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# ON THE RESOLVENT OF THE LAPLACE–BELTRAMI OPERATOR IN HYPERBOLIC SPACE

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**Abstract**

In this paper, a detailed description of the resolvent of the Laplace–Beltrami operator in $n$-dimensional hyperbolic space is given. The resolvent is an integral operator with the kernel (Green's function) being a solution of a hypergeometric differential equation. Asymptotic analysis of the solution of this equation is carried out.

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## 1. Introduction

Although geometric and algebraic ideas play an important role in the theory of linear operators, the solution of a number of fundamental questions of this theory was achieved only thanks to the application of various tools from the theory of analytic functions. The basic channel for the use of the methods of the theory of functions is the tradition of studying the spectral properties of a linear operator by studying its resolvent as an analytic operator-valued function (see [1, 4, 6]).

If $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is a linear operator on the Hilbert (or Banach) space $\mathcal{H}$, and $I$ the identity operator, then the resolvent of $A$ is the operator

$$R_\mu = (A - \mu I)^{-1}.$$

This operator is considered for those values of the complex parameter $\mu$ such that $R_\mu$ exists and is a bounded operator defined on the whole of the space $\mathcal{H}$. Investigating $R_\mu$ as an operator-valued function of the complex variable $\mu$ gives a powerful method (the so-called resolvent method) in the analysis of the operator $A$. If $A$ is a differential operator, its resolvent $R_\mu$ is usually an integral operator the kernel of which is called the Green's function of the operator $A - \mu I$.

In the present paper, we deal with the case where $A$ is the Laplace–Beltrami operator $L$ (an elliptic partial differential operator of special form) in the half-space model of $n$-dimensional hyperbolic space $H^n$. We give a thorough analysis of the resolvent of $L$ and describe explicitly the structure of the resolvent. The Green's function appears as a solution of a hypergeometric differential equation and the asymptotic behaviour of the solution of this equation is exploited. The resolvent kernel of the (free) Laplace–Beltrami operator can be used for constructing the resolvent kernel of the automorphic Laplace–Beltrami operator on the fundamental domain of a discrete group of motions in the hyperbolic space by means of an averaging over the discrete group [3, 5, 9, 10].

We choose the spectral parameter of the form $\mu = s(n-1-s)$ in constructing the Green's function of $-L-\mu I$ so that the function $w(z) = y^s$ is a simplest particular solution of the equation

$$-Lw = \mu w.$$

The minus sign in front of $L$ is chosen, as usual, to obtain an operator with nonnegative spectrum. Besides, such a choice of the spectral parameter converts the form of the solution given in (3.4) to the more convenient form given in (3.5). Let

$$\begin{aligned} L_1 &= L + \left(\frac{n-1}{2}\right)^2 I \\ &= y^2\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2}\right) + y^n \frac{\partial}{\partial y}\left(\frac{1}{y^{n-2}}\frac{\partial}{\partial y}\right) + \left(\frac{n-1}{2}\right)^2 I. \end{aligned}$$

If we put $s = (n-1)/2 + i\lambda$ or $s = (n-1)/2 - i\lambda$, then

$$-L - s(n-1-s)I = -L - \left(\frac{n-1}{2}\right)^2 I - \lambda^2 I = -L_1 - \lambda^2 I$$

which involves the spectral parameter $\lambda$ in a more usual form. Because of this property, in the literature the shifted Laplace–Beltrami operator $L_1$ is used more often rather than the operator $L$. Since $\Re s > (n-1)/2$ is equivalent to $\Im\lambda < 0$ in the case $s = (n-1)/2 + i\lambda$ and to $\Im\lambda > 0$ in the case $s = (n-1)/2 - i\lambda$, the Green's function $r_1(z, z'; \lambda)$ of $-L_1 - \lambda^2 I$ is expressed in terms of the Green's function $r(z, z'; s)$ of $-L - s(n-1-s)I$ by the formula

$$\begin{aligned} r_1(z, z'; \lambda) &= \begin{cases} r\left(z, z'; \dfrac{n-1}{2} + i\lambda\right) & \Im\lambda < 0, \\ r\left(z, z'; \dfrac{n-1}{2} - i\lambda\right) & \Im\lambda > 0, \end{cases} \\ &= \begin{cases} \omega\left(u(z, z'); \dfrac{n-1}{2} + i\lambda\right) & \Im\lambda < 0, \\ \omega\left(u(z, z'); \dfrac{n-1}{2} - i\lambda\right) & \Im\lambda > 0, \end{cases} \end{aligned} \tag{1.1}$$

where $\omega(u; s)$ is defined by (3.5), (3.6) and $u(z, z')$ by (2.3).

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Note that for real values of $\lambda$ the functions $\omega(u(z,z');(n-1)/2 \pm i\lambda)$ are still defined by (3.5) as for $s=(n-1)/2 \pm i\lambda$ with real $\lambda$ the integral in (3.5) converges. Therefore for real $\lambda$ we can define the limit kernels

$$r_1(z,z';\lambda^{\pm}) = \lim_{\varepsilon\to 0^+} \omega\Big(u(z,z'); \frac{n-1}{2} \pm i(\lambda \pm i\varepsilon)\Big) = \omega\Big(u(z,z'); \frac{n-1}{2} \mp i\lambda\Big)$$

according to (1.1). However, it turns out that the integral operator with the kernel $r_1(z,z';\lambda^+)$ or $r_1(z,z';\lambda^-)$ does not define a bounded operator on the Hilbert space $L^2(H^n, dv)$. This is because of the fact that for real $\lambda$ the number $\lambda^2$ is a point of continuous spectrum of the operator $-L_1$ and therefore the resolvent $(-L_1 - \lambda^2 I)^{-1}$ cannot exist to be a bounded operator.

If $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is a self-adjoint operator on the Hilbert space $\mathcal{H}$, then according to the spectral theorem [1] there is a unique family $E_\mu(-\infty < \mu < \infty)$ of spectral projections $E_\mu$ (resolution of the identity) for $A$ such that

$$Af = \int_{-\infty}^{\infty} \mu \, dE_\mu f, \quad f \in D(A).$$

Description of the spectral projection $E_\mu$ of a given self-adjoint operator $A$ is called the eigenfunction expansion (spectral expansion) problem due to the formula

$$f = \int_{-\infty}^{\infty} dE_\mu f, \quad f \in \mathcal{H},$$

which defines an expansion of the element $f$ in the form of a Stieltjes integral. By the well-known Stone formula

$$E_{\mu_2} - E_{\mu_1} = \lim_{\varepsilon\to 0^+} \int_{\mu_1}^{\mu_2} (R_{\mu+i\varepsilon} - R_{\mu-i\varepsilon}) \, d\mu,$$

the results of the present paper on the structure of resolvent of the Laplace–Beltrami operator can be used for description of $E_\mu$ for this operator. This problem will be considered by the author elsewhere.

This paper is organized as follows. In Section 2 a symmetric operator $T'$ generated in a Hilbert space of complex-valued measurable functions on $H^n$ by the Laplace–Beltrami operator $L$ is introduced and investigated. In Section 3 the asymptotic behaviour of a solution of the hypergeometric equation associated with the radial part of $L$ is examined. In Section 4 the Green's function of $-L - s(n-1-s)I$ is constructed. Finally, in Section 5, it is shown that the closure $T$ of the operator $T'$ is self-adjoint and the resolvent of the operator $T$ is described explicitly.

