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FAST DIFFUSION WITH LOSS AT INFINITY

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Abstract

We study the equation

$$\frac{\partial\theta}{\partial t} = r^{1-s}\frac{\partial}{\partial r}\left(r^{s-1}\theta^m\frac{\partial\theta}{\partial r}\right) \quad \text{in } 0 \le r \le \infty \quad \text{with } \theta \ge 0,\ s > 0. \qquad \text{(A)}$$

Here s is not necessarily integral; m is initially unrestricted. Material-conserving instantaneous source solutions of A are reviewed as an entrée to material-losing solutions. Simple physical arguments show that solutions for a finite slug losing material at infinity at a finite nonzero rate can exist only for the following m-ranges: $0 < s < 2$, $-2s^{-1} < m \le -1$; $s > 2$, $-1 < m < -2s^{-1}$. The result for $s = 1$ was known previously. The case $s = 2$, $m = -1$, needs further investigation. Three different similarity schemes all lead to the same ordinary differential equation. For $0 < s < 2$, parameter γ ($0 < \gamma < \infty$) in that equation discriminates between the three classes of solution: class 1 gives the concentration scale decreasing as a negative power of $(1 + t/T)$; 2 gives exponential decrease; and 3 gives decrease as a positive power of $(1 - t/T)$, the solution vanishing at $t = T < \infty$. Solutions for $s = 1$, $m = -\frac{3}{2}$ are presented graphically. The variation of concentration and flux profiles with increasing γ is physically explicable in terms of increasing flux at infinity. An indefinitely large number of exact solutions are found for $s = 1$, $\gamma = 1$. These demonstrate the systematic variation of solution properties as m decreases from -1 toward -2 at fixed γ.

1. Formulating the problems

We are concerned with the nonlinear diffusion equation

$$\frac{\partial\theta}{\partial t} = r^{1-s}\frac{\partial}{\partial r}\left(r^{s-1}\theta^m\frac{\partial\theta}{\partial r}\right) \tag{1.1}$$

in $0 \le r \le \infty$. Here θ (≥ 0) is concentration, t is time, r the radial space coordinate, s (> 0) is the number of space dimensions, 1, 2, or 3 in physical problems. We shall,

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initially, treat the index m as unrestricted: it may be positive, zero, or negative. The solutions examined here are, in fact, for $m < 0$, corresponding to fast diffusion.

We seek instantaneous source solutions describing the diffusion of a *finite* slug of material released at $t = 0$. The radial symmetry requires that θ be an even function of r, and that at $r = 0$, $\partial\theta/\partial r = 0$ and there is no flux of material. We consider two types of source: material-conserving sources, and sources which lose material at infinity. The first of these is well-known (for example, [3]), but we review it briefly here as an entrée into what follows.

1.1. Material-conserving sources This type concerns instantaneous releases at $(r, t) = (0, 0)$, with solutions of (1.1) satisfying initial condition (1.2):

$$t = 0, \qquad 0 \le r \le \infty, \qquad \theta = Q\delta(r), \tag{1.2}$$

with $\delta(\,)$ defined by a limiting process to be determined. Here Q is the *instantaneous source strength*, and for $t \ge 0$, (1.1) satisfies the condition

$$0 < \frac{2\pi^{s/2}}{\Gamma(s/2)} \int_0^\infty r^{s-1}\,\theta(r,t)\,dr = \text{constant} = Q < \infty. \tag{1.3}$$

1.2. Sources losing material at infinity For this type we explore solutions of (1.1) satisfying initial condition (1.4):

$$t = 0, \qquad 0 \le r \le \infty, \qquad \theta = \Theta(r), \tag{1.4}$$

with $\Theta(r)$ a nonincreasing function of r to be determined, and such that $\int_0^\infty r^{s-1}\,\Theta(r)\,dr$ is positive and finite. Here for $t \ge 0$ (1.1) satisfies the relation

$$0 < \frac{2\pi^{s/2}}{\Gamma(s/2)} \int_0^\infty r^{s-1}\,\theta(r,t)\,dr = q(t) < \infty, \tag{1.5}$$

with $q(t)$ a nonincreasing function of t. The *instantaneous source strength* here is $q(0)$; and the rate of loss of material is $-q'(t)$, the prime signifying differentiation with respect to t.

2. Source solutions with material conservation

Similarity solutions of (1.1) subject to (1.2) of the form

$$\theta = \Theta(\rho)\,t^{-\alpha}, \qquad \rho = rt^{-\alpha/s}, \qquad \alpha > 0, \tag{2.1}$$

ensure material conservation, and hence satisfy (1.3). Putting (2.1) into (1.1) shows that similarity requires

$$\alpha = s/(sm + 2), \tag{2.2}$$

and the condition $\alpha > 0$ restricts m to the range

$$m > -2s^{-1}. \tag{2.3}$$

The relevant solutions are then:

$$m > 0, \qquad \Theta = \Theta_0 \left[1 - \frac{m\rho^2}{2(sm+2)\,\Theta_0^m}\right]^{1/m}; \tag{2.4}$$

$$m = 0, \qquad \Theta = \Theta_0 \exp\left[-\frac{\rho^2}{4}\right]; \tag{2.5}$$

$$-2s^{-1} < m < 0, \qquad \Theta = \Theta_0 \left[1 + \frac{(-m)\,\rho^2}{2(sm+2)\,\Theta_0^m}\right]^{1/m}. \tag{2.6}$$

Here Θ_0 is the value of Θ at $\rho = 0$. Equation (2.4) gives

$$\Theta = 0, \qquad \rho = \left[\frac{2(sm+2)\,\Theta_0^m}{m}\right]^{1/2} < \infty \tag{2.7}$$

whereas (2.5) and (2.6) yield

$$\Theta \to 0 \quad \text{as} \quad \rho \to \infty. \tag{2.8}$$

Now (2.4) represents the source solutions of Barenblatt [1] and Pattle [4]; (2.5) is the well-known linear solution; and (2.6) extends the solutions to the negative m-range $-2s^{-1} < m < 0$. In this range there are infinite tails to the concentration profiles for all $t > 0$.

Evaluating Q completes these solutions. Combining (1.3), (2.1), (2.2) and (2.4) and integrating gives

$$m > 0, \qquad Q = \frac{\Gamma(m^{-1}+1)}{\Gamma\left(\frac{1}{2}s + m^{-1} + 1\right)} \left[\frac{2\pi(sm+2)}{m}\right]^{s/2} \Theta_0^{(sm+2)/2}. \tag{2.9}$$

Similarly, (2.6) yields

$$-2s^{-1} < m < 0, \qquad Q = \frac{\Gamma\left(-m^{-1} - \frac{1}{2}s\right)}{\Gamma(-m^{-1})} \left[-\frac{2\pi(sm+2)}{m}\right]^{s/2} \Theta_0^{(sm+2)/2}. \tag{2.10}$$

Note that Θ/Θ_0 is a function of $\rho[(sm+2)\,\Theta_0^m]^{-1/2}$. Figure 1 shows solutions in this form for $m = 2, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1, -\frac{3}{2}$. Figure 2 shows the dependence on m of $Q\Theta_0^{-(sm+2)/2}$ for $s = 1, 2, 3$. As m increases, this quantity decreases for $s = 1$, is constant ($= 4\pi$) for $s = 2$, and increases for $s = 3$.

