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Exceptional Howe Correspondences over Finite Fields

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Abstract. We consider the restriction of the reflection representation to various reductive dual pairs in exceptional groups, and determine the correspondence of generic representations.

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1. Introduction

Let k be a finite field with q elements and of characteristic p. Recall that if Sp is the symplectic group over k, then a pair of reductive subgroups (H_1, H_2) of Sp is a dual pair if they are mutual commutants in Sp. Examples of (H_1, H_2) are (Sp_{2n}, O_m), $(\mathrm{GL}_n, \mathrm{GL}_m)$ and (U_n, U_m). The Weil representation [Ge] then defines a correspondence between the irreducible representations of H_1 and H_2, called the Howe duality correspondence. A first study of the Howe correspondence over finite fields was carried out by Srinivasan [Sr], who determined the decomposition of (the uniform part of) the Weil representation into Deligne–Lusztig characters of the dual pair $H_1 \times H_2$. From her results, one sees that the Howe correspondence respects the correspondence of geometric conjugacy classes [D-L] with respect to some natural inclusion of dual groups $i: H_1^\vee \hookrightarrow H_2^\vee$. In particular, the unipotent representations correspond. In [AM], it was shown by Adams and Moy that the cuspidal unipotent representations correspond in the case of first occurrence in their terminology. Finally, the Howe correspondence was completely determined by Aubert *et al*. in their recent paper [A-M-R], except in the case (Sp_{2n}, O_m) where they have a conjecture. We remark that the Howe correspondence over local and global fields has also been extensively studied.

In this paper, we study an analogue of the Howe correspondence over finite fields in the exceptional groups. So assume that G is a split simply-laced simple linear algebraic group over k, and let F be the corresponding Frobenius. We first need an analogue of the Weil representation. This is the so-called reflection representation Π [Lu]. Π can be characterized in various ways, but for now, it suffices to say that Π is a unipotent principal series representation of G^F, and its dimension is the smallest among all nontrivial irreducible representations of G^F (that is, those of dimension greater than 1). Hence, it is the analogue of the minimal representation [MS] in the p-adic case. We recall that the character values of irreducible representations of G^F on semi-simple elements are known through the work of Lusztig [Lu2]. However, in Section 3, we shall see that the character values of Π on semi-simple elements s can be interpreted neatly in terms of the motive $M(s)$, introduced by Gross [Gr], of the connected centralizer $C_G(s)^0$ of s in G. More precisely, for s a semi-simple element in G^F, we have

PROPOSITION 1.1. $\mathrm{Tr}(s|\Pi) = \mathrm{Tr}(F|M(s))$.

We shall consider various dual pairs in exceptional groups. As an example, consider the case where G is a split adjoint group of type E_6. There is a dual pair [MS], $\mathrm{PGL}_3 \times G_2 \subset E_6$, which is interesting because there is a natural inclusion of dual groups $i\colon \mathrm{SL}_3 \hookrightarrow G_2$. For each irreducible representation π of PGL_3, let $\Theta(\pi)$ be the set of irreducible representations π' of G_2, counted with multiplicities, such that $\pi \otimes \pi'$ occurs in Π, regarded as a $\mathrm{PGL}_3 \times G_2$-module. Also let $\Theta_{\mathrm{gen}}(\pi) \subset \Theta(\pi)$ be the subset of generic representations. Then we shall determine $\Theta_{\mathrm{gen}}(\pi)$ for each generic representation π of PGL_3. We first prove

THEOREM 1.2. *Let U be the unipotent radical of a Borel subgroup of G_2, and let ψ be a character of U in general position. Then $\Pi_{U,\psi}$ is the Gelfand–Graev representation of* PGL_3. *In particular,*

$$\#\Theta_{\mathrm{gen}}(\pi) = \begin{cases} 1, & \text{if } \pi \text{ is generic;} \\ 0, & \text{otherwise.} \end{cases}$$

Now recall that the generic characters of a connected reductive group with connected center can be parametrized by semi-simple classes in the dual group. For a semi-simple class s in the dual group, we let $\chi(s)$ be the corresponding generic representation. Then we shall show

THEOREM 1.3. $\Theta_{\mathrm{gen}}(\chi(s)) = \{\chi(i(s))\}$ *for each semi-simple class s in* SL_3, *the dual group of* PGL_3.