## 2. The operator $T'$

The $n$-dimensional hyperbolic space $H^n$ can be realized as the upper half-space

$$H^n = \{z = (x_1, \dots, x_{n-1}, y) : -\infty < x_j < \infty \ (1 \le j \le n-1), 0 < y < \infty\}$$

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in the Euclidean space $\mathbb{R}^n$. $H^n$ is a homogeneous space of the group

$$G = \mathrm{SO}^+(1,n) = \{g \in \mathrm{GL}(n+1,\mathbb{R}) : g^t J g = J, \det g = 1, g_{00} > 0\},$$

where $\mathrm{GL}(n+1,\mathbb{R})$ is the group of all nonsingular real $(n+1)\times(n+1)$ matrices $g = [g_{jk}]_{j,k=0}^n$, $J$ is the $(n+1)\times(n+1)$ diagonal matrix whose first diagonal element equals $-1$ and the remaining diagonal elements are all equal to 1; and the symbol $t$ stands for matrix transposition (see [2]). The group $G = \mathrm{SO}^+(1,n)$ acts on $H^n$ as follows: if $g \in G$, $g = [g_{jk}]_{j,k=0}^n$ and $z = (x_1,\dots,x_{n-1},y)$, then the point $gz = z' = (x'_1,\dots,x'_{n-1},y')$ has coordinates

$$x'_j = \frac{(g_{j0}+g_{jn})|z|^2 + 2\sum_{k=1}^{n-1} g_{jk}x_k + g_{j0} - g_{jn}}{c_g|z|^2 + 2\sum_{k=1}^{n-1}(g_{0k}-g_{nk})x_k + d_g} \quad (1 \le j \le n-1),$$

$$y' = \frac{2y}{c_g|z|^2 + 2\sum_{k=1}^{n-1}(g_{0k}-g_{nk})x_k + d_g},$$

where $|z|^2 = x_1^2 + \cdots + x_{n-1}^2 + y^2$, $c_g = g_{00} + g_{0n} - g_{n0} - g_{nn}$, $d_g = g_{00} + g_{nn} - g_{0n} - g_{n0}$.

The invariant (under the action of $G$) Riemannian metric $ds^2$ and the invariant volume element $dv(z)$ associated with it have the form

$$ds^2 = \frac{dx_1^2 + \cdots + dx_{n-1}^2 + dy^2}{y^2}, \quad dv(z) = \frac{dx_1 \cdots dx_{n-1}\, dy}{y^n}. \tag{2.1}$$

Denote by $L$ the invariant differential operator (Laplace–Beltrami operator)

$$L = y^2\Big(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2}\Big) + y^n \frac{\partial}{\partial y}\Big(\frac{1}{y^{n-2}}\frac{\partial}{\partial y}\Big). \tag{2.2}$$

An invariant of a pair of points, $u(z,z')$, has the form

$$u(z,z') = \frac{|z-z'|^2}{yy'} = \frac{(x_1-x'_1)^2 + \cdots + (x_{n-1}-x'_{n-1})^2 + (y-y')^2}{yy'} \tag{2.3}$$

so that $u(gz,gz') = u(z,z')$ for all $g \in G$ and $z,z' \in H^n$. The geodesic distance $\rho(z,z')$ on $H^n$, generated by the metric $ds^2$, has the form

$$\rho(z,z') = \ln \frac{|z-\overline{z'}| + |z-z'|}{|z-\overline{z'}| - |z-z'|}, \tag{2.4}$$

where we put $\bar{z} = (x_1,\dots,x_{n-1},-y)$ for $z = (x_1,\dots,x_{n-1},y)$. It follows from (2.3) and (2.4) that

$$u = 2\cosh\rho - 2 = 4\sinh^2\frac{\rho}{2}. \tag{2.5}$$

Let us note that in what follows throughout the paper $z$ and $z'$ will stand for arbitrary points in $H^n$.

We denote by $L^2(H^n, dv)$ the Hilbert space of all complex-valued measurable functions $f(z)$ defined on $H^n$ such that

$$\int_{H^n} |f(z)|^2\, dv(z) < \infty,$$

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with the inner product

$$(f_1, f_2) = \int_{H^n} f_1(z)\overline{f_2(z)}\, dv(z),$$

where $dv(z)$ is the volume element given in (2.1).

In the sequel we will denote the space $L^2(H^n, dv)$ briefly by $\mathcal{H}$. Further, $C^\infty(H^n)$ will denote the space of infinitely differentiable functions on $H^n$ and $C_0^\infty(H^n)$ the space of infinitely differentiable functions on $H^n$ with compact (with respect to the geodesic distance) support.

Let $L$ be the Laplace–Beltrami operator given by (2.2). By $D'$ we denote the set of all functions $f \in C^\infty(H^n)$ with the following properties:

(i) $f$ and $Lf$ belong to $\mathcal{H}$;

(ii) for some $\varepsilon > (n-1)/2$, generally speaking, different for different functions, the inequalities

$$|f(z)| \le C_1 \frac{y^\varepsilon}{(1+|z|^2)^\varepsilon}, \tag{2.6}$$

$$\left|\frac{\partial f}{\partial x_j}\right| \le C_2 \frac{(1+|x_j|)y^\varepsilon}{(1+|z|^2)^{1+\varepsilon}}, \quad \left|\frac{\partial f}{\partial y}\right| \le C_3 \frac{y^{\varepsilon-1}}{(1+|z|^2)^\varepsilon} \quad (1 \le j \le n-1), \tag{2.7}$$

hold, where the constants $C_1, C_2, C_3$ are, in general, different for different functions.

Obviously, $D'$ is a linear subset in $\mathcal{H}$ and since $C_0^\infty(H^n) \subset D'$, we conclude that $D'$ is dense in $\mathcal{H}$. Let us define on $\mathcal{H}$ the linear operator $T'$ with the domain of definition $D'$, putting

$$T'f = -Lf \quad \text{for } f \in D'.$$

We want to show that the operator $T'$ is symmetric.

Theorem 2.1. *For arbitrary functions $f_1, f_2 \in D'$, the integral*

$$[f_1, f_2] := \int_{H^n} \left( \frac{\partial f_1}{\partial y} \frac{\partial \overline{f_2}}{\partial y} + \sum_{j=1}^{n-1} \frac{\partial f_1}{\partial x_j} \frac{\partial \overline{f_2}}{\partial x_j} \right) \frac{dx\, dy}{y^{n-2}}, \tag{2.8}$$

*where $dx = dx_1 \ldots dx_{n-1}$, converges absolutely and*

$$(T'f_1, f_2) = [f_1, f_2]. \tag{2.9}$$

Proof. Let $m$ be an arbitrary positive integer. Let us set

$$K_m = \{x = (x_1, \ldots, x_{n-1}) : -m \le x_k \le m\ (1 \le k \le n-1)\},$$
$$K_m^{(j)} = \{x^{(j)} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n-1}) : -m \le x_k \le m\ (1 \le k \le n-1, k \ne j)\},$$
$$dx = dx_1 \ldots dx_{n-1}, \quad dx^{(j)} = dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_{n-1}.$$

Then integrating by parts, one has, for $f_1, f_2 \in D'$,

$$\begin{aligned}-\int_0^m \int_{K_m} (Lf_1)\overline{f_2}\, dv(z) &\\ = -\sum_{j=1}^{n-1} \int_0^m \int_{K_m^{(j)}} &\Big\{ \frac{\partial f_1}{\partial x_j} \overline{f_2} \Big|_{x_j=-m}^{x_j=m} \Big\} \frac{dx^{(j)}\, dy}{y^{n-2}} - \int_{K_m} \Big\{ \frac{1}{y^{n-2}} \frac{\partial f_1}{\partial y} \overline{f_2} \Big|_{y=0}^{y=m} \Big\}\, dx \\ &+ \int_0^m \int_{K_m} \Big( \sum_{j=1}^{n-1} \frac{\partial f_1}{\partial x_j} \frac{\partial \overline{f_2}}{\partial x_j} + \frac{\partial f_1}{\partial y} \frac{\partial \overline{f_2}}{\partial y} \Big) \frac{dx\, dy}{y^{n-2}}. \end{aligned} \tag{2.10}$$

