lambda a)}\, d\lambda$$

vanishes.

The presence of the factor $e^{-\kappa\lambda^2 t}$ and the choice of the path (P) cause the integral

$$\int e^{-\kappa\lambda^2 t} \frac{\lambda J_m(\lambda r') J_m(\lambda r) U_m(\lambda a)}{J_m(\lambda a)} d\lambda$$

to converge uniformly to its value for $t=0$, and thus

$$\underset{t=0}{\text{Lt.}} \int \lambda e^{-\kappa\lambda^2 t} \frac{J_m(\lambda r') J_m(\lambda r) U_m(\lambda a)}{J_m(\lambda a)} d\lambda$$

vanishes.

The Initial and Boundary Conditions are thus both satisfied by the expressions we have obtained.

Expansion in Series.

We may replace the term

$$\frac{1}{i\pi^2} \int \lambda e^{-\kappa\lambda^2 t} \frac{J^m(\lambda r')}{J_m(\lambda a)} [U_m(\lambda r) J_m(\lambda a) - U_m(\lambda a) J_m(\lambda r)] \, d\lambda$$

over the path (P) by half this integrand over the path (Q), the integrand being a uniform, odd function of λ.

The poles of the integrand are the zeroes of $J_m(\lambda a)$, which lie symmetrically along the real axis and are not repeated.

Thus from this term in v we obtain $-\frac{1}{\pi}$ [sum of the residues along the real axis] which reduces to

$$\frac{2}{\pi} \Sigma \, \lambda e^{-\kappa\lambda^2 t} \frac{J_m(\lambda r') J_m(\lambda r) U_m(\lambda a)}{J_m'(\lambda a)},$$

the summation being taken over the positive roots of the equation $J_m(\lambda a) = 0$.

But $$U_m(x) J_m'(x) - J_m(x) U_m'(x) = \frac{1}{x}.*$$

and therefore the expression for v may be written

$$\frac{2}{\pi a^2} \Sigma \, a_m \cos m(\theta - \theta') \sum_{\lambda} e^{-\kappa\lambda^2 t} \frac{J_m(\lambda r') J_m(\lambda r)}{[J_m'(\lambda a)]^2}.$$

This is the value of the temperature at points (r, θ), $(r > r')$, in the infinite cylinder $r = a$, due to a source at $t=0$ at the points (r', θ').

* Cf. Weber. *Über die stationären Strömungen der Electricität in Cylindern.* Crelles' Journal. Bd. 76, p. 10.
Graf u. Gubler, loc. cit. Erstes Heft, pp. 43-45.

Since for the points $r<r'$, we have to interchange r and r' in the above work, as they enter symmetrically, this expression holds for both cases.

The Green's Function for this case is therefore given by the expression

$$\frac{2}{\pi a^2}\sum_{m=0}^{\infty} a_m \cos m(\theta-\theta')\sum_{\lambda} e^{-\kappa\lambda^2(t-\tau)}\frac{J_m(\lambda r')\,J_m(\lambda r)}{[J_m'(\lambda a)]^2}$$

where $a_0=\frac{1}{2}$ and $a_m=1$ $(m\geqq 1)$, and the summation takes place over the positive roots of the equation $J_m(\lambda a)=0$.

The solutions of the temperature problems in the Cylinder follow at once.

In particular when the initial temperature is $f(r)$ we obtain the temperature at (r, θ) by integration in the form

$$v=\frac{2}{a^2}\int_0^a r'f(r')\,\Sigma\, e^{-\kappa\lambda^2 t}\frac{J_0(\lambda r)\,J_0(\lambda r')}{[J_0'(\lambda a)]^2}\,dr',$$

which may be written

$$\frac{2}{a^2}\,\Sigma\,\frac{J_0(\lambda r)}{[J_0'(\lambda a)]^2}\,e^{-\kappa\lambda^2 t}\int_0^a r'f(r')\,J_0(\lambda r')dr'.$$