A check on the dimensions shows that this already accounts for a large part of Π. For non-cuspidal representations, this theorem is proved by the computation of Jacquet functors. For cuspidal representations, we resort to the use of base change or Shintani descent, under a suggestion of T. Uzawa. It is interesting to see if it can be adapted to the p-adic case.

We also have similar results for the dual pairs $G_2 \times \mathrm{PGSp}_6 \subset E_7$ and $G_2 \times \mathrm{PU}_3 \subset {}^2E_6$. The results of this paper are in a natural way complement to those of Magaard and Savin [MS], who considered the exceptional Howe correspond-

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ence over a p-adic field and showed that the correspondence of tempered spherical representations respects Langlands functoriality.

2. Weyl Group Invariants and Motive of G

In this section, k is any field, k^s a separable closure of k, and $\Gamma = \mathrm{Gal}(k^s/k)$.

Let G be a connected quasi-split reductive group over k, of rank l. Let T_0 be a maximal torus of G over k, which is contained in a Borel subgroup over k. The Weyl group of G is $W = N_G(T_0)/T_0$. Suppose that W has rank r as a reflection group, i.e. W can be generated by r simple reflections. Let $X_\bullet(T_0)$ be the cocharacter group of T_0, and let $E = X_\bullet(T_0) \otimes \mathbb{Q}$, a $\mathbb{Q}$-vector space of dimension l. Then W acts naturally on E, and as a W-module, E is the direct sum of $(l-r)$ copies of the trivial representation, and one copy of the natural reflection representation E_W of W.

The group Γ also acts naturally as automorphisms on E, and the action of Γ normalizes that of W. Hence the semi-direct product $W \rtimes \Gamma$ acts on E. Moreover, Γ preserves the decomposition

$$E = E^W \oplus E_W. \tag{2.1}$$

We shall denote the character of E as χ_E.

Let $S = \mathrm{Sym}^\bullet(E)$. Then $W \rtimes \Gamma$ acts on S, preserving degrees. Let S^W be the ring of W-invariants. As is well known, S^W is a polynomial ring, with l generators: $S^W = \mathbb{Q}[I_1, \dots, I_l]$. Let d_i be the degree of the primitive invariant I_i. Then the numbers d_i's are well-defined and the numbers $e_i = d_i - 1$ will be called the *exponents of the reductive group* G. Note that exactly $l-r$ of the e_i's, say $e_{r+1}, \dots, e_l$, are zero, and $e_1, \dots, e_r$ are the exponents of the Coxeter group W.

Note that Γ acts on S^W. By complexifying E, we can choose generators I_i's of $S^W \otimes \mathbb{C}$ such that, for a given $\sigma \in \Gamma$,

$$\sigma(I_i) = \varepsilon_i(I_i). \tag{2.2}$$

Now we have the following formula, which can be proved in a similar way to [Ca, p. 363]

LEMMA 2.3.

$$\frac{1}{|W|} \sum_{w \in W} \frac{\chi(\sigma w)}{\det_E(1 - t\sigma w)} = \frac{\sum_i \varepsilon_i t^{e_i}}{\prod_i (1 - \varepsilon_i t^{e_i+1})}.$$

Now we briefly recall the notion of the motive of G from [Gr]. Consider the graded Γ-module

$$\mathcal{J}/\mathcal{J}^2 := V = \oplus_d V_d, \tag{2.4}$$

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where $\mathcal{J}$ is the ideal of S^W generated by the invariants of positive degrees. Note that V is generated by the primitive invariants. It is a theorem of Steinberg [St] that

$$E \cong V \tag{2.5}$$

as a Γ-module.

For a prime number l not equal to the characteristic of k, let $\mathbb{Q}_l(1)$ be the Tate motive (cf. [Gr]) given by the action of Γ on the l-power roots of unity in k^s. Then the motive of G is defined to be the Artin–Tate motive

$$M_G = \oplus_d V_d(1-d) \tag{2.6}$$

where $V_d(1-d) = V_d \otimes \mathbb{Q}_l(1)^{\otimes(1-d)}$.

