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TRIPLET INVARIANCE AND PARALLEL SUMS

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Abstract

Let R be a semiprime ring with extended centroid C and let $I(x)$ denote the set of all inner inverses of a regular element x in R. Given two regular elements a, b in R, we characterise the existence of some $c \in R$ such that $I(a) + I(b) = I(c)$. Precisely, if $a, b, a + b$ are regular elements of R and a and b are parallel summable with the parallel sum $\mathcal{P}(a, b)$, then $I(a) + I(b) = I(\mathcal{P}(a, b))$. Conversely, if $I(a) + I(b) = I(c)$ for some $c \in R$, then $\mathrm{E}[c]a(a + b)^- b$ is invariant for all $(a + b)^- \in I(a + b)$, where $\mathrm{E}[c]$ is the smallest idempotent in C satisfying $c = \mathrm{E}[c]c$. This extends earlier work of Mitra and Odell for matrix rings over a field and Hartwig for prime regular rings with unity and some recent results proved by Alahmadi *et al.* ['Invariance and parallel sums', *Bull. Math. Sci.* **10**(1) (2020), 2050001, 8 pages] concerning the parallel summability of unital prime rings and abelian regular rings.

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1. Introduction

Throughout, rings are associative, not necessarily with unity. Given elements a, b in a ring R, the elements $(1 - a)b$ and $b(1 - a)$ always mean $b - ab$ and $b - ba$, respectively. An element a in a ring R is called von Neumann regular (or regular for short) if there exists $a^- \in R$ such that $aa^-a = a$. The element a^- is called an inner inverse of a. A ring R is called regular if each element of R is regular. We denote by $\mathrm{Reg}(R)$ the set of all regular elements in the ring R and by $I(a)$ the set of all inner inverses of a in R. Let $a, b \in R$ with $a + b$ regular. Given $(a + b)^- \in I(a + b)$, $a(a + b)^- b$ is called a parallel sum of a and b. If $a(a + b)^- b$ is invariant for all $(a + b)^- \in I(a + b)$, then a and b are called parallel summable. In this case, the common value of $a(a + b)^- b$ is called the parallel sum of a and b and is denoted by $\mathcal{P}(a, b)$.

Parallel sums originally arose in the study of network synthesis. The concept of parallel sum is analogous to the concept of connecting resistors either in series or in parallel, a basic concept in elementary network theory (see [11, Ch. 9]).

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This notion of parallel summability was also introduced by Anderson and Duffin using Moore–Penrose inverses (see [3]) and was extended by Rao and Mitra in a general setting replacing the Moore–Penrose inverse by an inner inverse (see [12]). Mitra and Odell [11] proved the following theorem (see also [10, Theorem 9.2.14]).

THEOREM 1.1. *For matrices a, b in a matrix ring over any field, if a and b are parallel summable, then $I(a) + I(b) = I(\mathcal{P}(a, b))$. Conversely, if a nonzero matrix c satisfies $I(a) + I(b) = I(c)$, then a and b are parallel summable and $c = \mathcal{P}(a, b)$.*

We remark that the converse part of Theorem 1.1 is not true if $c = 0$ (see [10, Remark 9.2.15] or [7, page 194]). A ring R is called semiprime if, for $a \in R$, $aRa = 0$ implies that $a = 0$. When R is a semiprime ring with $I(a) + I(b) = I(c)$, it follows from Theorem 2.8 below that c is uniquely determined. See also [1, Theorem 7] for semiprime rings with unity. A ring R is called a prime ring if, for $a, b \in R$, $aRb = 0$ implies that $a = 0$ or $b = 0$. It is known that any matrix ring over a field is a prime ring. Hartwig generalised Theorem 1.1 to prime regular rings with unity (see [7]). He also asked whether the prime condition on the prime regular ring can be dropped (see [7, page 197]). Note that every regular ring is semiprime.

In a recent paper [2], Alahmadi *et al.* showed the following result for unital prime rings.