Other material-conserving solutions exist in the range $m \leq -2s^{-1}$. We exclude them here, since they give infinite slugs.

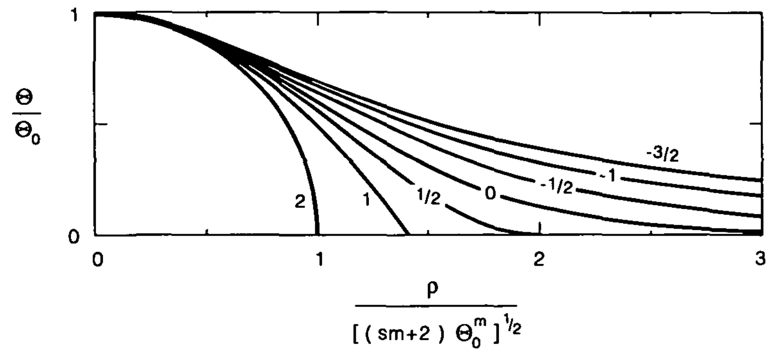

FIGURE 1. The material-conserving instantaneous source solutions in the reduced form Θ/Θ_0 as a function of $\rho\left[(sm+2)\,\Theta_0^m\right]^{-1/2}$. Numerals on the curves are values of m.

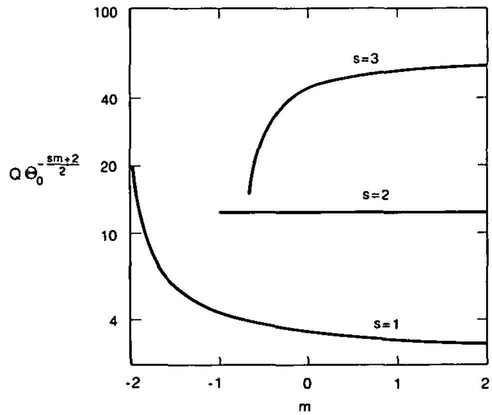

FIGURE 2. Material-conserving instantaneous sources. Slug magnitude in the reduced form $Q\,\Theta_0^{(sm+2)/2}$ as a function of m for the indicated values of s.

3. Source solutions with material loss at infinity: physical constraints

For material-conserving sources the physical requirement that the slug be finite limits m to the range $m > -2s^{-1}$. For sources losing material at infinity, on the other hand, two physical constraints operate. Not only must the slug be finite, but we require also that, at least in some finite nonzero interval in t, the total material flux at infinity is finite and positive.

3.1. Finite total flux at infinity The flux condition implies that, at least for any fixed t in the relevant nonzero t-interval,

$$\lim_{r\to\infty} r^{s-1}\,\theta^m\,\frac{\partial\theta}{\partial r} = -A, \tag{3.1}$$

with A a finite real positive function of t. We require also that

$$\lim_{r\to\infty} \theta = 0. \tag{3.2}$$

We must integrate separately the following four cases.

(a) $s \neq 2$, $m \neq -1$.

$$\text{At large } r, \qquad \theta \approx \left[-\frac{A(m+1)}{2-s}\right]^{1/(m+1)} r^{(2-s)/(m+1)}. \tag{3.3}$$

This case satisfies (3.2) only if $(2-s)/(m+1)$ is negative, that is,

$$\text{either} \quad s < 2, \quad m < -1 \quad \text{or} \quad s > 2, \quad m > -1. \tag{3.4}$$

(b) $s = 2$, $m \neq -1$.

$$\text{At large } r, \qquad \theta \approx [-A(m+1)\ \ln r]^{1/(m+1)}. \tag{3.5}$$

This satisfies (3.2) only if $m+1$ is negative, that is,

$$m < -1. \tag{3.6}$$

(c) $s \neq 2$, $m = -1$.

$$\text{At large } r, \qquad \theta \approx \exp\left[-\frac{A}{2-s} r^{2-s}\right]. \tag{3.7}$$

This satisfies (3.2) only if $2-s$ is positive, that is,

$$0 < s < 2. \tag{3.8}$$

(d) $s = 2$, $m = -1$.

$$\text{At large } r, \qquad \theta \approx r^{-A}, \tag{3.9}$$

which satisfies (3.2). We thus obtain the conditions on m:

$$\begin{aligned} 0 < s \leq 2, &\qquad m \leq -1; \\ s > 2, &\qquad m > -1. \end{aligned} \tag{3.10}$$

3.2. Finite slug The finite slug condition requires that $\int^{\infty} r^{s-1}\,\theta\, dr$ be finite. We examine the four cases separately.

(a) $s \neq 2$, $m \neq -1$. It follows from (3.3) that $\int^{\infty} r^{s-1}\,\theta\, dr$ is finite provided either

$$0 < s < 2, \quad -2s^{-1} < m < -1 \qquad \text{or} \qquad s > 2, \quad -1 < m < -2s^{-1}. \tag{3.11}$$

(b) $s = 2$, $m \neq -1$. It follows from (3.5) that in this case the integral is divergent and there is no physically acceptable solution.

(c) $s \neq 2$, $m = -1$. For (3.7) the integral is convergent for

$$0 < s < 2. \tag{3.12}$$

(d) $s = 2$, $m = -1$. For (3.9) the integral is convergent only if

$$A > 2. \tag{3.13}$$

Consolidating these results, we find that both physical constraints are satisfied, provided either

$$0 < s < 2, \quad -2s^{-1} < m \leq -1 \qquad \text{or} \qquad s > 2, \quad -1 < m < -2s^{-1}. \tag{3.14}$$

In the singular case $s = 2, m = -1$, convergence of the integral would require $A > 2$ and specific cases need further investigation. All that can be said here is that any acceptable solution would require to be limited to some finite nonzero interval in t with the flux at infinity $> 4\pi$.

We go on to develop three classes of similarity solutions of (1.1) giving material loss at infinity and satisfying the physical constraints embodied in (3.14).