Next, using the estimates (2.6), (2.7) for $f_1$ and $f_2$ (with $\varepsilon = \varepsilon_1$ and $\varepsilon = \varepsilon_2$, respectively) it is easy to check that

$$\int_0^m \int_{K_m^{(j)}} \frac{\partial f_1}{\partial x_j} \overline{f_2} \Big|_{x_j=\pm m} \frac{dx^{(j)}\, dy}{y^{n-2}} = O\Big( \frac{1}{m^{\varepsilon_1+\varepsilon_2}} \Big) \quad (m \to \infty),$$

$$\Big| \frac{1}{y^{n-2}} \frac{\partial f_1}{\partial y} \overline{f_2} \Big| \le C \frac{y^{\varepsilon_1+\varepsilon_2-(n-1)}}{(1+|z|^2)^{\varepsilon_1+\varepsilon_2}} \to 0 \quad \text{as } y \to 0,$$

$$\int_{K_m} \frac{1}{y^{n-2}} \frac{\partial f_1}{\partial y} \overline{f_2} \Big|_{y=m}\, dx = O\Big( \frac{1}{m^{\varepsilon_1+\varepsilon_2}} \Big) \quad (m \to \infty).$$

Therefore (2.10) takes the form

$$-\int_0^m \int_{K_m} (Lf_1)\overline{f_2}\, dv(z) = \int_0^m \int_{K_m} \Big( \frac{\partial f_1}{\partial y} \frac{\partial \overline{f_2}}{\partial y} + \sum_{j=1}^{n-1} \frac{\partial f_1}{\partial x_j} \frac{\partial \overline{f_2}}{\partial x_j} \Big) \frac{dx\, dy}{y^{n-2}} + O\Big( \frac{1}{m^{\varepsilon_1+\varepsilon_2}} \Big). \tag{2.11}$$

Taking $f_1 = f_2 = f$ in (2.11), we have

$$\int_0^m \int_{K_m} \Big( \Big| \frac{\partial f}{\partial y} \Big|^2 + \sum_{j=1}^{n-1} \Big| \frac{\partial f}{\partial x_j} \Big|^2 \Big) \frac{dx\, dy}{y^{n-2}} = -\int_0^m \int_{K_m} (Lf)\overline{f}\, dv(z) + O\Big( \frac{1}{m^{2\varepsilon}} \Big). \tag{2.12}$$

It follows that the left-hand side of (2.12) increases as $m \to \infty$ remaining bounded. Therefore

$$\int_{H^n} \Big( \Big| \frac{\partial f}{\partial y} \Big|^2 + \sum_{j=1}^{n-1} \Big| \frac{\partial f}{\partial x_j} \Big|^2 \Big) \frac{dx\, dy}{y^{n-2}} < \infty.$$

Then applying the Cauchy–Schwarz inequality, we find that the integral in (2.8) converges absolutely for $f_1, f_2 \in D'$. Consequently, passing to the limit in (2.11) as $m \to \infty$, one has (2.9). □

Theorem 2.1 yields the following result.

Corollary 2.2. *The operator $T'$ is symmetric:*

$$(T' f_1, f_2) = (f_1, T' f_2), \quad \forall f_1, f_2 \in D(T').$$

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From the same theorem one has that

$$(T'f,f)=\int_{H^n}\left(\left|\frac{\partial f}{\partial y}\right|^2+\sum_{j=1}^{n-1}\left|\frac{\partial f}{\partial x_j}\right|^2\right)\frac{dx\,dy}{y^{n-2}}\quad \forall f\in D(T'). \tag{2.13}$$

It follows from (2.13) that the operator $T'$ is positive. Actually, the following stronger statement holds.

THEOREM 2.3. *The inequality*

$$T'\geq\frac{(n-1)^2}{4}I$$

*holds, that is,*

$$(T'f,f)\geq\frac{(n-1)^2}{4}\|f\|^2\quad \forall f\in D(T'). \tag{2.14}$$

PROOF. Since the function $f\in D(T')$ tends to zero as $y\to 0$ by (2.6), we have, for all $z=(x,y)\in H^n$,

$$f(x,y)=\int_0^y\frac{\partial f(x,t)}{\partial t}\,dt.$$

Therefore

$$\int_0^N\frac{|f(x,y)|^2}{y^n}\,dy\leq\int_0^N\frac{1}{y^n}\left(\int_0^y\left|\frac{\partial f(x,t)}{\partial t}\right|dt\right)^2dy, \tag{2.15}$$

where $N$ is an arbitrary positive number. Further (note that the inequality (2.14) is obvious for $n=1$ from (2.13) so that we may assume that $n\geq 2$),

$$\begin{aligned}\int_0^N\frac{1}{y^n}\left(\int_0^y\left|\frac{\partial f(x,t)}{\partial t}\right|dt\right)^2dy&=-\frac{1}{(n-1)y^{n-1}}\left(\int_0^y\left|\frac{\partial f(x,t)}{\partial t}\right|dt\right)^2\Big|_{y=0}^{y=N}\\&\quad+\frac{2}{n-1}\int_0^N\frac{1}{y^{n-1}}\left(\int_0^y\left|\frac{\partial f(x,t)}{\partial t}\right|dt\right)\left|\frac{\partial f(x,y)}{\partial y}\right|dy.\end{aligned}$$

Hence, taking into account that, by (2.7),

$$\begin{aligned}\frac{1}{y^{n-1}}\left(\int_0^y\left|\frac{\partial f(x,t)}{\partial t}\right|dt\right)^2&\leq\frac{C}{y^{n-1}}\left(\int_0^y\frac{t^{\varepsilon-1}}{(1+|x|^2+t^2)^{\varepsilon}}\,dt\right)^2\\&\leq\frac{Cy^{2\varepsilon-(n-1)}}{\varepsilon^2(1+|x|^2)^{2\varepsilon}}\to 0\end{aligned}$$

as $y\to 0$, and

$$-\frac{1}{(n-1)y^{n-1}}\left(\int_0^N\left|\frac{\partial f(x,t)}{\partial t}\right|dt\right)^2\leq 0,$$

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one has that

$$\int_0^N \frac{1}{y^n}\Big(\int_0^y \Big|\frac{\partial f(x,t)}{\partial t}\Big|\,dt\Big)^2\,dy$$
$$\le \frac{2}{n-1}\int_0^N \frac{1}{y^{n-1}}\Big(\int_0^y \Big|\frac{\partial f(x,t)}{\partial t}\Big|\,dt\Big)\Big|\frac{\partial f(x,y)}{\partial y}\Big|\,dy$$
$$\le \frac{2}{n-1}\Big\{\int_0^N \frac{1}{y^n}\Big(\int_0^y \Big|\frac{\partial f(x,t)}{\partial t}\Big|\,dt\Big)^2\Big\}^{1/2}\Big\{\int_0^N \frac{1}{y^{n-2}}\Big|\frac{\partial f(x,y)}{\partial y}\Big|^2\,dy\Big\}^{1/2},$$

where we have used

$$y^{n-1} = y^{n/2}y^{(n-2)/2}.$$

Comparing the initial and terminal terms of the latter inequalities, we find that

$$\int_0^N \frac{1}{y^n}\Big(\int_0^y \Big|\frac{\partial f(x,t)}{\partial t}\Big|\,dt\Big)^2\,dy \le \frac{4}{(n-1)^2}\int_0^N \frac{1}{y^{n-2}}\Big|\frac{\partial f(x,y)}{\partial y}\Big|^2\,dy.$$

Integrating this inequality over $x \in \mathbb{R}^{n-1}$ and then letting $N \to \infty$, we arrive at the inequality

$$\int_{H^n}\Big|\frac{\partial f(x,y)}{\partial y}\Big|^2\frac{dx\,dy}{y^{n-2}} \ge \frac{(n-1)^2}{4}\|f\|^2,$$

taking into account (2.15). Now inequality (2.14) follows from (2.13). □

## 3. Asymptotic analysis of a hypergeometric differential equation

If we apply the operator $L$ to functions depending only on the invariant of a pair of points $u(z, z_0)$, then we can get for it an expression in the form of an ordinary differential operator with respect to the variable $u$. Namely, the following statement holds which can be verified directly.