THEOREM 1.2 [2, Theorem 10]. *Let a, b, c be regular elements of a prime ring R with unity. Suppose that $ab = ba$ and that one of the following conditions holds:*

(a) *$a, b \in U(R)$, the set of all units of R;*
(b) *$a = u$ and $b = e$, where $u \in U(R)$ and $e = e^2$;*
(c) *$2 \in R$ and a and b are commuting idempotents.*

Then $I(c) = I(a) + I(b)$ if and only if a and b are parallel summable and $c = \mathcal{P}(a, b)$.

Motivated by these results, it is natural to raise the following question.

QUESTION 1.3. Let R be a semiprime ring with elements $a, b, a + b$ regular. Can one characterise the existence of some $c \in R$ such that $I(a) + I(b) = I(c)$ if and only if a and b are parallel summable?

In the paper we answer this question. Precisely, let R be a semiprime ring and let $a, b, a + b$ be regular elements of R. If a and b are parallel summable with the parallel sum $\mathcal{P}(a, b)$, then $I(a) + I(b) = I(\mathcal{P}(a, b))$ (see Theorem 2.5). Conversely, if $I(a) + I(b) = I(c)$ for some element $c \in R$ and if $\mathrm{E}[c] = \mathrm{E}[a]\mathrm{E}[b]$, then a and b are parallel summable and $c = \mathcal{P}(a, b)$ (see Theorem 2.9). Here, given $x \in R$, $\mathrm{E}[x]$ is the smallest idempotent in the extended centroid of R satisfying $x = \mathrm{E}[x]x$ (see the next section for details).

As a consequence, the following result generalises Theorem 1.1, Hartwig's theorem (see [7]) and Theorem 1.2 to the context of prime rings.

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THEOREM 1.4. *Let R be a prime ring and let $a, b, a + b \in \mathrm{Reg}(R)$. If a and b are parallel summable, then $I(a) + I(b) = I(\mathcal{P}(a, b))$. Conversely, if $I(a) + I(b) = I(c)$ for some nonzero $c \in R$, then a and b are parallel summable and $c = \mathcal{P}(a, b)$.*

We remark that the prime ring R in Theorem 1.4 is not in general a regular ring. Our proof is thus different from that given in [7]. A ring R is called abelian if all idempotents of R are central. Clearly, every reduced ring is an abelian semiprime ring but there exists an abelian semisimple ring which is not reduced (see [6, Example 2.12]). Also, R is an abelian regular ring if and only if it is a strongly regular ring, that is, for any $x \in R$, there exists $y \in R$ such that $x = x^2y$ (see [5, Theorem 3.5]).

For abelian semiprime rings we obtain the following characterisation of the parallel summability of two given regular elements.

THEOREM 1.5. *Let R be an abelian semiprime ring and let $a, b, a + b \in \mathrm{Reg}(R)$. If a and b are parallel summable, then $I(a) + I(b) = I(\mathcal{P}(a, b))$. Conversely, if $I(a) + I(b) = I(c)$ for some $c \in R$, then a and b are parallel summable and $c = \mathcal{P}(a, b)$.*

We remark that Alahmadi *et al.* obtained the same conclusion when R is an abelian regular ring with unity and $\frac{1}{2} \in R$ (see [2, Theorem 13]).

2. Results

Let R be a semiprime ring with $Q_{mr}(R)$ the maximal right ring of quotients of R. It is known that $Q_{mr}(R)$ is also a semiprime ring. The centre of $Q_{mr}(R)$, denoted by C, is called the extended centroid of R. It is known that C is a regular self-injective ring and is a field if and only if R is a prime ring.

The set $\mathcal{B}$ of all idempotents of C forms a Boolean algebra with respect to the binary operations $e \dotplus h := e + h - 2eh$ and $e \cdot h := eh$ for $e, h \in \mathcal{B}$. It is complete with respect to the partial order $e \leq h$ (defined by $eh = e$) in the sense that any subset S of $\mathcal{B}$ has a supremum $\vee S$ and an infimum $\wedge S$. Given $a \in Q_{mr}(R)$, it is known that there exists the smallest central idempotent, denoted by $\mathrm{E}[a]$, in $\mathcal{B}$ such that $a = \mathrm{E}[a]a$. Clearly, $\mathcal{B} = \{0, 1\}$ if R is a prime ring. The notion of extended centroids is essential to the study of semiprime rings (see [4]).