4. Source solutions with material loss at infinity: class 1

We seek similarity solutions of (1.1) subject to (1.4) and (1.5) of the form

$$\begin{gathered} \theta = \Theta(\rho)(1 + t/T)^{-\alpha}, \qquad \rho = rT^{-1/2}\,(1 + t/T)^{-(\alpha-\beta)/s}, \\ 0 < T < \infty, \qquad 0 < \alpha < \infty, \qquad 0 < \beta < \infty. \end{gathered} \tag{4.1}$$

Equations (4.1) give

$$q(t) = q(0)(1 + t/T)^{-\beta}, \tag{4.2}$$

material being lost at infinity at the rate

$$-q'(t) = \frac{\beta q(0)}{T}\,(1 + t/T)^{-(1+\beta)}. \tag{4.3}$$

It follows from (1.5) and (4.1) that

$$q(0) = \frac{(\pi T)^{s/2}}{\Gamma(s/2)} \int_0^\infty \rho^{s-1}\,\Theta(\rho)\,d\rho. \tag{4.4}$$

Putting (4.1) into (1.1), we find that similarity requires

$$-\alpha - 1 = -\alpha(1 + m) - 2(\alpha - \beta)\,s^{-1},$$

that is,

$$\alpha = (2\beta + s)/(sm + 2). \tag{4.5}$$

The conditions $0 < \alpha < \infty$, $0 < \beta < \infty$, limit the range of m to

$$m > -2s^{-1}. \tag{4.6}$$

The constraint of (3.14) thus restricts permissible solutions to the ranges

$$0 < s < 2, \qquad -2s^{-1} < m \leq -1. \tag{4.7}$$

Problems with integral s are thus limited to those for which

$$s = 1, \qquad -2 < m \leq -1. \tag{4.8}$$

Acceptable solutions of this class do not exist for $s = 2$ or 3.

Substituting (4.1) in (1.1) gives the ordinary differential equation

$$-\alpha \frac{d}{d\rho} (\rho^s \Theta) + \beta \rho^s \frac{d\Theta}{d\rho} = s \frac{d}{d\rho} \left(\rho^{s-1} \Theta^m \frac{d\Theta}{d\rho} \right), \tag{4.9}$$

subject to conditions

$$\rho = 0, \qquad \Theta = \Theta_0, \qquad d\Theta/d\rho = 0. \tag{4.10}$$

We return to the solution of (4.9) subject to (4.10) in Section 7 below.

5. Source solutions with material loss at infinity: class 2

We next explore similarity solutions of (1.1) subject to (1.4) and (1.5) of the form

$$\begin{gathered} \theta = \Theta(\rho)\, e^{-\alpha t/T}, \qquad \rho = r T^{-1/2} e^{(\beta-\alpha)t/(sT)} \\ 0 < T < \infty, \qquad 0 < \alpha < \infty, \qquad 0 < \beta < \infty. \end{gathered} \tag{5.1}$$

Equations (5.1) give

$$q(t) = q(0)\, e^{-\beta t/T}, \tag{5.2}$$

$$-q'(t) = \frac{\beta q(0)}{T} e^{-\beta t/T}. \tag{5.3}$$

It follows from (1.5) and (5.1) that (4.4) holds here also. Putting (5.1) in (1.1) indicates that similarity requires

$$-\alpha = -\alpha(1 + m) - 2(\alpha - \beta)\, s^{-1},$$

that is,

$$\alpha = 2\beta/(sm+2). \tag{5.4}$$

Here also the conditions $0 < \alpha < \infty$, $0 < \beta < \infty$, combined with (3.4), limit permissible solutions to the ranges of (4.7), and solutions with integral s are restricted to (4.8). Again there are no acceptable solutions for $s = 2$ or 3.

Substituting (5.1) in (1.1) gives (4.9) here also; and again it is subject to (4.10).

6. Source solutions with material loss at infinity: class 3

Finally, we consider similarity solutions of (1.1) subject to (1.4) and (1.5) of the form

$$\begin{gathered}\theta = \Theta(\rho)(1-t/T)^{\alpha}, \qquad \rho = rT^{-1/2}(1-t/T)^{(\alpha-\beta)/s}, \\ 0 < T < \infty, \qquad 0 < \alpha < \infty, \qquad 0 < \beta < \infty.\end{gathered} \tag{6.1}$$

Equations (6.1) give for $0 \le t \le T$,

$$q(t) = q(0)(1-t/T)^{\beta} \qquad -q'(t) = \frac{\beta q(0)}{T}(1-t/T)^{\beta-1}. \tag{6.2}$$

and for $t > T$,

$$q(t) = 0 \qquad \text{and} \qquad -q'(t) = 0. \tag{6.3}$$

Putting (6.1) in (1.5), we see that (4.4) holds again here.

Equations (6.1) give a diffusion process ceasing at $t = T$, at which time all material has been lost at infinity. There is nothing left to happen for $t > T$.

Putting (6.1) in (1.1), we find that similarity requires

$$\alpha - 1 = \alpha(1+m) + 2(\alpha-\beta)s^{-1},$$

that is,

$$\alpha = (2\beta - s)/(sm+2). \tag{6.4}$$

The conditions $0 < \alpha < \infty$, $0 < \beta < \infty$, require that either

$$0 < \beta < s/2, \quad m < -2s^{-1} \qquad \text{or} \qquad s/2 < \beta < \infty, \quad m > -2s^{-1}. \tag{6.5}$$

Combining (6.5) with (3.14), we find that physically acceptable solutions are restricted to the ranges

$$\begin{array}{rcccccccl} 0 < s < 2, & & s/2 < \beta < \infty, & & -2s^{-1} < m \le 1; \\ s > 2, & & 0 < \beta < s/2, & & -1 < m < -2s^{-1}. \end{array} \tag{6.6}$$

We thus find that for $s = 1, 2$, or 3, acceptable solutions are for the ranges

$$\begin{array}{lll} s = 1, & \tfrac{1}{2} < \beta < \infty, & -2 < m \le -1; \\ s = 3, & 0 < \beta < \tfrac{3}{2}, & -1 < m < -\tfrac{2}{3}. \end{array} \tag{6.7}$$

For this class acceptable solutions may exist not only for $s = 1$, but also for $s = 3$.

Substituting (6.1) in (1.1) once again gives (4.9), and here also it is subject to (4.10).

7. The solution of (4.9), (4.10)

We have found the useful result that the various similarity forms giving material loss at infinity, (4.1), (5.1) and (6.1) all lead to the same ordinary differential equation (4.9) subject to (4.10). We reduce the number of parameters in (4.9), (4.10) to 3 (γ, m, and s) through the substitutions

$$\vartheta = \Theta/\Theta_0, \quad \gamma = \beta/\alpha, \quad \sigma = \rho\,\alpha^{1/2}\,\Theta_0^{-m/2}. \tag{7.1}$$

In terms of these quantities, (4.9), (4.10) become

$$-\frac{d}{d\sigma}(\sigma^s\vartheta) + \gamma\sigma^s\frac{d\vartheta}{d\sigma} = s\frac{d}{d\sigma}\left(\sigma^{s-1}\vartheta^m\frac{d\vartheta}{d\sigma}\right), \tag{7.2}$$

$$\sigma = 0, \qquad \vartheta = 1, \qquad d\vartheta/d\sigma = 0. \tag{7.3}$$

It is convenient to introduce reduced forms of the instantaneous source strength (that is, slug magnitude) and the flux at infinity, expressible in terms of ϑ and σ.