Lemma 3.1. *Suppose that $\omega$ is a smooth function on the positive semi-axis and that $z_0$ is a fixed point in $H^n$. Let us set $f(z) = \omega(u(z, z_0))$. Then $Lf = l\omega$, where*

$$l\omega = (u^2 + 4u)\omega''(u) + (nu + 2n)\omega'(u). \tag{3.1}$$

Remark 3.2. Suppose that $\psi$ is a smooth function on the positive semi-axis and that $z_0$ is a fixed point in $H^n$. Let us set $f(z) = \psi(\rho(z, z_0))$. Then $Lf = l_1\psi$, where

$$l_1\psi = \psi''(\rho) + (n-1)\frac{\cosh\rho}{\sinh\rho}\psi'(\rho).$$

This statement can easily be derived from Lemma 3.1 on the basis of the connection (2.5) between $u$ and $\rho$.

Now we investigate the solution of the linear homogeneous differential equation

$$-l\omega = s(n-1-s)\omega,$$

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that is,

$$-(u^2+4u)\omega''(u)-(nu+2n)\omega'(u)=s(n-1-s)\omega(u), \tag{3.2}$$

where $s$ is a complex variable.

As is known [7, Ch. 15], one solution of the hypergeometric equation

$$\zeta(1-\zeta)\frac{d^2f}{d\zeta^2}+[c-(a+b+1)\zeta]\frac{df}{d\zeta}-abf=0, \tag{3.3}$$

where $a$, $b$, and $c$ are independent of $\zeta$, is given by the Gauss hypergeometric function (with Euler's integral representation)

$$f_1(\zeta)=F(a,b,c,\zeta)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t\zeta)^{-a}\,dt,$$

where $\Gamma(s)$ is the gamma function [7, Ch. 5]. Another solution of (3.3) is (see [7, (15.10.15)])

$$\begin{aligned} f_2(\zeta)&=(-\zeta)^{-a}F(a,a+1-c;a+1-b;\zeta^{-1})\\ &=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}(-\zeta)^{-a}\int_0^1 t^{a-c}(1-t)^{c-b-1}\Big(1-\frac{t}{\zeta}\Big)^{-a}\,dt\\ &=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{a-c}(1-t)^{c-b-1}(t-\zeta)^{-a}\,dt. \end{aligned} \tag{3.4}$$

If we set

$$\zeta=-\frac{u}{4},\quad f(\zeta)=f\Big(-\frac{u}{4}\Big)=\omega(u),\quad a=s,\quad b=n-1-s,\quad c=\frac{n}{2},$$

then (3.3) turns into (3.2). Therefore one solution of (3.2) is given, according to (3.4), by the integral

$$\omega(u;s)=c_n(s)\int_0^1 [t(1-t)]^{s-(n/2)}\Big(t+\frac{u}{4}\Big)^{-s}\,dt, \tag{3.5}$$

where the constant $c_n(s)$ we choose to be

$$c_n(s)=\frac{\Gamma(s)}{2^n\pi^{n/2}\Gamma(s-\frac{n}{2}+1)}. \tag{3.6}$$

Note that the integral in (3.5) converges absolutely for $u>0$ and complex $s=\sigma+i\tau$ with $\sigma>(n-2)/2$. The constant $c_n(s)$ given in (3.6) can be represented in the form

$$c_n(s)=\frac{1}{sb_{n+2}(s+1)2^{n-1}\sigma_n},\quad \sigma_n=\frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\quad (n=1,2,3,\ldots),$$

with

$$b_n(s)=\int_0^\infty\Big(1+\frac{1}{\xi}\Big)^{-s}\xi^{-n/2}d\xi=\frac{\Gamma(s-\frac{n}{2}+1)\Gamma(\frac{n}{2}-1)}{\Gamma(s)}\quad (n\ge 3). \tag{3.7}$$

Note that $\sigma_n$ is the surface area of the unit sphere in $\mathbb{R}^n$. Note also that

$$b_4(s+1) = \frac{1}{s}, \quad \text{and therefore } c_2(s) = \frac{1}{4\pi}.$$

Let us set

$$\varphi(u; s) = \omega(4u; s) = c_n(s) \int_0^1 [t(1-t)]^{s-(n/2)} (t+u)^{-s}\, dt, \tag{3.8}$$

so that

$$\omega(u; s) = \varphi\Big(\frac{u}{4}; s\Big).$$

Now we derive from (3.8) with (3.6) the following important result.

THEOREM 3.3. *The function $\varphi(u; s)$ is analytic in the region $\Re s > (n-2)/2$, $u > 0$ with respect to $s$, belongs to the class $C^\infty$ with respect to $u$, and*

(i) $-(u^2+u)\varphi''(u) - (nu + n/2)\varphi'(u) = s(n-1-s)\varphi(u)$;

(ii) *the estimates*

$$\varphi(u; s) = \begin{cases} O(1+|s|^{3/2}) \text{ and is continuous in } u & (n=1), \\ -\dfrac{1}{4\pi}\ln u + O(|s|^2) & (n=2), \\ \dfrac{1}{8\pi}u^{-1/2} + O(|s|^{5/2}) & (n=3), \\ \dfrac{1}{16\pi}u^{-1} + O(|s(s-2)|\ln u + |s|^3) & (n=4), \\ \dfrac{1}{(n-2)2^{n-2}\sigma_n}u^{-(n-2)/2} + O(|s|^{(n+2)/2}u^{-((n-2)/2)+1}) & (n \geq 5), \end{cases} \tag{3.9}$$

$$\varphi'(u; s) = \begin{cases} -\dfrac{1}{2}u^{-1/2} + O(|s|^{3/2}) & (n=1), \\ -\dfrac{1}{4\pi}u^{-1} + O(|s-1|\ln u + |s|^2) & (n=2), \\ -\dfrac{1}{2^{n-1}\sigma_n}u^{-n/2} + O(|s|^{(n+2)/2}u^{-(n/2)+1}) & (n \geq 3), \end{cases} \tag{3.10}$$

*hold as $u \to 0$ uniformly with respect to $s$ with $\Re s \geq (n-2)/2 + \delta_0$, where $\delta_0$ is any fixed positive number, $\sigma_n = 2\pi^{n/2}/\Gamma(n/2)$;*

(iii) *the estimates*

$$\varphi(u; s) = O(|s|^{(n-2)/2}u^{-\sigma}), \quad \varphi'(u; s) = O(|s|^{n/2}u^{-\sigma-1}), \tag{3.11}$$

*hold as $u \to \infty$ uniformly with respect to $s$ ($\Re s \geq (n-2)/2 + \delta_0$, $\delta_0 > 0$).*

PROOF. (i) The first statement of the theorem follows from (3.2) and the equality $\varphi(u) = \omega(4u)$.

(ii) To prove the second statement of the theorem, let us show that

$$
\begin{aligned}
\Phi_{s,n}(u) &:= \int_0^1 [t(1-t)]^{s-(n/2)}(t+u)^{-s}\,dt \\
&= \begin{cases} O(1+|s|^2) \text{ and is continuous in } u & (n=1), \\ -\ln u + O(|s|^2) & (n=2), \\ b_3(s)u^{-(1/2)} + O(|s|^2) & (n=3), \\ b_4(s)u^{-1} + O(|s-2|\ln u + |s|^2) & (n=4), \\ b_n(s)u^{-(n-2)/2} + O(|s|^2 u^{-((n-2)/2)+1}) & (n\ge 5), \end{cases}
\end{aligned}
\tag{3.12}
$$

as $u \to 0$ uniformly with respect to $s$ with $\Re s \ge (n-2)/2 + \delta_0$, where $\delta_0$ is any fixed positive number; $b_n(s)(n \ge 3)$ is defined by (3.7).