Throughout, unless specially stated, R *always denotes a semiprime ring*. We begin with the following well-known result.

LEMMA 2.1. *Given $a, b \in Q_{mr}(R)$, we have $aRb = 0$ if and only if $\mathrm{E}[a]\mathrm{E}[b] = 0$ if and only if $a\mathrm{E}[b] = 0$ if and only if $\mathrm{E}[a]b = 0$.*

The following is also well known (see, for instance, [1, Lemma 3]).

LEMMA 2.2. *Let R be an arbitrary ring with $a \in \mathrm{Reg}(R)$. Given a fixed $a^- \in I(a)$,*

$$I(a) = \{a^- + (1 - a^-a)x + y(1 - aa^-) \mid x, y \in R\}.$$

Let $b, c \in Q_{mr}(R)$ and $a \in \mathrm{Reg}(R)$. We say that the triplet ba^-c is invariant for all $a^- \in I(a)$ if there exists $z \in Q_{mr}(R)$ such that $ba^-c = z$ for all $a^- \in I(a)$, that is, $bI(a)c = \{z\}$.

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THEOREM 2.3 [9, Theorem 18]. *Let R be a semiprime ring and let $a, b, c \in R$ with $a \in \mathrm{Reg}(R)$. Then the triplet ba^-c is invariant for all $a^- \in I(a)$ if and only if $\mathrm{E}[c]b = xa$ and $\mathrm{E}[b]c = ay$ for some $x \in \mathrm{E}[c]R$ and $y \in \mathrm{E}[b]R$.*

The following result will play a key role in the proof of Theorem 2.5 below.

THEOREM 2.4. *Let R be a semiprime ring and let $a, b \in R$ with $a + b \in \mathrm{Reg}(R)$. Then the following are equivalent:*

(i) *a and b are parallel summable;*
(ii) *$a, b \in R(a + b) \cap (a + b)R$;*
(iii) *b and a are parallel summable.*

In this case, we have $\mathcal{P}(a, b) = \mathcal{P}(b, a)$.

PROOF. (i) $\Rightarrow$ (ii). Since $a(a + b)^-b$ is invariant for all $(a + b)^- \in I(a + b)$, it follows from Theorem 2.3 that there exist $x \in \mathrm{E}[b]R$ and $y \in \mathrm{E}[a]R$ such that

$$\mathrm{E}[b]a = x(a + b) \quad \text{and} \quad \mathrm{E}[a]b = (a + b)y. \tag{2.1}$$

By (2.1),

$$\mathrm{E}[b]a = \mathrm{E}[b]a(a + b)^-(a + b) \quad \text{and} \quad \mathrm{E}[a]b = (a + b)(a + b)^-\mathrm{E}[a]b. \tag{2.2}$$

We compute

$$a + b = (a + b)(a + b)^-(a + b) = a(a + b)^-(a + b) + b(a + b)^-(a + b). \tag{2.3}$$

Multiplying (2.3) by $\mathrm{E}[b]$ and applying the first equality of (2.2),

$$b = b(a + b)^-(a + b) \in R(a + b). \tag{2.4}$$

It follows from (2.3) and (2.4) that $a = a(a + b)^-(a + b) \in R(a + b)$. We next consider $a + b = (a + b)(a + b)^-a + (a + b)(a + b)^-b$. Applying the same argument as above gives $a, b \in (a + b)R$.

(ii) $\Rightarrow$ (i). Since $a, b \in R(a + b) \cap (a + b)R$, there exist $x, y \in R$ such that $a = x(a + b)$ and $b = (a + b)y$. Therefore, for $(a + b)^- \in I(a + b)$,

$$a(a + b)^-b = x(a + b)(a + b)^-(a + b)y = x(a + b)y,$$

implying that a and b are parallel summable.