As an example, consider the case when $k = \mathbb{F}_q$ is a finite field, and $\Gamma = \langle F \rangle$. Then (2.5) says that:

$$\mathrm{Tr}(F|E) = \mathrm{Tr}(F|V). \tag{2.7}$$

Also, if we take $\sigma = F$, then

$$\mathrm{Tr}(F|M_G) = \sum_i \varepsilon_i q^{e_i}, \tag{2.8}$$

where the e_i's are the exponents of the reductive group G.

3. The Reflection Representation

Henceforth, unless otherwise stated, we assume that $k = \mathbb{F}_q$ is a finite field, of characteristic p, so that $\Gamma = \langle F \rangle$. Also, assume that G is a split, simply-laced, simple linear algebraic group over $\mathbb{F}_q$ of rank l. Let $F\colon G \to G$ be the corresponding Frobenius map, so that $G^F = G(\mathbb{F}_q)$. For each $w \in W$, T_w will denote an F-stable maximal torus of G which is obtained from T_0 by twisting with w.

The reflection representation Π of G is the unipotent principal series representation corresponding to the reflection representation E_W of W. It was shown by Kilmoyer in his thesis that Π is the unique representation of G satisfying

$$\langle \mathrm{Ind}_P^G 1, \Pi \rangle = l - l', \tag{3.1}$$

where P is any parabolic subgroup, and l' is the semi-simple rank of the Levi factor of P.

Let χ_Π be the character of Π. It was shown by Lusztig ([Lu] and [Lu2]) that

$$\chi_\Pi = \frac{1}{|W|} \sum_{w \in W} \chi_E(w) R_w := R_{\chi_E}, \tag{3.2}$$

where $R_w = R_{T_w,1}$ is the character of Deligne–Lusztig [D-L]. From this, he deduced that if $s \in G^F$ is regular semi-simple, so that s is contained in a unique maximal torus T_{w_0}, then

$$\chi_\Pi(s) = \chi_E(w_0). \tag{3.3}$$

We shall use (3.2) to obtain a formula, which may be already well known, for $\chi_\Pi(s)$, where s is any semi-simple element.

Let $C^0(s)$ denote the connected component of the centralizer of s in G. The value of R_w on semisimple elements s is given by [Ca p. 233]

$$\begin{aligned} R_w(s) &= \frac{\varepsilon_{T_w}\varepsilon_{C^0(s)}}{|T_w^F||C^0(s)^F|_q} \sum_{g\in G^F} 1(g^{-1}sg \in T_w^F) \\ &= \frac{\varepsilon_{T_w}\varepsilon_{C^0(s)}}{|C^0(s)^F|_q} \cdot |W(T_w)^F| \cdot |\Delta(s,w)|, \end{aligned}$$

where we let

$$\Delta(s,w) = \{T : T \subset C^0(s) \text{ and } T \text{ is } G^F\text{-conjugate to } T_w\}. \tag{3.4}$$

Note that $W(T_w)^F \cong C_W(w)$, the centralizer of w in W.

Now suppose that T_{w_0} is a maximally split torus in $C^0(s)$. The Weyl group $W_{C^0(s)}$ of $C^0(s)$ can be identified with a reflection subgroup $W^0(s)$ of W, so that the action of F on $W_{C^0(s)}$ is given by the action of w_0^{-1} on $W^0(s)$ by conjugation. Any $T \subset C^0(s)$ can be obtained from T_{w_0} by twisting with an element of $W^0(s)$. Hence, $R_w(s)$ is nonzero only if the conjugacy class C_w of w in W has nonzero intersection with the coset $w_0W^0(s)$. Moreover, for $w_1, w_2 \in W^0(s)$, w_0w_1 and w_0w_2 are conjugate in W if and only if w_1 and w_2 are w_0-conjugate in W. Thus we see that there is a uniquely determined w_0-conjugacy class Y_{C_w} in W such that

$$C_w \cap w_0W^0(s) = w_0(Y_{C_w} \cap W^0(s)).$$

Note that $Y_{C_w} \cap W^0(s)$ is then a union of w_0-conjugacy classes of $W^0(s)$.