Note that the integral from $1/2$ to $1$ is estimated easily:

$$
\begin{aligned}
&\left|\int_{1/2}^1 [t(1-t)]^{s-(n/2)}(t+u)^{-s}\,dt\right| \\
&\quad\le \int_{1/2}^1 \frac{[t(1-t)]^{\sigma-(n/2)}}{(t+u)^{\sigma}}\,dt \\
&\quad\le 2^{n/2}\int_0^1 (1-t)^{\sigma-(n/2)}\,dt = \frac{2^{n/2}}{\sigma - \frac{n-2}{2}} = O(1).
\end{aligned}
$$

Let us estimate the integral

$$
J = \int_0^{1/2} (1-t)^{s-(n/2)}\frac{t^{s-(n/2)}}{(t+u)^s}\,dt.
$$

To do this, consider the function

$$
\Psi(r,u) = \int_0^r \frac{t^{s-(n/2)}}{(t+u)^s}\,dt.
$$

Making the change of variables $t = u\xi$, we get

$$
\Psi(r,u) = u^{-(n-2/2)}\Psi_1\Big(\frac{r}{u}\Big), \quad \text{where } \Psi_1(r) = \int_0^r \Big(1+\frac{1}{\xi}\Big)^{-s}\xi^{-n/2}d\xi. \tag{3.13}
$$

It is not difficult to check that, for $r \le 1$,

$$
\Psi_1(r) = \frac{r^{s-(n-2)/2}}{s - \frac{n-2}{2}} + O(|s| r^{\sigma-((n-2)/2)+1}), \tag{3.14}
$$

and, for $r \ge 1$,

$$
\Psi_1(r) = \begin{cases} 2\sqrt{r} + O(|s|) & (n=1), \\ \ln r + O(|s|) & (n=2), \\ b_n(s) + \dfrac{2}{n-2}r^{-(n-2)/2} + O(|s| r^{-(n/2)}) & (n\ge 3), \end{cases} \tag{3.15}
$$

where $b_n(s)$ is defined by (3.7). The above estimates as well as those below are satisfied uniformly with respect to $s(\Re s \ge (n-2)/2 + \delta_0, \delta_0 > 0)$.

We have

$$
\begin{aligned}
J &= \int_0^{1/2} (1-t)^{s-(n/2)} \frac{d\Psi(t,u)}{dt}\, dt \\
&= \Big(1-\frac{1}{2}\Big)^{s-(n/2)} \Psi\Big(\frac{1}{2}, u\Big) - \int_0^{1/2} \frac{d(1-t)^{s-(n/2)}}{dt} \Psi(t,u)\, dt \\
&= \Big(1-\frac{1}{2}\Big)^{s-(n/2)} \Psi\Big(\frac{1}{2}, u\Big) + J_1 + J_2, \qquad (3.16)
\end{aligned}
$$

where

$$
J_1 = -\int_0^{u} \frac{d(1-t)^{s-(n/2)}}{dt} \Psi(t,u)\, dt, \quad J_2 = -\int_u^{1/2} \frac{d(1-t)^{s-(n/2)}}{dt} \Psi(t,u)\, dt.
$$

Further, using (3.13)–(3.15), it can be shown that

$$
\Psi(\tfrac{1}{2}, u) = \begin{cases} \sqrt{2} + O(|s|\sqrt{u}) & (n=1), \\ -\ln u + O(|s|) & (n=2), \\ b_n(s) u^{-(n-2)/2} + O(|s|) & (n \geq 3), \end{cases}
$$

$$
J_1 = O(|s|^2 u^{-((n-2)/2)+1}),
$$

$$
J_2 = \begin{cases} O(|s|^2) & (n=1), \\ (\ln u)[(1-\frac{1}{2})^{s-1} - (1-u)^{s-1}] + O(|s|^2) & (n=2), \\ -b_3(s) u^{-1/2}[(1-\frac{1}{2})^{s-(3/2)} - (1-u)^{s-(3/2)}] + O(|s|^2) & (n=3), \\ -b_4(s) u^{-1}[(1-\frac{1}{2})^{s-2} - (1-u)^{s-2}] + O(|s-2|\ln u + |s|^2) & (n=4), \\ -b_n(s) u^{-(n-2)/2}[(1-\frac{1}{2})^{s-(n/2)} - (1-u)^{s-(n/2)}] + O(|s|^2 u^{-((n-2)/2)+1}) & (n \geq 5). \end{cases}
$$

Therefore one has from (3.16) that

$$
J = \begin{cases} O(1+|s|^2) & (n=1), \\ -\ln u + O(|s|^2) & (n=2), \\ b_3(s) u^{-(1/2)} + O(|s|^2) & (n=3), \\ b_4(s) u^{-1} + O(|s-2|\ln u + |s|^2) & (n=4), \\ b_n(s) u^{-(n-2)/2} + O(|s|^2 u^{-((n-2)/2)+1}) & (n \geq 5). \end{cases}
$$

Notice that we have used the fact that due to

$$
\frac{\Gamma(s+a)}{\Gamma(s)} = s^a\Big[1 + O\Big(\frac{1}{s}\Big)\Big], \qquad |\arg s| \leq \pi - \delta,
$$

we have from (3.7) that

$$
b_n(s) = O(|s|^{-(n-2)/2}) \quad (|s| \to \infty).
$$

Thus, the estimate (3.12) is established. Now the estimate (3.9) follows from (3.12) if we note that

$$
\varphi(u; s) = c_n(s)\Phi_{s,n}(u), \quad c_n(s) = O(|s|^{(n-2)/2}) \quad (n = 1, 2, 3, \ldots),
$$

$$
c_2(s) = \frac{1}{4\pi}, \quad c_n(s) b_n(s) = \frac{1}{(n-2)2^{n-2}\sigma_n} \quad (n \geq 3).
$$

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The estimate (3.10) for $\varphi'(u;s)$ also follows from (3.12) because

$$\varphi'(u;s) = c_n(s)\frac{d\Phi_{s,n}(u)}{du} = -sc_n(s)\Phi_{s+1,n+2}(u),$$
$$sc_n(s)b_{n+2}(s+1) = \frac{1}{2^{n-1}\sigma_n}, \quad c_n(s) = O(|s|^{(n-2)/2}) \quad (n = 1, 2, 3, \ldots).$$

(iii) The last statement of the theorem can be proved simply:

$$\begin{aligned}|\varphi(u;s)| &\le |c_n(s)| \int_0^1 \frac{[t(1-t)]^{\sigma-(n/2)}}{(t+u)^\sigma}\,dt \\ &\le \frac{|c_n(s)|}{u^\sigma}\int_0^1 [t(1-t)]^{\sigma-(n/2)}\,dt = O(|s|^{(n-2)/2}u^{-\sigma}).\end{aligned}$$

The function $\varphi'(u,s)$ is estimated similarly. □

## 4. The Green's function of $-L - s(n-1-s)I$

In this section, we introduce an integral operator $\widehat{R}(s)$ with the kernel

$$r(z,z';s) = \omega(u(z,z');s) = \varphi\Big(\frac{u(z,z')}{4};s\Big) \tag{4.1}$$

and investigate its necessary properties.

THEOREM 4.1. *Let $f \in C_0^\infty(H^n)$ and $\Re s = \sigma > (n-1)/2$. Let us set*

$$\widehat{R}(s)f(z) = \int_{H^n} r(z,z';s)f(z')\,dv(z'). \tag{4.2}$$

*Then the function $\widehat{R}(s)f(z)$ possesses the following properties:*

(i) $\widehat{R}(s)f(z) \in D'$, *where $D'$ is the domain of definition of the operator $T'$ (see Section 2).*

(ii) *The following equality holds:*

$$[T' - s(n-1-s)I]\widehat{R}(s)f = f. \tag{4.3}$$

The proof will be provided in the form of two lemmas.