7.1. Reduced source strength The reduced instantaneous source strength V is defined by

$$V = \frac{2\pi^{s/2}}{\Gamma(s/2)}\int_0^\infty \sigma^{s-1}\vartheta\,d\sigma. \tag{7.4}$$

It then follows from (7.1) that

$$V = \frac{2\pi^{s/2}}{\Gamma(s/2)}\alpha^{s/2}\,\Theta_0^{-(1+\frac{1}{2}sm)}\int_0^\infty \rho^{s-1}\,\Theta\,d\rho. \tag{7.5}$$

Putting (4.4) in (7.5) we then find, for all class 1, 2, and 3 solutions that

$$V = \left(\frac{\alpha}{T}\right)^{s/2}\,\Theta_0^{-(1+\frac{1}{2}sm)}\,q(0). \tag{7.6}$$

7.2. Reduced flux at infinity The reduced flux at infinity Φ is defined by

$$\Phi = -\frac{2\pi^{s/2}}{\Gamma(s/2)}\lim_{\sigma\to\infty}\sigma^{s-1}\,\vartheta^m\,\frac{\partial\vartheta}{\partial\sigma}. \tag{7.7}$$

Integrating (7.2) with respect to σ from $\sigma = 0$ to ∞, we find that

$$\lim_{\sigma\to\infty}\sigma^{s-1}\,\vartheta^m\,\frac{\partial\vartheta}{\partial\sigma} = -\gamma\int_0^\infty \sigma^{s-1}\,\vartheta\,d\sigma, \tag{7.8}$$

so that

$$\Phi = \gamma V. \tag{7.9}$$

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Now it follows from (4.3), (5.3), (6.2) that, for all class 1, 2, and 3 solutions, the initial total flux at infinity

$$-q'(0) = \beta q(0)/T. \tag{7.10}$$

Combining (7.6) and (7.10) with (7.9), we find

$$\Phi = -\left(\frac{\alpha}{T}\right)^{\frac{1}{2}s-1} \Theta_0^{-(1+\frac{1}{2}sm)} q'(0), \tag{7.11}$$

the connection between Φ and $-q'(0)$.

7.3. Expanding the solutions about $\sigma = 0$ Repeated differentiation of (7.2) and use of (7.3) gives Taylor series expansions of $\vartheta(\sigma)$ about $\sigma = 0$. Odd derivatives at the origin are zero, and we find the leading terms of the expansion as:

$$\begin{aligned} s = 1, \qquad & \vartheta(\sigma) = 1 - \frac{\sigma^2}{2} + (3 - 3m - 2\gamma)\frac{\sigma^4}{24} - \mathrm{O}(\sigma^6); \\ s = 3, \qquad & \vartheta(\sigma) = 1 - \frac{\sigma^2}{2} + (5 - 5m - 2\gamma)\frac{\sigma^4}{360} - \mathrm{O}(\sigma^6). \end{aligned} \tag{7.12}$$

These expansions tell us how the solutions leave the origin, but are of relatively little use for practical computation of solutions. Their radius of practical convergence tends to be of order 0.1, whereas we require the solutions in $0 \le \sigma < \infty$.

7.4. Asymptotic expansions for large σ To develop asymptotic forms of the solution at large σ, we integrate (7.2) with respect to σ and obtain

$$m \neq -1, \qquad -\sigma^s \vartheta - \gamma \int_\vartheta^1 \sigma^s \, d\vartheta = \frac{s}{m+1} \sigma^{s-1} \frac{d\vartheta^{m+1}}{d\sigma}. \tag{7.13}$$

Since physically acceptable solutions have finite slugs with

$$0 < \int_0^1 \sigma^s \, d\vartheta = I < \infty, \tag{7.14}$$

we rewrite (7.13) as

$$-\sigma^s \vartheta + \gamma \int_0^\vartheta \sigma^s \, d\vartheta - \gamma I = \frac{s}{m+1} \sigma^{s-1} \frac{d\vartheta^{m+1}}{d\sigma}. \tag{7.15}$$

Note that

$$I = s\Gamma(s/2)\pi^{-s/2} V. \tag{7.16}$$

As we have seen already [and as follows from (7.15)], the finiteness of the total flux at infinity translates into the requirement that

$$\lim_{\sigma\to\infty} \frac{s}{m+1} \sigma^{s-1} \frac{d\vartheta^{m+1}}{d\sigma} = -\gamma I. \tag{7.17}$$

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The fact that the first and second terms on the left of (7.15) are of smaller order than the other terms as $\sigma \to \infty$ then leads to a procedure for generating an asymptotic expansion for ϑ^{m+1}. Compare [7]. It follows from (7.17) that, as long as s and m satisfy conditions (3.14), the leading term of the asymptotic expansion is

$$\vartheta^{m+1} \sim -\frac{\gamma(m+1)I}{s(2-s)}\sigma^{2-s}. \tag{7.18}$$

The higher terms of the expansion prove to be complicated and not especially useful. Our numerical method of solution (Section 7.5), however, makes use of (7.18) in the limit as $\vartheta \to 0$.

In the case $m = -1$, $0 < s < 2$, we replace (7.13), (7.17), (7.18), with (7.19), (7.20), (7.21), respectively:

$$-\sigma^s\vartheta - \gamma\int_\vartheta^1 \sigma^s\, d\vartheta = s\sigma^{s-1}\frac{d\ln\vartheta}{d\sigma}. \tag{7.19}$$

$$\lim_{\sigma\to\infty} s\,\sigma^{s-1}\frac{d\ln\vartheta}{d\sigma} = -\gamma I; \tag{7.20}$$

$$\ln\vartheta \sim -\frac{\gamma I}{s(2-s)}\sigma^{2-s}. \tag{7.21}$$

7.5. Solution by forward integration We have seen that neither Taylor nor asymptotic expansions provide much help in solving (7.2), (7.3) throughout the interval $0 \le \sigma q < \infty$. A numerical method of solving second-order nonlinear ordinary differential equations [5], however, proves useful, at least for $s = 1$. The method employs the integrodifferential form (7.13) of (7.2), and takes ϑ as the independent variable and σ as the dependent one. Integration starts at $(\vartheta, \sigma) = (1, 0)$ and goes backward in ϑ (forward in σ). The asymptotic form (7.18) is useful only at rather large σ (small ϑ), so the numerical work was carried close to $\vartheta = 0$, with smaller steps in ϑ in that neighbourhood.

7.6. Exact solutions for $s = 1$, $\gamma = 1$ When $s = 1$, $\gamma = 1$, (7.2) reduces to the simple

$$\frac{d}{d\sigma}\left(\vartheta^m\frac{d\vartheta}{d\sigma}\right) + \vartheta = 0. \tag{7.22}$$

For $m = -1$, the solution of (7.22) subject to (7.3) is

$$\sigma = \sqrt{2}\cosh^{-1}\vartheta^{-1/2}. \tag{7.23}$$

For $-2 < m < -1$ the solution is

$$\sigma = \left(\frac{m+2}{2}\right)^{1/2}\frac{1}{m+1}\int_1^{\vartheta^{m+1}}\frac{du}{(1-u^{(m+2)/(m+1)})^{1/2}}. \tag{7.24}$$

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For arbitrary m in this range, the solution is evaluated simply by numerical quadrature. On the other hand, we see by rewriting (7.24) as

$$\frac{2(m+1)}{m+2} = -n, \qquad \sigma = \left(n+\frac{1}{2}\right)^{1/2} \int_0^{\cosh^{-1}\vartheta^{-1/(n+2)}} \cosh^n v\, dv, \tag{7.25}$$

that the solution is integrable in closed form when n is a positive integer or half a positive integer. Eleven additional exact solutions are:

$n = \frac{1}{2}$, that is, $m = -\frac{6}{5}$,

$$\sigma = \sqrt{5}\left[F\left(\cos^{-1}\vartheta^{1/5}, \frac{1}{\sqrt{2}}\right) - 2E\left(\cos^{-1}\vartheta^{1/5}, \frac{1}{\sqrt{2}}\right)\right] + \sqrt{10}\left(\vartheta^{-2/5} - \vartheta^{2/5}\right)^{1/2}, \tag{7.26}$$

with $F(\ ,\ )$, $E(\ ,\ )$, the elliptic integrals of the first and second kinds, respectively.