Let Θ be the set of conjugacy classes of W which have nonzero intersection with $w_0W^0(s)$. For each class C in Θ, w_C will denote an element of C such that $T_{w_C} \subset C^0(s)$. Also, for each $C \in \Theta$, let θ_C be the set of w_0-conjugacy classes of $W^0(s)$ which are contained in $Y_C \cap W^0(s)$. For each $J \in \theta_C$, w_J will denote an element of J, and $C_{W^0(s),w_0}(w_J)$ will denote the w_0-centralizer of w_J in $W^0(s)$.

Now we have

$$\begin{aligned} R_{\chi_E}(s) &= \frac{1}{|W|}\sum_{C\in\Theta} |C|\cdot\chi(w_C)\cdot\frac{\varepsilon_{T_{w_C}}\varepsilon_{C^0(s)}}{|C^0(s)^F|_q}\cdot|C_W(w_C)|\cdot|\Delta(s,w_C)| \\ &= \sum_{C\in\Theta}\chi(w_C)\cdot\frac{\varepsilon_{T_{w_C}}\varepsilon_{C^0(s)}}{|C^0(s)^F|_q}\cdot\left\{\sum_{J\in\theta_C}\frac{|C^0(s)^F|}{|N_{C^0(s)}(T_{w_0w_J})^F|}\right\} \end{aligned}$$

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$$= |C^0(s)^F|_{q'} \cdot \sum_J \det_E(w_J) \cdot \frac{1}{|C_{W^0(s),w_0}(w_J)|} \cdot \frac{\chi(w_0 w_J)}{|T^F_{w_0 w_J}|}$$

$$= |C^0(s)^F|_{q'} \cdot \frac{1}{|W^0(s)|} \sum_{w \in W^0(s)} \det_E(w) \cdot \frac{\chi(w_0 w)}{|T^F_{w_0 w}|}$$

$$= (-1)^l \det_E(w_0) \cdot |C^0(s)^F|_{q'} \times$$

$$\times \frac{1}{|W^0(s)|} \sum_{w \in W^0(s)} \frac{\chi(w_0 w)}{\det_E(1 - q w^{-1} w_0^{-1})}.$$

Now if we denote $E_0 = X_\bullet(T_{w_0}) \otimes \mathbb{Q}$, and let $f_1, \ldots, f_l$ be the exponents of the reductive group $C^0(s)$, we have

$$R_{\chi_E}(s) = \prod (1 - \varepsilon_i q^{f_i + 1}) \cdot \frac{1}{|W^0(s)|} \cdot \sum_{w \in W^0(s)} \frac{\chi_{E_0}(Fw)}{\det_{E_0}(1 - qFw)}.$$

By Lemma (2.3), and (2.8), we then get

PROPOSITION 3.5. $\chi_\Pi(s) = R_{\chi_E}(s) = \mathrm{Tr}(F|M(s))$, *where $M(s)$ is the motive of the reductive group $C^0(s)$.*

EXAMPLES. If $s = 1$, then $\chi_\Pi(1) = \sum q^{e_i}$. If s is regular semi-simple, then $\chi_\Pi(s) = \chi(w_0)$, as shown in [Lu].

Remarks. We can regard the above formula as a q-deformation of (2.7). Indeed, if we set $q = 1$ on the right-hand side of the above proposition, we get $\mathrm{Tr}(F|V_{C^0(s)})$. On the other hand, it is 'reasonable' to regard the reflection representation E_W of W as a degeneration, as $q \to 1$, of the reflection representation Π. Viewing

$$\{\text{semi-simple classes in } G^F\} \to \{\text{conjugacy classes in } W\}$$

$$(s) \mapsto (w_0)$$

as a way of deforming semi-simple classes in G^F to classes in W, we see that the left-hand side of the above Proposition becomes: $\mathrm{Tr}(w_0|E) = \mathrm{Tr}(F|E_0)$. So Proposition (3.5) becomes, on letting $q = 1$, $\mathrm{Tr}(F|E_0) = \mathrm{Tr}(F|V_{C^0(s)})$, which is (2.7) for the reductive group $C^0(s)$.