LEMMA 4.2. *Let $f \in C_0^\infty(H^n)$ and $\Re s = \sigma > (n-1)/2$. Then the function $h_s(z) = \widehat{R}(s)f(z)$ is bounded on $H^n$, belongs to the space $L^2(H^n, dv) \cap C^\infty(H^n)$, and satisfies the following inequalities:*

$$|h_s(z)| \le C_0(1 + |s|^{(n+2)/2}), \tag{4.4}$$

$$|h_s(z)| \le C_1\frac{y^\sigma}{(1+|z|^2)^\sigma}, \tag{4.5}$$

$$\Big|\frac{\partial h_s(z)}{\partial x_j}\Big| \le C_2\frac{(1+|x_j|)y^\sigma}{(1+|z|^2)^{1+\sigma}} \quad (1 \le j \le n-1), \tag{4.6}$$

$$\Big|\frac{\partial h_s(z)}{\partial y}\Big| \le C_3\frac{y^{\sigma-1}}{(1+|z|^2)^\sigma}, \tag{4.7}$$

*where $C_0$ does not depend both on $s$ and on $z$ whereas $C_1$, $C_2, C_3$ do not depend only on $z$ (these constants are, generally speaking, different for different functions $f$).*

Proof. Let $\operatorname{supp} f \subset Q \Subset H^n$, where $Q$ is a bounded domain which lies strictly inside $H^n$. We have

$$h_s(z) = \int_{H^n} r(z, z'; s) f(z')\, dv(z') = \int_Q r(z, z'; s) f(z')\, dv(z'). \tag{4.8}$$

Hence

$$|h_s(z)| \le C_f \int_Q |r(z, z'; s)|\, dv(z'), \quad C_f = \max_{z\in H^n} |f(z)|. \tag{4.9}$$

First we show that $h_s(z)$ is bounded and the estimate (4.4) is fulfilled. To this end we take a small number $\varepsilon_0 > 0$ and a large number $R_0 > 0$ such that the support $Q$ of the function $f$ lies strictly inside the parallelepiped

$$\Pi = \{z = (x_1, \ldots, x_{n-1}, y) : \varepsilon_0 \le y \le R_0, -R_0 \le x_j \le R_0 (1 \le j \le n-1)\}.$$

Let us estimate the function $h_s$ on $\Pi$ and on $H^n \backslash \Pi$ separately.

As follows from Theorem 3.3, for $z \in \Pi$, $z' \in Q$, the function $r(z, z'; s)$ defined by (4.1) admits the estimation

$$r(z, z'; s) = \begin{cases} O(1 + |s|^{3/2}) & (n = 1), \\ O(|s|^2 \ln u) & (n = 2), \\ O(|s|^{(n+2)/2} u^{-(n-2)/2}) & (n \ge 3), \end{cases}$$

uniformly on $s(\Re s > (n-1)/2)$, and for $z \in H^n \backslash \Pi$, $z' \in Q$, the estimate

$$r(z, z'; s) = O(|s|^{(n-2)/2} 4^{\sigma} u^{-\sigma}) \tag{4.10}$$

also uniformly on $s(\Re s > (n-1)/2)$.

Therefore, for $z \in \Pi$ (we assume that $n \ge 3$; for $n = 1, 2$ the reasoning is similar),

$$\begin{aligned} \int_Q |r(z, z'; s)|\, dv(z') &\le C|s|^{(n+2)/2} \int_Q [u(z, z')]^{-(n-2)/2}\, dv(z') \\ &= C|s|^{(n+2)/2} \int_Q \frac{(yy')^{(n-2)/2}}{|z - z'|^{n-2}} \frac{dx'\, dy'}{(y')^n} \le C|s|^{(n+2)/2} \frac{R_0^{n-2}}{\varepsilon_0^n} \int_Q \frac{dx'\, dy'}{|z - z'|^{n-2}} \le C_1 |s|^{(n+2)/2}, \end{aligned}$$

because the last integral converges and is bounded for $z \in \Pi$.

Consider $z \in H^n \backslash \Pi$. Since for $z \in H^n \backslash \Pi$ and $z' \in Q$ we may assume that $u(z, z')$ is large enough, it can be assumed that $u/4 > 1 + u/8$. Therefore, for $z \in H^n \backslash \Pi$ we have, by (4.10),

$$|r(z, z'; s)| \le C|s|^{(n-2)/2} \Big(1 + \frac{u}{8}\Big)^{-\sigma},$$

$$\int_Q |r(z, z'; s)|\, dv(z') \le C|s|^{(n-2)/2} \int_Q \Big(1 + \frac{u(z, z')}{8}\Big)^{-\sigma} dv(z') \le C v(Q) |s|^{(n-2)/2},$$

because the integrand is less than or equal to 1.

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Thus, the boundedness of the function $h(z, s)$ with respect to $z \in H^n$ and the estimate (4.4) are established. Now we show that $h(z, s) \in L^2(H^n, dv)$. Since, as has already been proved, the function $h(z, s)$ is bounded, it is sufficient to show that

$$\int_{H^n \setminus \Pi} |h_s(z)|^2 \, dv(z) < \infty, \tag{4.11}$$

where $\Pi$ is the above defined parallelepiped. From (4.8) we have, applying the Cauchy–Schwarz inequality,

$$|h_s(z)|^2 \le \|f\|^2 \int_Q |r(z, z'; s)|^2 \, dv(z').$$

Therefore

$$\int_{H^n \setminus \Pi} |h_s(z)|^2 \, dv(z) \le \|f\|^2 \int_Q \Big\{ \int_{H^n \setminus \Pi} |r(z, z'; s)|^2 \, dv(z) \Big\} \, dv(z'). \tag{4.12}$$

Next, taking into account (4.10), one has

$$\begin{aligned} \int_{H^n \setminus \Pi} |r(z, z'; s)|^2 \, dv(z) &\le C|s|^{n-2} 4^{2\sigma} \int_{H^n \setminus \Pi} \frac{dv(z)}{[1 + u(z, z')]^{2\sigma}} \\ &= C|s|^{n-2} 4^{2\sigma} \int_{H^n \setminus \Pi} \frac{(yy')^{2\sigma}}{[yy' + |x - x'|^2 + (y - y')^2]^{2\sigma}} \frac{dx\, dy}{y^n}. \end{aligned}$$

Obviously, there is a constant $\alpha$ not depending on $z \in H^n \setminus \Pi$ and $z' \in Q$ such that

$$\frac{(y')^{2\sigma}}{[yy' + |x - x'|^2 + (y - y')^2]^{2\sigma}} \le \alpha \frac{1}{(|x|^2 + y^2)^{2\sigma}} \quad (z \in H^n \setminus \Pi, z' \in Q).$$

Therefore from the last inequality we obtain

$$\int_{H^n \setminus \Pi} |r(z, z'; s)|^2 \, dv(z) \le C_1 \int_{H^n} \frac{y^{2\sigma - n} \, dx\, dy}{(1 + |x|^2 + y^2)^{2\sigma}},$$

where the constant $C_1$ depends on $s$. Further,

$$\begin{aligned} \int_{H^n} \frac{y^{2\sigma - n} \, dx\, dy}{(1 + |x|^2 + y^2)^{2\sigma}} &= \int_{\mathbb{R}^{n-1}} dx \int_0^1 \frac{y^{2\sigma - n} \, dy}{(1 + |x|^2 + y^2)^{2\sigma}} \\ &\quad + \int_{\mathbb{R}^{n-1}} dx \int_1^\infty \frac{y^{2\sigma - n} \, dy}{(1 + |x|^2 + y^2)^{2\sigma}} =: J_1 + J_2. \end{aligned}$$

Let us estimate the integrals $J_1$ and $J_2$. For $\sigma > (n-1)/2$,

$$J_1 \le \int_{\mathbb{R}^{n-1}} \frac{dx}{(1 + |x|^2)^{2\sigma}} \int_0^1 y^{2\sigma - n} \, dy < \infty.$$

In the integral $J_2$ we make first the change of variables $x = rv, v \in \mathbb{R}^{n-1}$, $|v| = 1$, $0 \le r < \infty$, and then $r = ty$, to get

$$J_2 = \sigma_{n-1} \int_0^\infty \frac{t^{n-2}}{(1 + t^2)^{2\sigma}} \, dt \int_1^\infty \frac{dy}{y^{2\sigma + 1}} < \infty, \tag{4.13}$$

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where $\sigma_{n-1}$ is the area of the unit sphere in $\mathbb{R}^{n-1}$. Note that in the case $n = 1$ the integration on $x$ is absent and the integral $J_2$ becomes

$$J_2 = \int_1^\infty \frac{dy}{y^{2\sigma+1}} < \infty.$$

Therefore in (4.13) we may assume that $n \geq 2$ (so that the function $t^{n-2}$ has no singularity at zero). Now inequality (4.11) follows from (4.12).

