EULER GRAPHS ON LABELLED NODES

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Introduction. In this paper we shall derive a concise formula for the number of Euler graphs on n labelled nodes and k edges. An Euler graph is a connected graph in which every node has even valency, where by the valency of a node is meant the number of edges which are incident with that node. Throughout most of the paper we shall be dealing with graphs whose nodes have even valencies but which may or may not be connected. For convenience we shall refer to these graphs as Euler graphs, although the usage is not, strictly speaking, correct. We shall impose the condition of connectedness in § 4.

1. The main result. Let $\mathfrak{G}$ be the set of all graphs on n labelled nodes and k edges. Unless otherwise stated we do not allow loops (an edge joining a node to itself) or multiple edges (more than one edge joining two nodes). Let us take any graph $G \in \mathfrak{G}$, and to each node allocate either the number $+1$ or the number -1. The nodes can then be referred to as "positive" or "negative" nodes respectively. The product of the numbers allocated to the end-nodes of an edge will be called the "sign" of the edge, and is clearly ± 1. The "sign," $\sigma(G)$, of the whole graph is then defined as the product of the signs of its edges.

In this product, the number allocated to a node will occur as a factor v times, where v is the valency of the node. From this we see that

$$\sigma(G) = (-1)^V \tag{1}$$

where V is the sum of the valencies of the negative nodes.

On the other hand, from the definition of $\sigma(G)$, we have

$$\sigma(G) = (-1)^\nu \tag{2}$$

where ν is the number of edges in G which join a positive to a negative node.

We now consider the expression $\sum\sigma(G)$, where the summation is for all graphs in $\mathfrak{G}$ and over the set S of all the 2^n possible allocations of $+1$ or -1 to the nodes. From (1) and (2) we then have

$$\sum_{G\in\mathfrak{G}} \left\{ \sum_{S} (-1)^V \right\} = \sum_{S} \left\{ \sum_{G\in\mathfrak{G}} (-1)^\nu \right\} \tag{3}$$

these being different ways of writing $\sum\sigma(G)$.

We look first at the left-hand side of (3). If G is an Euler graph then V is

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even, whatever the allocation in S. Hence $\sum_S(-1)^V = 2^n$, and G contributes 2^n to the left-hand side of (3). If G is not an Euler graph then it has at least one node A of odd valency. The allocations in S for which A is positive, and those for which it is negative are equinumerous, and contribute equal and opposite amounts to $\sum_S(-1)^V$. Hence G contributes nothing to the left-hand side of (3).

It follows that the left-hand side of (3) is 2^n times the number of Euler graphs in $\mathfrak{G}$.

We now look at the right-hand side of (3), and consider an allocation in S for which p nodes are positive and $q(=n-p)$ nodes are negative. There are $\binom{n}{p}$ such allocations. If there are ν edges which join positive to negative nodes these may be disposed in $\binom{pq}{\nu}$ different ways; the remaining $k-\nu$ edges may then be disposed in

$$\binom{\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1)}{k-\nu}$$

different ways. Summing from $\nu = 0$ to k we obtain

$$\sum_{\nu=0}^{k} (-1)^\nu \binom{pq}{\nu}\binom{\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1)}{k-\nu}$$

as the contribution towards the right-hand side of (3) for each allocation with given p and q. This contribution is the coefficient of t^k in

$$(4) \qquad (1-t)^{pq}(1+t)^{\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1)}.$$

Hence the right-hand side of (3) is the coefficient of t^k in

$$(5) \qquad \sum_{p=0}^{n} \binom{n}{p}(1-t)^{pq}(1+t)^{\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1)},$$

and this coefficient is therefore 2^n times the number of Euler graphs in $\mathfrak{G}$. Observing that

$$\tfrac{1}{2}p(p-1)+\tfrac{1}{2}q(q-1) = \tfrac{1}{2}n(n-1) - p(n-p)$$

we obtain the final result, that the required number of Euler graphs is the coefficient of t^k in

$$(6) \qquad \left(\frac{1}{2}\right)^n (1+t)^{\frac{1}{2}n(n-1)} \sum_{p=0}^{n} \binom{n}{p}\left(\frac{1-t}{1+t}\right)^{p(n-p)}.$$

We may note in passing that the total number of Euler graphs on n nodes (obtained by putting $t = 1$ in (6)) is

$$\left(\frac{1}{2}\right)^n 2^{\frac{1}{2}n(n-1)}(1+1) = 2^{\frac{1}{2}(n-1)(n-2)}.$$

This number is also the total number of graphs on $n-1$ labelled nodes.

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We can see this directly by setting up a one-to-one correspondence between these two sets of graphs in the following manner. We choose a particular label, and from each Euler graph on n labelled nodes we remove the node bearing that label, together with all edges incident to it. We obtain a graph on $n-1$ labelled nodes corresponding to every Euler graph on n nodes. It is not difficult to show that this correspondence is one-to-one.*

2. Graphs with loops and multiple edges. The corresponding formulae for Euler graphs with loops or multiple edges allowed are easily obtained by straightforward modifications of the argument in § 1. If loops are allowed, then in forming graphs corresponding to a given allocation in S and totalling their contributions to the right-hand side of (3), we have $\frac{1}{2}p(p+1)+\frac{1}{2}q(q+1)$ possible "places" in which to put the $k-\nu$ edges joining pairs of nodes with the same allocated number, instead of only $\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1)$.