By symmetry, (ii) $\Leftrightarrow$ (iii).

Finally, suppose that (i), (ii) and (iii) hold. Then, for $(a + b)^- \in I(a + b)$,

$$\begin{aligned}
\mathcal{P}(b, a) &= b(a + b)^-a \\
&= (a + b)(a + b)^-a - a(a + b)^-a \\
&= a - a(a + b)^-a && \text{(by (ii), } a \in (a + b)R) \\
&= a(a + b)^-(a + b) - a(a + b)^-a && \text{(by (ii), } a \in R(a + b)) \\
&= a(a + b)^-b = \mathcal{P}(a, b).
\end{aligned}$$

□

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For an element w in a ring R, we denote by $r_R(w)$ and $l_R(w)$ respectively the right and left annihilators of w in R. We are now ready to prove our first main theorem.

THEOREM 2.5. *Let R be a semiprime ring and let $a, b, a+b$ be regular elements in R. If a and b are parallel summable, then $I(a) + I(b) = I(\mathcal{P}(a, b))$.*

PROOF. Suppose that a and b are parallel summable. Let $c := \mathcal{P}(a, b)$. From Theorem 2.4, $c = a(a+b)^- b = b(a+b)^- a$ for all $(a+b)^- \in I(a+b)$.

Step 1: $I(a) + I(b) \subseteq I(\mathcal{P}(a, b))$. In view of Theorem 2.4(ii), $c \in Ra \subseteq R(a+b)$ and so $c = c(a+b)^-(a+b)$. Therefore, for $a^- \in I(a)$ and $b^- \in I(b)$,

$$\begin{aligned} c(a^- + b^-)c &= ca^-c + cb^-c \\ &= (b(a+b)^-a)a^-(a(a+b)^-b) + (a(a+b)^-b)b^-(b(a+b)^-a) \\ &= b(a+b)^-a(a+b)^-b + a(a+b)^-b(a+b)^-a \\ &= c(a+b)^-b + c(a+b)^-a \\ &= c(a+b)^-(a+b) = c. \end{aligned}$$

Therefore, $a^- + b^- \in I(c)$. This proves that $I(a) + I(b) \subseteq I(\mathcal{P}(a, b))$.

Step 2: $r_R(c) \subseteq r_R(a) + r_R(b)$ and $l_R(c) \subseteq l_R(a) + l_R(b)$. We only give the proof of the first inclusion. The other one has a similar argument. In view of Theorem 2.4(ii), $b \in (a+b)R$. Therefore, $b = (a+b)(a+b)^-b$ for all $(a+b)^- \in I(a+b)$.

Let $z \in r_R(c)$ and let $(a+b)^- \in I(a+b)$. Then $a(a+b)^-bz = cz = 0$. Moreover,

$$bz = (a+b)(a+b)^-bz = a(a+b)^-bz + b(a+b)^-bz = b(a+b)^-bz,$$

implying that $z - (a+b)^-bz \in r_R(b)$. Hence, $z \in r_R(a) + r_R(b)$ since $(a+b)^-bz \in r_R(a)$. So, $r_R(c) \subseteq r_R(a) + r_R(b)$, as desired.

Step 3: $I(c) \subseteq I(a) + I(b)$. Fix $a^- \in I(a)$ and $b^- \in I(b)$. By Step 1, $c^- = a^- + b^- \in I(c)$. Let $w \in I(c)$. It follows from Lemma 2.2 that there exist $x, y \in R$ such that

$$w = a^- + b^- + (1 - c^-c)x + y(1 - cc^-).$$

Note that $(1 - c^-c)x \in r_R(c)$ and $y(1 - cc^-) \in l_R(c)$. Since $r_R(c) \subseteq r_R(a) + r_R(b)$ and $l_R(c) \subseteq l_R(a) + l_R(b)$, it follows from Step 2 that