$n = 1$, that is, $m = -\frac{4}{3}$,

$$\sigma = \sqrt{3}\left(\vartheta^{-2/3} - 1\right)^{1/2}. \tag{7.27}$$

$n = \frac{3}{2}$, that is, $m = -\frac{10}{7}$,

$$\sigma = \frac{\sqrt{7}}{3} F\left(\cos^{-1}\vartheta^{1/2}, \frac{1}{\sqrt{2}}\right) + \frac{\sqrt{14}}{3}\left(\vartheta^{-6/7} - \vartheta^{-2/7}\right)^{1/2}. \tag{7.28}$$

$n = 2$, that is, $m = -\frac{3}{2}$,

$$\sigma = \cosh^{-1}\vartheta^{-1/4} + \vartheta^{-1/4}\left(\vartheta^{-1/2} - 1\right)^{1/2}. \tag{7.29}$$

$n = \frac{5}{2}$, that is, $m = -\frac{14}{9}$,

$$\sigma = \tfrac{9}{5}\left[F\left(\cos^{-1}\vartheta^{1/9}, \frac{1}{\sqrt{2}}\right) - 2E\left(\cos^{-1}\vartheta^{1/9}, \frac{1}{\sqrt{2}}\right)\right] + \frac{3\sqrt{2}}{5}\left(\vartheta^{-4/9} - 1\right)^{1/2}\left(3\vartheta^{1/9} + \vartheta^{-1/3}\right). \tag{7.30}$$

$n = 3$, that is, $m = -\frac{8}{5}$,

$$\sigma = \frac{\sqrt{5}}{3}\left(\vartheta^{-2/5} - 1\right)^{1/2}\left(2 + \vartheta^{-2/5}\right). \tag{7.31}$$

$n = \frac{7}{2}$, that is, $m = -\frac{18}{11}$,

$$\sigma = \frac{5\sqrt{11}}{21} F\left(\cos^{-1}\vartheta^{1/11}, \frac{1}{\sqrt{2}}\right) + \frac{\sqrt{22}}{21}\left(\vartheta^{-4/11} - 1\right)^{1/2}\left(5\vartheta^{-1/11} + 3\vartheta^{-5/11}\right). \tag{7.32}$$

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$n = 4$, that is, $m = -\frac{5}{3}$,

$$\sigma = \sqrt{\frac{3}{32}}\left[3\cosh^{-1}\vartheta^{-1/6} + \vartheta^{-1/6}\left(\vartheta^{-1/3} - 1\right)^{1/2}\left(3 + 2\vartheta^{-1/3}\right)\right]. \tag{7.33}$$

$n = 5$, that is, $m = -\frac{12}{7}$,

$$\sigma = \frac{\sqrt{7}}{15}\left(\vartheta^{-2/7} - 1\right)^{1/2}\left[8 + 4\vartheta^{-2/7} + 3\vartheta^{-4/7}\right]. \tag{7.34}$$

$n = 6$, that is, $m = -\frac{7}{4}$,

$$\sigma = \frac{1}{12\sqrt{2}}\left[15\cosh^{-1}\vartheta^{-1/8} + \vartheta^{-1/8}\left(\vartheta^{-1/4} - 1\right)^{1/2}\left(15 + 10\vartheta^{-1/4} + 8\vartheta^{-1/2}\right)\right]. \tag{7.35}$$

$n = 7$, that is, $m = -\frac{16}{9}$,

$$\sigma = \frac{3}{35}\left(\vartheta^{-2/9} - 1\right)^{1/2}\left[16 + 8\vartheta^{-2/9} + 6\vartheta^{-4/9} + 5\vartheta^{-2/3}\right]. \tag{7.36}$$

7.7. Limiting and integral properties of the exact solutions Inspection of the solutions of (7.23) and (7.26)– (7.36) reveals the limiting behaviour of $\sigma(\vartheta)$ as $\vartheta \to 0$. It follows at once that we may evaluate Φ from (7.7) and V from (7.9). With $s = 1$, $\gamma = 1$, $V = \Phi$ here. Table 1 displays results obtained in this way.

The bottom lines of the table express the results for arbitrary n (and m). Combining (7.4) and (7.25) and comparing the result with the general expression for V in Table 1, we infer the following definite double integral

$$n \geq 0, \qquad \int_0^1 \int_0^{\cosh^{-1}\vartheta^{-1/(n+2)}} \cosh^n v\, dv\, d\vartheta = 1, \tag{7.37}$$

which does not seem to be known.

8. Comparing the various solutions

Before we examine some illustrative solutions, we bring together various results on the permissible ranges of the parameters and the properties of the similarity solutions we have developed. Table 2 gives an overall classification of the solutions and includes the ranges for γ, not discussed to this point. It is of interest that the value of γ suffices to discriminate between the classes of material-losing solutions for $0 < s < 2$. The total range of γ covers the 3 classes: class 1 solutions for $0 < \gamma < (sm + 2)/2$, class 2 for $\gamma = (sm + 2)/2$ and class 3 for $(sm + 2)/2 < \gamma < \infty$.

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TABLE 1. Limiting and integral properties of exact solutions.

n	m	$\lim_{\vartheta\to 0}\ \sigma$	$V = \Phi$
0	-1	$\sqrt{2}\ln 2 - 1/\sqrt{2}\ln\vartheta$	$\sqrt{8}$
1/2	$-6/5$	$\sqrt{10}\,\vartheta^{-1/5}$	$\sqrt{10}$
1	$-4/3$	$\sqrt{3}\,\vartheta^{-1/3}$	$\sqrt{12}$
3/2	$-10/7$	$\sqrt{14/3}\,\vartheta^{-3/7}$	$\sqrt{14}$
2	$-3/2$	$\vartheta^{-1/2}$	4
5/2	$-14/9$	$3\sqrt{2}\,5^{-1}\,\vartheta^{-5/9}$	$\sqrt{18}$
3	$-8/5$	$\sqrt{5}\,3^{-1}\,\vartheta^{-3/5}$	$\sqrt{20}$
7/2	$-18/11$	$\sqrt{22}\,7^{-1}\,\vartheta^{-7/11}$	$\sqrt{22}$
4	$-5/3$	$\sqrt{3}\,8^{-1}\,\vartheta^{-2/3}$	$\sqrt{24}$
5	$-12/7$	$\sqrt{7}\,5^{-1}\,\vartheta^{-5/7}$	$\sqrt{28}$
6	$-7/4$	$\sqrt{2}\,3^{-1}\,\vartheta^{-3/4}$	$\sqrt{32}$
7	$-16/9$	$3\,7^{-1}\,\vartheta^{-7/9}$	6
$n \geq 0$	$-1 \leq m < 2$		$(4n+8)^{1/2} = \left(\frac{8}{m+2}\right)^{1/2}$
$n > 0$	$-1 < m < 2$	$n^{-1}(n+2)^{1/2}\vartheta^{-\frac{n}{(n+2)}}$	
		$= -(m+1)^{-1}(\frac{1}{2}m+1)^{1/2}\vartheta^{1+m}$	