Let us show that $h_s \in C^\infty(H^n)$. To do this, note that there exists a smooth mapping $H^n \ni z \mapsto g_z \in G$ such that $g_z e_n = z$, where $e_n = (0, \ldots, 0, 1)$. Indeed, as such a $g_z$ one can take

$$g_z = \begin{bmatrix} \dfrac{|z|^2+1}{2y} & x_1 & \cdots & x_{n-1} & \dfrac{y^2-|x|^2-1}{2y} \\ \dfrac{x_1}{y} & 1 & \cdots & 0 & -\dfrac{x_1}{y} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \dfrac{x_{n-1}}{y} & 0 & \cdots & 1 & -\dfrac{x_{n-1}}{y} \\ \dfrac{|z|^2-1}{2y} & x_1 & \cdots & x_{n-1} & \dfrac{y^2-|x|^2+1}{2y} \end{bmatrix}, \quad z = (x, y).$$

Evidently, the entries of the matrix $g_z$ are infinitely differentiable functions of the variables $(x_1, \ldots, x_{n-1}, y) \in H^n$. Next, we have

$$\begin{aligned} h_s(z) &= \int_{H^n} r(z, z'; s) f(z')\, dv(z') = \int_{H^n} r(g_z e_n, z'; s) f(z')\, dv(z') \\ &= \int_{H^n} r(e_n, g_z^{-1} z'; s) f(z')\, dv(z') = \int_{H^n} r(e_n, z'; s) f(g_z z')\, dv(z'), \end{aligned}$$

where we have used the invariance of the volume element $dv$. Since the function $f$ is smooth and has a compact support, it follows from the last formula that $h(z, s)$ is a smooth function of the variables $(x_1, \ldots, x_{n-1}, y)$ on $H^n$.

It remains to prove the estimates (4.5)–(4.7). As we have already proved, $h_s(z)$ is bounded and smooth on $H^n$ for fixed $s$. Hence $h_s(z)$ and its partial derivatives are bounded on compact subregions of $H^n$. Therefore it is sufficient to prove the estimates for $z$ such that $\rho(z, Q) \to \infty$, where $\rho(z, Q)$ denotes the geodesic distance of the point $z$ from the set $Q$.

If $z' \in Q$ and $\rho(z, Q) \to \infty$, then $u(z, z') = 2\cosh \rho(z, z') - 2 \to \infty$. Since for $u \to \infty$ and fixed $s(\Re s > (n-1)/2)$ we have $\omega(u; s) = O(u^{-\sigma})$ by (4.1) and (3.11), it follows from (4.9) that

$$|h_s(z)| \leq C \int_Q \frac{dv(z')}{[1 + u(z, z')]^\sigma},$$

where $C$ is a constant depending on $s$. Next,

$$1 + u(z, z') = \frac{yy' + |x - x'|^2 + (y - y')^2}{yy'} \geq a \frac{1 + |x|^2 + y^2}{y}, \tag{4.14}$$

where the positive constant $a$ does not depend on $z$ and $z' \in Q$. Therefore

$$|h_s(z)| \le \frac{C}{a^\sigma} v(Q)\Big(\frac{y}{1+|x|^2+y^2}\Big)^\sigma,$$

that is, the estimate (4.5) holds.

To derive the estimates (4.6) and (4.7), we differentiate equality (4.8). Taking into account (4.1), we get

$$\frac{\partial h_s(z)}{\partial x_j} = \frac{1}{4}\int_Q \varphi'\Big(\frac{u(z,z')}{4}; s\Big)\frac{\partial u(z,z')}{\partial x_j}\, dv(z'), \tag{4.15}$$

$$\frac{\partial h_s(z)}{\partial y} = \frac{1}{4}\int_Q \varphi'\Big(\frac{u(z,z')}{4}; s\Big)\frac{\partial u(z,z')}{\partial y}\, dv(z'). \tag{4.16}$$

Next, for $z' \in Q$ and $\rho(z, Q) \to \infty$,

$$\Big|\frac{\partial u(z,z')}{\partial x_j}\Big| = 2\Big|\frac{x_j - x'_j}{yy'}\Big| \le a_1 \frac{1+|x_j|}{y},$$

$$\Big|\frac{\partial u(z,z')}{\partial y}\Big| = \Big|\frac{y^2 - y'^2 - |x-x'|^2}{y^2 y'}\Big| \le a_2 \frac{1+|x|^2+y^2}{y^2},$$

and by Theorem 3.3 and (4.14),

$$\Big|\varphi'\Big(\frac{u(z,z')}{4}; s\Big)\Big| \le \frac{C}{[1+u(z,z')]^{\sigma+1}} \le \frac{C}{a^{\sigma+1}}\Big(\frac{y}{1+|x|^2+y^2}\Big)^{\sigma+1}.$$

Therefore the estimates (4.6) and (4.7) follow from (4.15) and (4.16). □

Lemma 4.3. *If $f \in C_0^\infty(H^n)$ and $\Re s > (n-1)/2$, then*

$$[-L - s(n-1-s)I]\widehat{R}(s) f = f. \tag{4.17}$$

Proof. By Lemma 4.2 the function

$$h_s(z) = \widehat{R}(s) f(z) = \int_{H^n} r(z, z'; s) f(z')\, dv(z')$$

belongs to $C^\infty(H^n)$. Next, note that if $\chi(t)$ is an arbitrary smooth function on the semi-axis $0 < t < \infty$, then

$$L_z \chi(u(z,z')) = L_{z'}\chi(u(z,z')). \tag{4.18}$$

Indeed, by Lemma 3.1,

$$L_z\chi(u(z,z')) = (u^2+4u)\chi''(u) + (nu+2n)\chi'(u).$$

Since $u(z,z') = u(z',z)$, the same expression is obtained for $L_{z'}\chi(u(z,z'))$. Therefore (4.18) holds.

Now let $f \in C_0^\infty(H^n)$ and assume that supp $f \subset Q \Subset H^n$. Taking into account (4.1), using property (4.18) and integrating by parts, one has that

$$L_z \int_{H^n} r(z,z';s) f(z')\, dv(z') = \int_{H^n} r(z,z';s)(L_{z'} f(z'))\, dv(z').$$

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Therefore to prove (4.17) it is enough to establish that

$$\int_{H^n} r(z,z';s)\{[-L_{z'} - s(n-1-s)I]f(z')\}\,dv(z') = f(z). \tag{4.19}$$

We shall prove (4.19) by using the methods of potential theory (cf. [5, Ch. 8]). Let $U_\varepsilon$ be the exterior of the small $n$-dimensional Euclidean ball of radius $\varepsilon$ centred at the point $z$ so that $\partial U_\varepsilon = S_\varepsilon$ is an $(n-1)$-sphere. According to integration by parts formulae,

$$\begin{aligned}\int_{U_\varepsilon} & r(z,z';s)\{[-L_{z'} - s(n-1-s)I]f(z')\}\,dv(z') \\ &- \int_{U_\varepsilon} \{[-L_{z'} - s(n-1-s)I]r(z,z';s)\}f(z')\,dv(z') \\ &= -\int_{S_\varepsilon} \frac{1}{(y')^{n-2}}\Big[\frac{\partial r(z,z';s)}{\partial l}f(z') - r(z,z';s)\frac{\partial f(z')}{\partial l}\Big]\,dS_\varepsilon,\end{aligned}$$

where $l$ is the exterior normal to the surface $S_\varepsilon$ and $dS_\varepsilon$ is the (Euclidean) surface element on $S_\varepsilon$.