and when the initial distribution is $f(r, \theta)$ the solution is given by

$$v=\frac{2}{\pi a^2}\int_0^a\int_0^{2\pi} f(r', \theta')r'\left[\sum_{m=0}^{\infty} a_m \cos m(\theta-\theta')\sum_{\lambda} e^{-\kappa\lambda^2 t}\frac{J_m(\lambda r)J_m(\lambda r')}{[J_m'(\lambda a)]^2}\right]dr'd\theta',$$

the summation extending over the positive roots of

$$J_m(\lambda a)=0.$$

If we assume that this series may be integrated term by term we have for the co-efficient of $J_m(\lambda r)\cos m\theta$ the expression

$$\frac{2}{\pi a^2}\frac{e^{-\kappa\lambda^2 t}}{[J_m'(\lambda a)]^2}\int_0^a\int_0^{2\pi} r'f(r', \theta')\,J_m(\lambda r')\cos m\theta' dr'd\theta'.$$

These two series correspond with the expansions obtained for the arbitrary functions $f(r)$ and $f(r, \theta)$ by the Fourier Method, and occur here as the limiting cases of the expressions obtained for the temperature when t vanishes.*

---

* Cf. Gray and Mathews, Chapter VI.

§8.

Infinite Cylinder $r=a$. Radiation at Boundary into a Medium at Zero.

Starting with the expression for the source in the infinite solid, we transformed it as before into

$$V=\frac{1}{i\pi^2}\Sigma\, a_m \cos m(\theta-\theta')\int \lambda e^{-\kappa\lambda^2 t} J_m(\lambda r')\, U_m(\lambda r) d\lambda \qquad (r>r')$$

the integral being taken over the path (P).

We then obtain the supplementary solution

$$V_1=-\frac{1}{i\pi^2}\Sigma a_m \cos m(\theta-\theta')$$

$$\int \lambda e^{-\kappa\lambda^2 t}\frac{J_m(\lambda r)J_m(\lambda r')(\lambda U'_m(\lambda a)+hU_m(\lambda a))}{\lambda J_m'(\lambda a)+h J_m(\lambda a)}d\lambda,$$

over the same path (P), and we prove that

$$v=V+V_1,$$

which satisfies the Boundary Condition

$$\frac{\partial v}{\partial r}+hv=0 \text{ at } r=a,$$

also satisfies the Initial Conditions for a source at (r', θ').

The proof follows exactly the same lines as before. We examine

$$\int \lambda J_m(\lambda r) J_m(\lambda r')\,\frac{\lambda U_m'(\lambda a)+h\, U_m'(\lambda a)}{\lambda J_m'(\lambda a)+h J_m(\lambda a)}\,.\,d\lambda$$

over the path (P) and show that this vanishes, when $r+r'-2a<0$, using the fact that the roots of the equation

$$\lambda J_m'(\lambda a)+hJ_m(\lambda a)=0 \, *$$

are real and simple.

The choice of the path (P) then allows us to deduce that

$$\underset{t=0}{\mathrm{Lt}}\int \lambda e^{-\kappa\lambda^2 t} J_m(\lambda r')\, J_m(\lambda r)\,\frac{\lambda\, U_m'(\lambda a)+h\, U_m(\lambda a)}{\lambda\, J_m'(\lambda a)+h\, J_m(\lambda a)}\, d\lambda,$$

vanishes.

This expression for the temperature, involving Contour Integrals, may be reduced to a Double Infinite Series by taking the path (Q),

* Cf. Heine. *Einige Anwendungen der Residuen-Rechnung.* Crelle's Journal, Bd. 89.

as before, instead of the path (P). By this means the co-efficient of $a_m \cos m(\theta - \theta')$ in the expression for v becomes

$$\frac{2}{\pi} \Sigma \lambda e^{-\kappa\lambda^2 t} J_m(\lambda r) J_m(\lambda r') \frac{\lambda U_m'(\lambda a) + h U_m(\lambda a)}{a\lambda J_m''(\lambda a) + (1 + ha) J_m'(\lambda a)},$$

the summation extending over the positive roots of the equation

$$\lambda J_m'(\lambda a) + h J_m(\lambda a) = 0,$$

and substituting for $J_m''(\lambda a)$ we obtain for this term the series

$$\frac{2}{\pi a^2} \Sigma \lambda^2 e^{-\kappa\lambda^2 t} \frac{J_m(\lambda r) J_m(\lambda r')}{\left(h^2 + \lambda^2 - \frac{m^2}{a^2}\right)[J_m(\lambda a)]^2}$$

which holds for $r \gtreqless r'$.