C. J. van Duijn has pointed out that for $s > 2$, solving (4.9), (4.10) becomes an eigenvalue problem. Following van Duijn's remark, M. A. Peletier and H. Zhang have found that, for $s > 2$, solutions exist only for $m = m(\gamma, s)$, with $m(\gamma, s)$ continuous and such that $-4/(s+2) < m < -2/s$ for $0 < \gamma < 1$, $m = -4/(s+2)$ for $\gamma = 1$, $-1 < m < -4/(s+2)$ for $1 < \gamma < \infty$, and $\lim_{\gamma\to\infty} m = -1$.

It should be noted that all our similarity solutions, both material-preserving and material-losing, give a θ-scale (as measured, for example, by θ at $r = 0$) decreasing monotonically with increasing t (except, of course, for class 3 solutions with $t > T$). In addition, all solutions except class 3 give an r-scale increasing monotonically with t. However, for class 3 solutions, in $0 < t \leq T$, behaviour of the r-scale depends on γ. For $0 < \gamma < 1$ (A), it increases monotonically with increasing t; for $\gamma = 1$ (B), it is constant; and for $1 < \gamma < \infty$ (C) it decreases monotonically with increasing t. Table 3 gives details of subclasses A, B, and C of class 3. Note that, for $s = 1$, the 3B solutions are the exact solutions of Section 7.6.

Greatest physical interest attaches to the integral s-values 1, 2, 3. Figure 3 gives maps of the (γ, m) plane showing the regions giving solutions of the various types, classes, and subclasses. Note that for $s = 2$ the only acceptable solutions are the material-preserving ones with $\gamma = 0$, $m > -1$. The Peletier-Zhang study showed

TABLE 2. Comparison of the various solutions.

	s	class	α	β	m	γ
material-conserving	$0 < s < \infty$		$s/(sm+2)$	0	$-2s^{-1} < m < \infty$	0
material-losing	$0 < s < 2$	1	$(2\beta+s)/(sm+2)$	$0 < \beta < \infty$	$-2s^{-1} < m \leq -1$	$0 < \gamma < (sm+2)/2$
		2	$2\beta/(sm+2)$	$0 < \beta < \infty$	$-2s^{-1} < m \leq -1$	$(sm+2)/2$
		3	$(2\beta-s)/(sm+2)$	$s/2 < \beta < \infty$	$-2s^{-1} < m \leq -1$	$(sm+2)/2 < \gamma < \infty$
	$s = 2$		no solutions			
	$2 < s < \infty$	3	$(2\beta-s)/(sm+2)$	$0 < \beta < s/2$	$-1 < m < -2s^{-1}$	$0 < \gamma < \infty$

TABLE 3. The subclasses of class 3 solutions.

s	subclass	β	γ	variation of r-scale with t
$0 < s < 2$	A	$-m^{-1} < \beta < \infty$	$(sm+2)/2 < \gamma < 1$	increasing
	B	$\beta = -m^{-1}$	$\gamma = 1$	constant
	C	$s/2 < \beta < -m^{-1}$	$1 < \gamma < \infty$	decreasing
$2 < s < \infty$	A	$-m^{-1} < \beta < s/2$	$0 < \gamma < 1$	increasing
	B	$\beta = -m^{-1}$	$\gamma = 1$	constant
	C	$0 < \beta < -m^{-1}$	$1 < \gamma < \infty$	decreasing

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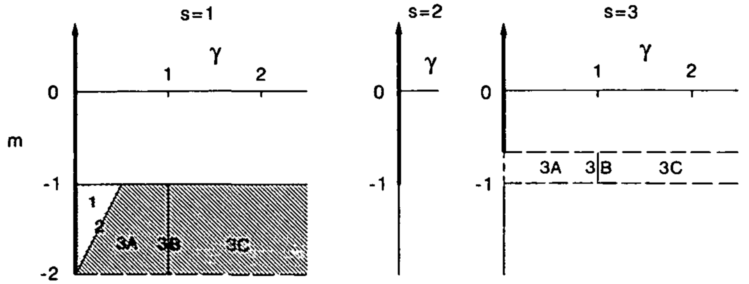

FIGURE 3. Maps of the (γ, m) plane for $s = 1, 2, 3$, showing regions giving similarity solutions of the various types, classes, and subclasses. See text for explanation of material-losing solutions with $s = 3$. The bold lines along (part of) $\gamma = 0$ represent the material-preserving solutions.

TABLE 4. Illustrative solutions for $s = 1, m = -\frac{3}{2}$. * Calculated to lesser accuracy.

α	β	γ	class or subclass
2	0	0	material-conserving
4	1/2	1/8	1
1	1/4	1/4	2
2	1	1/2	3A
2/3	2/3	1	3B
2/7	4/7	2	3C
2/15	8/15	4	3C*
2/31	16/31	8	3C*

that material-losing solutions for $s = 3$ exist only on the continuous curve $m = m(\gamma)$ such that $-\frac{4}{5} < m < -\frac{2}{3}$ for $0 < \gamma < 1$ (3A solutions), $m = -\frac{4}{5}$ for $\gamma = 1$ (3B), $-1 < m < -\frac{4}{5}$ for $1 < \gamma < \infty$ (3C), and $\lim_{\gamma\to\infty} = -1$.

9. Illustrative solutions

Table 4 sets out details of the illustrative solutions for $s = 1, m = -\frac{3}{2}$, presented here. Calculation of most solutions was carried out for ϑ-steps $\delta\vartheta = 0.1$ and 0.05 in the intervals $0 \le \vartheta \le 1 - \delta\vartheta$, combined with calculations for $\delta\vartheta = 0.01$ and 0.005 in $0.1 \le \vartheta \le 1 - \delta\vartheta$. The exceptions were calculations for $\gamma = 4$ and 8: these were done only for $\delta\vartheta = 0.1$. The h^2-extrapolation of Richardson [9] was used at various stages to secure final estimates of the solutions.

Computations were made also of seven of the exact solutions for $s = 1, \gamma = 1$, found in Section 7.6. We discuss these in detail in Section 9.3 below. The significant point here is that the exact solution for $s = 1$, $m = -\frac{3}{2}$, $\gamma = 1$ provided a check on the foregoing numerical work. The discrepancy between the numerical and exact values of $\sigma(\vartheta)$ was never greater than 0.03%.