Since $[-L_{z'} - s(n-1-s)I]r(z,z';s) = 0$ outside the diagonal, the last equation takes the form

$$\begin{aligned}\int_{U_\varepsilon} & r(z,z';s)\{[-L_{z'} - s(n-1-s)I]f(z')\}\,dv(z') \\ &= -\int_{S_\varepsilon} \frac{1}{(y')^{n-2}}\Big[\frac{\partial r(z,z';s)}{\partial l}f(z') - r(z,z';s)\frac{\partial f(z')}{\partial l}\Big]\,dS_\varepsilon.\end{aligned} \tag{4.20}$$

Let us show that

$$\lim_{\varepsilon\to 0}\int_{S_\varepsilon} \frac{1}{(y')^{n-2}} r(z,z';s)\frac{\partial f(z')}{\partial l}\,dS_\varepsilon = 0, \tag{4.21}$$

$$-\lim_{\varepsilon\to 0}\int_{S_\varepsilon} \frac{1}{(y')^{n-2}} \frac{\partial r(z,z';s)}{\partial l}f(z')\,dS_\varepsilon = f(z). \tag{4.22}$$

To this end we put $z' = z + t\nu$, where $0 \le t < \infty$, $\nu = (\nu_1,\dots,\nu_n) \in \mathbb{R}^n$, $|\nu| = 1$. We have $dS_t = t^{n-1}\,dS_1$, where $dS_1$ is the area element of the unit $(n-1)$-sphere, and

$$\begin{aligned} u(z,z')|_{z'=z+t\nu} &= \frac{|z-z'|^2}{yy'}\Big|_{z'=z+t\nu} \\ &= \frac{t^2}{y(y+t\nu_n)} = \frac{t^2}{y^2}[1+O(t)] \quad (t\to 0).\end{aligned} \tag{4.23}$$

Further, we will assume that $n \ge 3$ (in the case $n = 2$ the reasoning is similar; the case $n = 1$ is elementary and should be considered separately on the basis of the same reasoning). By Theorem 3.3 and equality (4.1), we have (for fixed $s$)

$$r(z,z';s) = O(u^{-(n-2)/2}) \quad \text{as } u \to 0,$$

and according to (4.23),

$$u(z,z')|_{z'=z+\varepsilon\nu} = O(\varepsilon^2).$$

Therefore

$$\left|\int_{S_\varepsilon} \frac{1}{(y')^{n-2}} r(z,z';s) \frac{\partial f(z')}{\partial l}\, dS_\varepsilon\right| \le C\varepsilon^{n-1}\varepsilon^{-(n-2)} = C\varepsilon \to 0 \quad \text{as } \varepsilon \to 0,$$

that is, (4.21) holds.

Next,

$$\frac{\partial r(z,z';s)}{\partial l} = \frac{\partial r(z,z';s)}{\partial t} = \frac{1}{4}\varphi'\Big(\frac{u}{4};s\Big)\frac{\partial u}{\partial t},$$
$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\Big\{\frac{t^2}{y(y+t\nu_n)}\Big\} = \frac{2ty + t^2\nu_n}{y(y+t\nu_n)^2},$$

and by Theorem 3.3,

$$\varphi'\Big(\frac{u}{4};s\Big) = -\frac{2}{\sigma_n}u^{-n/2} + O(u^{-(n/2)+1}) \quad \text{as } u \to 0.$$

Therefore, taking into account (4.23),

$$\frac{\partial r(z,z';s)}{\partial l}\bigg|_{z'=z+\varepsilon\nu} = -\frac{y^{n-2}}{\sigma_n \varepsilon^{n-1}} + O\Big(\frac{1}{\varepsilon^{n-2}}\Big) \quad \text{as } \varepsilon \to 0.$$

Now (4.22) becomes obvious. Finally, passing to the limit in (4.20) as $\varepsilon \to 0$, we get (4.19). The lemma is proved. □

Now note that the statement of Theorem 4.1 follows from Lemmas 4.2 and 4.3. Let us also remark that (4.17) and (4.2) show that $r(z,z';s)$ defined by (4.1) is the Green's function for $-L - s(n-1-s)I$.

As is well known (see, for example, [8, Ch. 13]), the Green's function $G(x,x';\mu)$ of $-\Delta - \mu I$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

is the Laplace operator, has a logarithmic singularity at $x = x'$ of the type $\ln|x - x'|^{-1}$ for $n = 2$ and a singularity of the type $|x - x'|^{-(n-2)}$ for $n \ge 3$. Since for $z$ close to $z'$ the behaviour of $u^{-1}$ is like the behaviour of $|z - z'|^{-2}$ by (2.3), it follows from formula (3.9) of Theorem 3.3 and (4.1) that for the Laplace–Beltrami operator $L$ on $H^n$ the Green's function $r(z,z';s)$ of $-L - s(n-1-s)I$ has the same type of singularity as the Green's function of the Laplace operator on $\mathbb{R}^n$. This circumstance is consistent with the fact that the hyperbolic space $H^n$, being a Riemannian manifold, must behave locally like Euclidean space $\mathbb{R}^n$.

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## 5. The operator $T$ and its resolvent

Denote by $T$ the closure of the operator $T'$ in the Hilbert space $\mathcal{H}$.

THEOREM 5.1. *The operator T is self-adjoint.*

PROOF. It follows from Theorem 4.1 that $C_0^\infty(H^n)$ is contained in the range of the operator $T' - s(n-1-s)I(\Re s > (n-1)/2)$. Since $C_0^\infty(H^n)$ is dense in $\mathcal{H}$, it follows that the range of the operator $T' - s(n-1-s)I(\Re s > (n-1)/2)$ is dense in $\mathcal{H}$. Therefore the closure $\overline{T'} = T$ of the symmetric operator $T'$ is a self-adjoint operator (see [1]). □

It follows from (2.14) that

$$T \geq \frac{(n-1)^2}{4} I, \quad \text{i.e. } (Tf, f) \geq \frac{(n-1)^2}{4} \|f\|^2 \quad \forall f \in D(T). \tag{5.1}$$

Next, since the operator $T$ is self-adjoint and inequality (5.1) is true, we get that the resolvent $R(s) := [T - s(n-1-s)I]^{-1}$ exists for $\Re s = \sigma > (n-1)/2$ and is a bounded operator defined on the whole of space $\mathcal{H}$.

THEOREM 5.2. *If $f \in C_0^\infty(H^n)$ and $\Re s > (n-1)/2$, then*

$$R(s)f(z) = \int_{H^n} r(z, z'; s) f(z')\, dv(z'),$$

*where $r(z, z'; s) = \omega(u(z, z'); s)$ in which $\omega(u; s)$ is defined by (3.5) and (3.6).*

PROOF. By the definition of $R(s)$ we have $[T - s(n-1-s)I]R(s)f = f$, $\forall f \in \mathcal{H}$. On the other hand, since $T' \subset T$, it follows from (4.3) that

$$[T - s(n-1-s)I]\widehat{R}(s)f = f, \quad \forall f \in C_0^\infty(H^n).$$

Subtracting the last two equations, we get $[T - s(n-1-s)I][R(s)f - \widehat{R}(s)f] = 0$, that is, $\lambda = s(n-1-s)$ is an eigenvalue and $R(s)f - \widehat{R}(s)f$ a corresponding eigenvector of the operator $T$. But for $\Re s > (n-1)/2$ the number $\lambda = s(n-1-s)$ lies outside the interval $[((n-1)^2/4), \infty)$ in the $\lambda$-plane and hence cannot be an eigenvalue for the operator $T$ by (5.1). Therefore $R(s)f - \widehat{R}(s)f = 0$ and the proof is complete. □

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