We are thus led to the following expression for the temperature at (r, θ) in the cylinder $r = a$, due to the source at (r', θ') at $t = 0$:—

$$v = \frac{2}{\pi a^2} \sum_{m=0}^{\infty} a_m \cos m(\theta - \theta') \sum_{\lambda} \lambda^2 e^{-\kappa\lambda^2 t} \frac{J_m(\lambda r') J_m(\lambda r)}{\left(h^2 + \lambda^2 - \frac{m^2}{a^2}\right)[J_m(\lambda a)]^2}$$

the summation extending over the positive roots of the equation

$$\lambda J_m'(\lambda a) + h J_m(\lambda a) = 0.$$

The results of the general problems with arbitrary initial temperature and arbitrary temperature for the surrounding medium may be at once deduced. In particular when the initial temperature is $f(r)$ and the medium is at zero, the temperature at (r, θ) at time t is given by

$$v = \frac{2}{a^2} \sum_{\lambda} \lambda^2 e^{-\kappa\lambda^2 t} \frac{J_0(\lambda r)}{(h^2 + \lambda^2)[J_0(\lambda a)]^2} \int_0^a r' f(r') J_0(\lambda r') dr' ;$$

and this gives the expansion of the arbitrary function $f(r)$ in the form,

$$f(r) = \operatorname*{Lt.}_{t=0} \frac{2}{a^2} \sum_{\lambda} \lambda^2 e^{-\kappa\lambda^2 t} \frac{J_0(\lambda r)}{(h^2 + \lambda^2)[J_0(\lambda a)]^2} \int_0^a r' f(r') J_0(\lambda r') dr' ;$$

while in the case of an arbitrary initial temperature $f(r, \theta)$ we obtain

$$v = \frac{2}{\pi a^2} \int_0^{2\pi} \int_0^a f(r', \theta') r'$$

$$\left[\sum_{m=0}^{\infty} a_m \cos m(\theta - \theta') \sum_{\lambda} \lambda^2 e^{-\kappa\lambda^2 t} \frac{J_m(\lambda r) J_m(\lambda r')}{\left(h^2 + \lambda^2 - \frac{m^2}{a^2}\right)[J_m(\lambda a)]^2} \right] dr' d\theta'.$$

If we assume that we may integrate this Double Series term by term, this expression gives for the co-efficient of the term in

$$J_m(\lambda r)\cos m\theta$$

the value

$$\frac{2\lambda^2 e^{-\kappa\lambda^2 t}}{\pi a^2\left(h^2+\lambda^2-\dfrac{m^2}{a^2}\right)[J_m(\lambda a)]^2}\int_0^{2\pi}\int_0^a r' f(r',\ \theta')\, J_m(\lambda r')\cos m\theta' dr' d\theta',$$

and we obtain for the expansion of $f(r',\ \theta')$ a series which corresponds with the Fourier-Bessel Series obtained by Fourier's method.*

§ 9.

The solution of the three Dimensional Problems discussed in Hobson's paper §§ 5, 6 follow from Green's Functions which may be at once written down. The case of the sphere may be treated as we have done the cylinder, and the problem of a source between two planes meeting at an angle a admits of a corresponding treatment. This latter problem has been discussed for special cases of a by the Method of Images in a Riemann's Space in my paper in the Proceedings of the London Mathematical Society.† The extension to a solid bounded by planes, cylinders, and spheres offers no special difficulty. I propose to return to these questions in a later paper.

---

* Cf. Gray and Mathews, Chapter VI.

† Proc. Lond. Math. Soc., Vol. XXX., pp. 151-161.