If multiple edges are allowed, then the k edges with sign -1 are put in the pq possible "places" with repetitions allowed. Hence we have

$$\binom{pq+\nu-1}{\nu}$$

instead of $\binom{pq}{\nu}$. Similarly we have

$$\binom{\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1)+k-r-1}{k-\nu}$$

instead of

$$\binom{\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1)}{k-\nu}.$$

These are the only modifications, and the formulae corresponding to (6) above are found to be as follows:

When loops are allowed but multiple edges are not;

$$\left(\frac{1}{2}\right)^n(1+t)^{\frac{1}{2}n(n+1)}\sum_{p=0}^{n}\binom{n}{p}\left(\frac{1-t}{1+t}\right)^{p(n-p)}.$$

When multiple edges are allowed but loops are not:

$$\left(\frac{1}{2}\right)^n(1-t)^{-\frac{1}{2}n(n-1)}\sum_{p=0}^{n}\binom{n}{p}\left(\frac{1-t}{1+t}\right)^{p(n-p)}.$$

When loops and multiple edges are allowed:

$$\left(\frac{1}{2}\right)^n(1-t)^{-\frac{1}{2}n(n+1)}\sum_{p=0}^{n}\binom{n}{p}\left(\frac{1-t}{1+t}\right)^{p(n-p)}.$$

*I am indebted to the referee for pointing out this correspondence between the two sets of graphs.

3. A generalization. Let us say that a graph is *m-ply-Eulerian* if the valencies of all its nodes are multiples of the integer m. This generalizes the concept of Euler graphs, for which $m = 2$. We can enumerate m-ply-Eulerian graphs on n labelled nodes and k edges by the same sort of procedure as that by which we have enumerated Euler graphs, but the formulae obtained are not so tractable. We shall, therefore, consider only the case $m = 3$, and that very briefly.

To every node of a graph $G \in \mathfrak{G}$ we shall allocate one of the numbers 1, ω, or ω^2, where ω is a complex cube root of unity. To each edge of G we allocate the product of the numbers allocated to its end-nodes, and to the graph G itself we allocate the product $\tau(G)$ of the numbers allocated to its edges.

If V is the sum of the valencies of nodes allocated ω, and W is the sum of the valencies of the nodes allocated ω^2, then we have

$$\tau(G) = \omega^V\omega^{2W}$$

as the analogue of (1).

If α is the number of edges in G allocated the number 1, β the number allocated ω, and γ the number allocated ω^2; then we have, as analogue of (2), the equation

$$\tau(G) = \omega^{\beta+2\gamma}.$$

The analogue of (3) becomes

$$\sum_{G\in\mathfrak{G}}\left\{\sum_T \omega^V\omega^{2W}\right\} = \sum_T\left\{\sum_{G\in\mathfrak{G}}\omega^{\beta+2\gamma}\right\} \tag{7}$$

where T denotes the set of 3^n possible allocations of 1, ω, or ω^2 to the n nodes.

It is readily verified that every triply-Eulerian graph in $\mathfrak{G}$ contributes 3^n to the left-hand side of (7), while any other graph contributes nothing. For allocations in which p, q, and r nodes are allocated 1, ω, and ω^2 respectively (there are

$$\frac{n!}{p!q!r!}$$

such) we consider the contribution to the right-hand side of (7) from those graphs for which α, β, and γ are given ($\alpha + \beta + \gamma = k$). This contribution is found to be

$$\omega^{\beta+2\gamma}\binom{\frac{1}{2}p(p-1)+qr}{\alpha}\binom{\frac{1}{2}r(r-1)+pq}{\beta}\binom{\frac{1}{2}q(q-1)+pr}{\gamma}.$$

Thus the total contribution for given p, q, r is

$$\frac{n!}{p!q!r!}\sum_{\alpha+\beta+\gamma=k}\omega^{\beta+2\gamma}\binom{\frac{1}{2}p(p-1)+qr}{\alpha}\binom{\frac{1}{2}r(r-1)+pq}{\beta}\binom{\frac{1}{2}q(q-1)+pr}{\gamma}$$

which is the coefficient of t^k in

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$$\frac{n!}{p!q!r!}(1+t)^{\frac{1}{2}p(p-1)+qr}(1+\omega t)^{\frac{1}{2}q(q-1)+pr}(1+\omega^2 t)^{\frac{1}{2}r(r-1)+pq}.$$

Summing for $p+q+r=n$ we obtain a series in which the coefficient of t^k is the right-hand side of (7). Hence the number of triply-Eulerian graphs in $\mathfrak{G}$ is the coefficient of t^k in

$$\left(\frac{1}{3}\right)^n \sum_{\substack{p+q+r\\=n}} \frac{n!}{p!q!r!}(1+t)^{\frac{1}{2}p(p-1)+qr}(1+\omega t)^{\frac{1}{2}q(q-1)+pr}(1+\omega^2 t)^{\frac{1}{2}r(r-1)+pq}$$

—a formula which does not appear to simplify very much.

The extension of this method to higher values of m is obvious.

4. Connected graphs. Formula (6) is the counting series for Euler graphs on a given number n of nodes. The counting series for all Euler graphs with any number of nodes and edges will be

$$(8)\qquad E(x,t)=\sum_{n=0}^{\infty}\left\{\left(\frac{1}{2}\right)^n(1+t)^{\frac{1}{2}n(n-1)}\sum_{p=0}^{n}\binom{n}{p}\left(\frac{1-t}{1+t}\right)^{p(n-p)}\right\}\frac{x^n}{n!}$$

the "$n!$" being introduced because the nodes are labelled. It is a well-known result that the counting series for connected graphs corresponding to a counting series such as (8) is given by its formal logarithm. Hence the counting series for "true" Euler graphs (that is, with the condition of connectedness imposed) is the series $\log E(x,t)$, and the number of Euler graphs on n labelled nodes and k edges is $n!$ times the coefficient of $x^n t^k$ in this series.

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