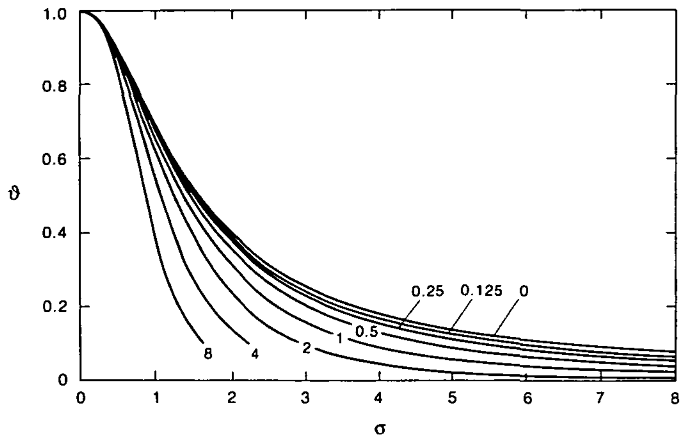

FIGURE 4. Solutions for $s = 1, m = -\frac{3}{2}$ in the reduced form $\vartheta(\sigma)$. Numerals on the curves are values of γ.

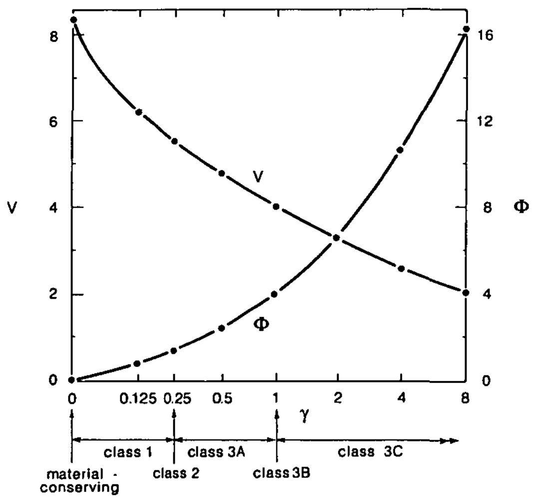

FIGURE 5. Integral properties of solutions for $s = 1$, $m = -\frac{3}{2}$. Reduced slug magnitude V and reduced flux at infinity Φ as functions of γ. Dots indicate the calculated points. The scale of the abscissa is linear in $\ln(\gamma + 0.125)$.

9.1. Concentration profiles and integral properties Figure 4 shows the basic solutions in the form of $\vartheta(\sigma)$ for the 8 values of γ. It will be seen that the solutions follow each other closely for θ near 1, diverging as θ decreases. Cf. (7.12).

Figure 5 compares integral properties of the solutions. The $V(\gamma)$ curve shows the dependence on γ of the reduced slug magnitude, and the $\Phi(\gamma)$ curve shows that of the reduced flux at infinity.

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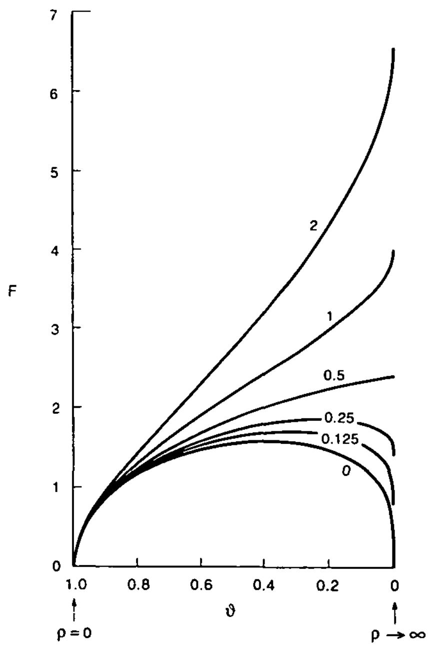

FIGURE 6. Flux-concentration relations for $s = 1, m = -\frac{3}{2}$. Reduced flux F as a function of ϑ for the indicated values of γ.

9.2. The flux-concentration relation Figure 6 is of some physical interest. It depicts the reduced flux F as a function of dimensionless concentration ϑ. Note that F is defined by

$$F(\vartheta) = -\frac{2\pi^{s/2}}{\Gamma(s/2)} \sigma^{s-1} \vartheta^m \frac{d\vartheta}{d\sigma}, \tag{9.1}$$

so that

$$\Phi = \lim_{\vartheta \to 0} F(\vartheta). \tag{9.2}$$

The useful procedure of characterizing a diffusion (or convection-diffusion) process by the dependence of reduced flux on dimensionless concentration appears to have originated in [6]. The "flux-concentration relation" has proved fruitful in calculations on water flows in swelling materials and in unsaturated soils (for example, [8], [11], [12], [14], [15]). Deeper mathematical use of the flux-concentration relation was made by Bouillet and Gomes [2], and subsequently by van Duijn *et al.* [13].

In Figure 6 we run the abscissa of ϑ left to right from 1 to 0, with σ running from 0 to ∞. The graphs of $F(\vartheta)$ illustrate vividly the increasing dominance on the diffusion process of the flux at infinity as γ increases.

For $s = 1, m = -\frac{3}{2}$, it follows from (7.12) and (9.1) that

$$\text{near } \vartheta = 1, \qquad F(\vartheta) \approx [32(1-\vartheta)]^{1/2}\, \vartheta, \tag{9.3}$$

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independent of γ. Accordingly, the $F(\vartheta)$ curves for various γ are virtually indistinguishable near $\vartheta = 1$, the singularity there behaving like $(1-\vartheta)^{1/2}$.

For the same parameter values, it follows from (2.8) that, for $\gamma = 0$,

$$\text{near } \vartheta = 0, \qquad F(\vartheta) \approx \frac{4}{\sqrt{3}}\vartheta^{1/4}. \tag{9.4}$$

For $\gamma > 0$, however, the singularity at $\vartheta = 0$ assumes a different character. For $s = 1, m = -\frac{3}{2}$, (7.15), (7.16), and (9.1) give

$$F(\vartheta) = -2\vartheta^{-3/2}\frac{d\vartheta}{d\sigma} = \gamma V + 2\sigma\vartheta - 2\gamma\int_0^{\vartheta}\sigma\,d\vartheta; \tag{9.5}$$

and (7.18) shows that

$$\text{near } \vartheta = 0, \qquad \sigma \approx \frac{4}{\gamma V \vartheta^{1/2}}. \tag{9.6}$$

Putting (9.6) in (9.5), we obtain

$$\text{near } \vartheta = 0, \qquad F(\vartheta) \approx \gamma V + \frac{8\vartheta^{1/2}[1-2\gamma]}{\gamma V}. \tag{9.7}$$

There are various modes of approach of $F(\vartheta)$ to Φ as $\vartheta \to 0$. For $\gamma = 0$, F decreases to zero like $\vartheta^{1/4}$. For $0 < \gamma < \frac{1}{2}$, the decrease of $(F - \Phi)$ to zero is like $\vartheta^{1/2}$. For $\gamma = \frac{1}{2}$ there is no singularity and the approach of F to Φ is regular. For $\gamma > \frac{1}{2}$, the decrease of $(\Phi - F)$ to zero is like $\vartheta^{1/2}$. Figure 6 is consistent with these inferences from (9.4) and (9.7).