$$(1 - c^-c)x = r_1 + r_2 \quad \text{and} \quad y(1 - cc^-) = t_1 + t_2,$$

where $r_1 \in r_R(a)$, $r_2 \in r_R(b)$, $t_1 \in l_R(a)$ and $t_2 \in l_R(b)$. Therefore,

$$w = (a^- + r_1 + t_1) + (b^- + r_2 + t_2).$$

Clearly, $a^- + r_1 + t_1 \in I(a)$ and $b^- + r_2 + t_2 \in I(b)$. Therefore, $w \in I(a) + I(b)$. This proves that $I(c) \subseteq I(a) + I(b)$.

By Steps 1 and 3, $I(a) + I(b) = I(c)$. □

LEMMA 2.6. *Let $a, b, c \in \mathrm{Reg}(R)$. If $I(a) + I(b) \subseteq I(c)$, then*

$$c = ca^-a = aa^-c = cb^-b = bb^-c$$

for all $a^- \in I(a)$ and $b^- \in I(b)$.

PROOF. Suppose that $I(a) + I(b) \subseteq I(c)$. Let $a^- \in I(a)$ and $b^- \in I(b)$. By assumption,

$$c(a^- + b^-)c = c. \tag{2.5}$$

Replacing a^- by $a^- + (1 - a^-a)x + y(1 - aa^-) \in I(a)$ in (2.5),

$$c(a^- + (1 - a^-a)x + y(1 - aa^-) + b^-)c = c \tag{2.6}$$

for all $x, y \in R$. It follows from (2.5) and (2.6) that $c((1 - a^-a)x + y(1 - aa^-))c = 0$ for all $x, y \in R$. Therefore, $c(1 - a^-a)Rc = 0$ and $cR(1 - aa^-)c = 0$. In view of Lemma 2.1, $c = ca^-a = aa^-c$. Similarly, $c = cb^-b = bb^-c$. □

As noted in [10, Remark 9.2.15] or [7, page 194], there exist a, b in a matrix ring over a field such that $I(a) + I(b) = I(0)$ but $a(a + b)^-b$ is not invariant under all $(a + b)^- \in I(a + b)$. We are now ready to prove the second main theorem in the paper.

THEOREM 2.7. *Let R be a semiprime ring and let $a, b, a + b$ be regular elements in R. If $I(a) + I(b) = I(c)$ for some $c \in R$, then $\mathrm{E}[c]a(a + b)^-b$ is invariant for all $(a + b)^- \in I(a + b)$.*

PROOF. Suppose that $I(a) + I(b) = I(c)$ for some $c \in R$. We claim that

$$cc^-a \in R(a + b) \tag{2.7}$$

for all $c^- \in I(c)$. Let $c^- \in I(c)$. In view of Lemma 2.6,

$$c = ca^-a = aa^-c = cb^-b = bb^-c \tag{2.8}$$

for all $a^- \in I(a)$ and $b^- \in I(b)$. Given $c^- \in I(c)$, we have $c^- = a^- + b^-$ for some $a^- \in I(a)$ and $b^- \in I(b)$. Applying (2.8),

$$cc^-a = ca^-a + cb^-a = c + cb^-a = cb^-b + cb^-a = cb^-(a + b) \in R(a + b).$$

Since c^- in (2.7) is arbitrary in $I(c)$, replacing c^- by $c^- + x(1 - cc^-) \in I(c)$ in (2.7),

$$c(c^- + x(1 - cc^-))a \in R(a + b) \tag{2.9}$$

for all $c^- \in I(c)$ and $x \in R$. Since $cc^-a, cxcc^-a \in R(a + b)$, it follows from (2.9) that $cxa \in R(a + b)$ for all $x \in R$. This means that

$$cRa(1 - (a + b)^-(a + b)) = 0.$$

for all $(a + b)^- \in I(a + b)$. Hence, by Lemma 2.1, $\mathrm{E}[c]a(1 - (a + b)^-(a + b)) = 0$ for all $(a + b)^- \in I(a + b)$. Thus, $\mathrm{E}[c]a \in R\mathrm{E}[c](a + b)$. An analogous argument proves that $\mathrm{E}[c]b \in (a + b)\mathrm{E}[c]R$.