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4. Generic Representations

We continue to assume that $k = \mathbb{F}_q$ is a finite field. The remainder of this paper will be devoted to the study of Howe correspondence. Before that, we review some material about generic representations. In this section, G will denote any connected reductive group over k, which has connected center. Then any Levi subgroup of G will also have connected center [Ca, p. 260], and G^F has a unique Gelfand–Graev representation Δ_G. Recall that $\Delta_G \cong \mathrm{Ind}_U^G \psi$, where ψ is a character in general position of a maximal unipotent subgroup U of G. An irreducible representation of G^F is said to be generic if it is a component of Δ_G, and then it occurs with multiplicity one. Moreover, G^F has exactly q^l generic characters, where l is the rank of G.

The following is a result of Rodier [Ca, p. 261]

PROPOSITION 4.1. *Let $P = L \cdot U$ be an F-stable standard parabolic subgroup. For any character χ_L of L^F, denote by χ_L also its lift to P^F. Then, $\langle \mathrm{Ind}_{P^F}^{G^F} \chi_L, \Delta_G \rangle = \langle \chi_L, \Delta_L \rangle$.*

COROLLARY 4.2. (1) *If χ_L is generic, then $\mathrm{Ind}_{P^F}^{G^F} \chi_L$ has a unique generic summand.*

(2) *If χ is a generic character of G^F, then any irreducible component of χ^U is also generic.*

We shall denote the unique generic summand in the above corollary by $\widetilde{\chi_L}^G$. By Corollary (4.2), $\chi_L \mapsto \widetilde{\chi_L}^G$ defines a map (not injective)

$$\{\text{generic characters of } L^F\} \to \{\text{generic characters of } G^F\}. \tag{4.5}$$

We can understand this map using the parametrization of Lusztig. From [D-L], we know that the irreducible characters of G^F can be partitioned into geometric conjugacy classes, which is in turn parametrized by semi-simple conjugacy classes in G^{*F^*}, where (G^*, F^*) is the dual group of (G, F). In each geometric conjugacy class, there is a unique generic character. The generic character corresponding to the class of s^* is denoted $\chi(s^*)$. For example, $\chi(1)$ is the Steinberg representation of G^F.

If L is a Levi factor of G, then there is a Levi factor $L^* \subset G^*$ which is in duality with L. Now if χ_L is generic with parameter $s^* \in L^{*F^*}$, then the parameter of $\widetilde{\chi_L}^G$ is just the class of s^* in G^{*F^*}.

Note that the generic characters are characterized by the fact that their dimensions have the form

$$q^N + (\text{terms involving lower powers of } q), \tag{4.6}$$

where N is the number of positive roots of G. Hence, the generic characters are the biggest representations of G^F in that their dimensions grow the fastest as q

becomes large. Indeed, if $s^* \in G^{*F^*}$, and $G^*_{s^*}$ the centralizer of s^* in G^*, then the dimension of $\chi(s^*)$ is given by

$$\dim(\chi(s^*)) = q^{\dim(U_{G^*_{s^*}})} \cdot \frac{\det(F-1|M^\vee_{G^*}(1))}{\det(F-1|M^\vee_{G^*_{s^*}}(1))}, \tag{4.7}$$

where $U_{G^*_{s^*}}$ is the maximal unipotent subgroup of $G^*_{s^*}$, and $M^\vee$ is the motive dual to M.

5. Dual Pairs

Henceforth, we consider the split adjoint groups of type E_6 and E_7. In [MS], the following dual pairs were constructed

$$G_2 \times \mathrm{PGL}_3 \subset E_6, \qquad G_2 \times \mathrm{PGSp}_6 \subset E_7, \tag{5.1}$$

and the representation correspondence arising from the restriction of the minimal representation of G over a non-Archimedean local field was studied. The analogue of the minimal representation in the finite field situation is exactly the reflection representation Π. Indeed, the dimension of Π is the smallest among all nontrivial irreducible representations of G^F (that is, those of dimension greater than 1). Hence, we shall be interested in the restriction of Π to these pairs. In this section, we describe the results we expect.