Note that with $\gamma = 0$, F has a maximum of about 1.588 with $\vartheta \approx 0.39$. As γ increases the maximum of F increases and moves toward $\vartheta = 0$. At $\gamma = \frac{1}{2}$ it is at $\vartheta = 0$. For $\gamma > \frac{1}{2}$, there is no turning point of $F(\vartheta)$, but a point of inflexion emerges, which moves to an increasing value of ϑ as γ increases. For $\gamma = 1$ it is about 0.39, and for $\gamma = 2$ about 0.68.

In the more general case of $-2 < m < -1$, (9.4) is replaced by

$$\text{near } \vartheta = 0, \qquad F(\vartheta) = (-8/m)^{1/2}\,\vartheta^{(m+2)/2}; \tag{9.8}$$

and (9.7) is replaced by

$$\text{near } \vartheta = 0, \qquad F(\vartheta) \approx \gamma V - \frac{4\vartheta^{m+2}}{\gamma(m+1)V}\left[1 - \frac{\gamma}{m+2}\right]. \tag{9.9}$$

We thus have the following more general results: 1. The index of the power law approach of $F - \Phi$ to zero as $\vartheta \to 0$ *doubles* as we switch from $\gamma = 0$ to $\gamma > 0$ for all $-2 < m < -1$. 2. The critical γ-value giving F maximum (and a regular increase to Φ) at $\vartheta = 0$ is $\gamma = m + 2$ for all $-2 < m < -1$.

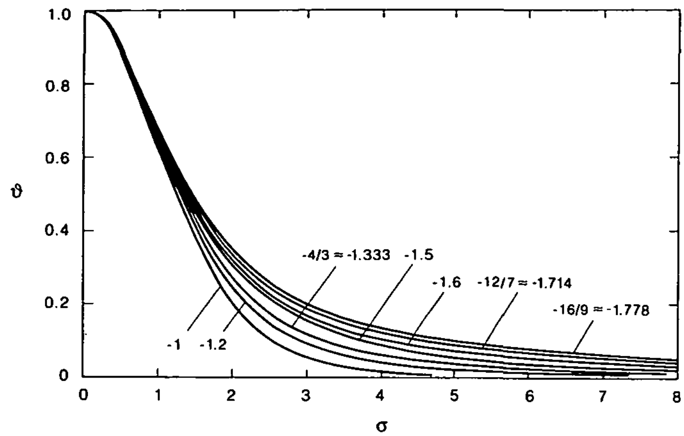

FIGURE 7. Solutions for $s = 1, \gamma = 1$ in the reduced form $\vartheta(\sigma)$. Numerals on the curves are values of m.

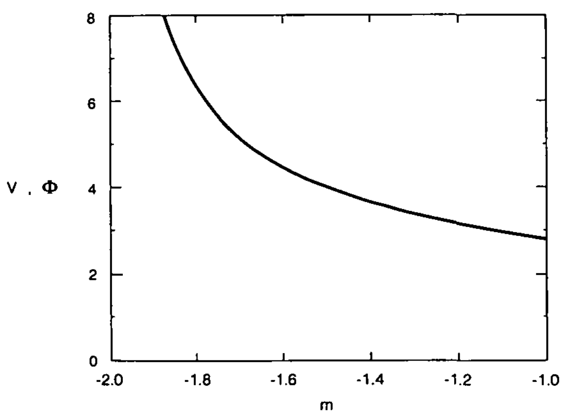

FIGURE 8. Integral properties of solutions for $s = 1, \gamma = 1$. V and Φ as functions of m. Note that here $V = \Phi$.

9.3. Exact solutions for $s = 1, \gamma = 1$ Figure 7 shows a selection of seven of the exact solutions of Section 7.6. The figure depicts the concentration profiles in the form of $\vartheta(\sigma)$. The profiles coalesce near $\vartheta = 1$, separating systematically as ϑ decreases toward 0. The approach of ϑ to zero as $\sigma \to \infty$ becomes slower as m decreases from -1 toward -2. Compare Table 1 and also (7.18) and (7.21).

Figure 8 shows, for these solutions, the dependence on m of both the reduced slug magnitude V and the reduced flux at infinity Φ. Note that $V = \Phi$ here, since $\gamma = 1$.

10. Concluding discussion

In an elaborate, study Rodríguez and Vázquez [10] proved our result (3.14) for the special case $s = 1$. In the present work, we have used the elementary, physically motivated, analysis of Section 3 to develop the generalization to arbitrary $s > 0$ (with $s = 2$ a partial exception). Specifically, we have found that solutions of (1.1), (1.4), (1.5) satisfying the physical constraints of finite slug magnitude and finite material flux at infinity (at least for some finite interval in T) can exist only for the following ranges of s and m:

$$\begin{aligned} 0 < s < 2, &\qquad -2s^{-1} < m \leq -1; \\ s > 2, &\qquad -1 < m < -2s^{-1}. \end{aligned}$$

The singular case $s = 2$, $m = 1$ requires further investigation.

It is of interest that for $0 < s < 2$ the allowable m-range lies *within* that permissible for the material-conserving solutions of Section 2. On the other hand, the allowable range for $s > 2$ lies totally *outside* the material-conserving m-range.

We observe that the material-losing similarity solutions developed in sections 4, 5, 6 cover the s- and m-ranges of (3.14). Notably, they do not exist for $s = 2$, but may for all other $s > 0$. It is of interest that three different similarity schemes all yield the same ordinary differential equation (4.9), restated as (7.2). For $0 < s < 2$, the value of parameter γ in (7.2) discriminates between three classes of solution. For class 1, with $0 < \gamma < (sm + 2)/2$, the concentration scale decreases as a negative power of $(1 + t/T)$; for class 2, with $\gamma = (sm + 2)/2$, it decreases exponentially; and for class 3, with $(sm + 2)/2 < \gamma < \infty$, it decreases as a positive power of $(1 - t/T)$ to zero when t reaches the finite value T.

A detailed range of solutions have been presented for $s = 1$, $m = -\frac{3}{2}$. They demonstrate the manner in which concentration and flux profiles change as the flux at infinity (parametrized by γ) increases.

These instantaneous source solutions for "fast diffusion" (diffusion with $m < 0$) requiring material loss at infinity are physically instructive. The finite rate of loss at infinity is unavoidable because the rate of approach to zero of $r^{s-1}\,\partial\theta/\partial r$ as $r \to \infty$ just balances the rate of increase without limit of θ^m as ($\theta \to 0$ and) $r \to \infty$. This suggests the exercise of caution before an actual physical transport process where $\theta \to 0$ is described as fast diffusion with diffusivity proportional to a negative power of concentration.

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