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There exist $u, v \in R\mathrm{E}[c]$ such that $\mathrm{E}[c]a = u(a+b)$ and $\mathrm{E}[c]b = (a+b)v$. Therefore,

$$\mathrm{E}[c]a(a+b)^{-}b = \mathrm{E}[c]a(a+b)^{-}\mathrm{E}[c]b = u(a+b)(a+b)^{-}(a+b)v = u(a+b)v \quad (2.10)$$

for all $(a+b)^{-} \in I(a+b)$, as desired. □

We remark that, in Theorem 2.7, from $I(a) + I(b) = I(c)$ with $c \in \mathrm{Reg}(R)$, we cannot conclude that a and b are parallel summable even if $c \neq 0$. For instance, let $R := \mathrm{M}_2(F) \oplus \mathrm{M}_2(F)$, where F is a field. Let $a := (a_1, a_2)$ and $b := (b_1, b_2) \in R$, where

$$a_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad b_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $c_1 = 0$, $c_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $c := (c_1, c_2)$. Then we have $I(a_1) + I(b_1) = \mathrm{M}_2(F) = I(c_1)$ and $I(a_2) + I(b_2) = I(c_2)$. This implies that $I(a) + I(b) = I(c)$. Note that a_1 and b_1 are not parallel summable since $a_1 \notin R(a_1 + b_1)$ (see [10, Remark 9.2.15]). Therefore, a and b are not parallel summable. Note that $\mathrm{E}[c] = (0, 1)$, $\mathrm{E}[a] = (1, 1)$ and $\mathrm{E}[b] = (1, 1)$. Hence, $\mathrm{E}[c] \neq \mathrm{E}[a]\mathrm{E}[b]$.

The following theorem was first proved by Alahmadi *et al.* for rings with unity (see [1, Theorem 7]).

THEOREM 2.8 [8, Corollary 2.2]. *Let R be a semiprime ring and $a, b \in \mathrm{Reg}(R)$. If $I(a) = I(b)$, then $a = b$.*

THEOREM 2.9. *Let R be a semiprime ring and let $a, b, a+b$ be regular elements in R. If $I(a) + I(b) = I(c)$ for some $c \in R$ and if $\mathrm{E}[c] = \mathrm{E}[a]\mathrm{E}[b]$, then a and b are parallel summable and $c = \mathcal{P}(a, b)$.*

PROOF. Suppose that $I(a) + I(b) = I(c)$ and $\mathrm{E}[c] = \mathrm{E}[a]\mathrm{E}[b]$ for some $c \in R$. In view of Theorem 2.7, $\mathrm{E}[c]a(a+b)^{-}b$ is invariant for all $(a+b)^{-} \in I(a+b)$. Then

$$a(a+b)^{-}b = \mathrm{E}[a]a(a+b)^{-}\mathrm{E}[b]b = \mathrm{E}[a]\mathrm{E}[b]a(a+b)^{-}b = \mathrm{E}[c]a(a+b)^{-}b$$

for all $(a+b)^{-} \in I(a+b)$. This proves that a and b are parallel summable. In view of Theorem 2.5, $I(a) + I(b) = I(\mathcal{P}(a, b))$. Therefore, $I(c) = I(\mathcal{P}(a, b))$. In view of Theorem 2.8, $c = \mathcal{P}(a, b)$, as desired. □

The converse of Theorem 2.9 is not true in general. Indeed, let $R := \mathrm{M}_2(F) \oplus \mathrm{M}_2(F)$, where F is a field. Let $a := (a_1, a_2)$ and $b := (b_1, b_2) \in R$, where

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad b_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $c_1 = 0$, $c_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $c := (c_1, c_2)$. Then a_i and b_i are parallel summable for $i = 1, 2$. Hence, a and b are parallel summable with $\mathcal{P}(a, b) = c$. Clearly, $\mathrm{E}[a] = (1, 1)$, $\mathrm{E}[b] = (1, 1)$ and $\mathrm{E}[c] = (0, 1)$. Therefore, $\mathrm{E}[c] \neq \mathrm{E}[a]\mathrm{E}[b]$.

We are now ready to prove Theorem 1.4.

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PROOF OF THEOREM 1.4. Since R is a prime ring, we recall that $\mathcal{B} = \{0, 1\}$. Therefore, $E[x] = 1$ for $x \in R \setminus \{0\}$. Suppose that a and b are parallel summable. In view of Theorem 2.5, $I(a) + I(b) = I(\mathcal{P}(a, b))$. Conversely, suppose that $I(a) + I(b) = I(c)$ with c nonzero. By Lemma 2.6, $c = ca^-a = aa^-c = cb^-b = bb^-c$ for all $a^- \in I(a)$ and $b^- \in I(b)$. In particular, neither a nor b is zero. Hence, $E[a] = E[b] = E[c] = 1$. It follows from Theorem 2.7 that $aI(a + b)b = \{c\}$. □

Before giving the proof of Theorem 1.5, we need the following observation.

LEMMA 2.10. *Let R be an abelian semiprime ring and suppose that $y \in \mathrm{Reg}(R)$. Then $E[y] = yy^- = y^-y$ for any $y^- \in I(y)$.*

PROOF. Since $y \in \mathrm{Reg}(R)$, $yy^-y = y$ for any $y^- \in I(y)$. Since R is abelian, $yy^- \in \mathcal{B}$ and hence $yy^- \geq E[y]$. Also, $E[y]yy^- = (E[y]y)y^- = yy^-$, implying that $yy^- \leq E[y]$. Therefore, $E[y] = yy^-$. Similarly, we have $E[y] = y^-y$. □

PROOF OF THEOREM 1.5. Suppose first that a and b are parallel summable. In view of Theorem 2.5, $I(a) + I(b) = I(\mathcal{P}(a, b))$, as desired.

Conversely, assume that $I(a) + I(b) = I(c)$ for some $c \in R$. We claim that $E[c] = E[a]E[b]$. Fix $a^- \in I(a)$ and $b^- \in I(b)$. Then $c^- := a^- + b^- \in I(c)$ since $I(a) + I(b) = I(c)$. Let $x \in R$. Clearly, $c^- + (1 - c^-c)x \in I(c)$. In view of Lemma 2.2, there exist $u, v, w, z \in R$ such that

$$c^- + (1 - c^-c)x = a^- + (1 - a^-a)u + v(1 - aa^-) + b^- + (1 - b^-b)w + z(1 - bb^-). \tag{2.11}$$

Since R is an abelian semiprime ring, by Lemma 2.10, $E[a] = aa^- = a^-a$, $E[b] = bb^- = b^-b$ and $E[c] = c^-c$. We rewrite (2.11) as

$$(1 - E[c])x = (1 - E[a])u + v(1 - E[a]) + (1 - E[b])w + z(1 - E[b]). \tag{2.12}$$

Multiplying (2.12) by $E[a]E[b]$, we get $E[a]E[b](1 - E[c])x = 0$. Since $x \in R$ is arbitrary, we have $E[a]E[b](1 - E[c])R = 0$, implying that $E[a]E[b] = E[a]E[b]E[c]$. That is, $E[a]E[b] \leq E[c]$.

On the other hand, since $I(a) + I(b) \subseteq I(c)$, we have $c = ca^-a = cb^-b$ for $a^- \in I(a)$ and $b^- \in I(b)$ (see Lemma 2.6). In particular, it follows that $c = E[a]c$ and $c = E[b]c$. Hence, $E[c] \leq E[a]E[b]$. So, $E[c] = E[a]E[b]$, as claimed. In view of Theorem 2.9, a and b are parallel summable and $c = \mathcal{P}(a, b)$. □

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