Consider, for example, the pair $G_2 \times \mathrm{PGL}_3$. Note that there is a natural inclusion of dual groups

$$\mathrm{SL}_3 \to G_2. \tag{5.2}$$

Using the parametrization of generic characters discussed in the previous section, we see that we have a map

$$\begin{aligned} &i_*\colon \{\text{generic characters of } \mathrm{PGL}_3\} \to \{\text{generic characters of } G_2\},\\ &\chi(s)_{\mathrm{PGL}_3} \mapsto \chi(i(s))_{G_2}. \end{aligned} \tag{5.3}$$

Note that this map is neither surjective nor injective. Indeed it is usually two-to-one; if $\chi(s)^* = \chi(s^{-1})$ denotes the contragradient character of $\chi(s)$, then $\chi(i(s)) = \chi(i(s^{-1})) = \chi(i(s))^*$.

Now we guess that

$$\langle \chi(i(s))_{G_2} \otimes \chi(s)_{\mathrm{PGL}_3}, \Pi\rangle_{G_2 \times \mathrm{PGL}_3} = 1, \tag{5.4}$$

for any generic character $\chi(s)_{\mathrm{PGL}_3}$ of PGL_3. Notice that PGL_3 has q^2 generic characters, and their dimensions, given by (4.7), have the form

$$q^3 + (\text{terms involving lower powers of } q),$$

whereas the generic characters of G_2 have dimensions:

$$q^6 + (\text{terms involving lower powers of } q).$$

So if our guess is correct, then we would have accounted for a subspace of dimension

$$q^{11} + (\text{terms involving lower powers of } q)$$

of Π, which has dimension $q^{11} + q^8 + q^7 + q^5 + q^4 + q$. Hence we would have accounted for a large piece of Π. Indeed, one can check that the dimension of the space unaccounted has leading term $2q^9$.

For a generic character χ_{PGL_3}, we let

$$\Theta(\chi_{\mathrm{PGL}_3}) = \{\chi_{G_2} : \langle \Pi, \chi_{\mathrm{PGL}_3} \otimes \chi_{G_2}\rangle \neq 0\}. \tag{5.5}$$

Here, the representations χ_{G_2} are counted with multiplicities. Also, let $\Theta_{\mathrm{gen}}(\chi_{\mathrm{PGL}_3})$ be the subset of generic representations in $\Theta(\chi_{\mathrm{PGL}_3})$. Thus, (5.4) says that

$$\Theta_{\mathrm{gen}}(\chi(s)_{\mathrm{PGL}_3}) = \{\chi(i(s))_{G_2}\}. \tag{5.6}$$

Similarly, in the case E_7, we have a natural inclusion of dual groups

$$i : G_2 \to \mathrm{Spin}_7. \tag{5.7}$$

Again, we guess that

$$\Theta_{\mathrm{gen}}(\chi(s)_{G_2}) = \{\chi(i(s))_{\mathrm{PGSp}_6}\}, \tag{5.8}$$

for every semi-simple class (s) in G_2, where the set $\Theta_{\mathrm{gen}}(\chi(s)_{G_2})$ is similarly defined.

Notice that G_2 has q^2 generic characters of dimension $(q^6 + \cdots)$, whereas the generic characters of PGSp_6 have dimension $(q^9 + \cdots)$. Hence, if our guess is true, we would have accounted for a subspace of dimension $(q^{17} + \cdots)$ in Π, which has dimension $q^{17} + q^{13} + q^{11} + q^9 + q^7 + q^5 + q$.

Remark. The inclusion (5.7) is realized by regarding G_2 as the stabilizer of a nonisotropic vector in the eight-dimensional Spin representation of Spin_7. If we work over an algebraic closure $\bar{k}$ of k, all such embeddings, which a priori depend on the choice of the nonisotropic vector, are in fact conjugate in Spin_7. This is, however, not true over k, since the norm $\langle v, v\rangle$ of the nonisotropic vector v is an invariant, with values in $k^\times/k^{\times 2}$, of the conjugacy class of an embedding. Fortunately, for our purposes, this does not matter, since any two semi-simple elements of $\mathrm{Spin}_7(k)$ which are conjugate over $\bar{k}$ are already conjugate over k.

The remainder of this paper is devoted to proving (5.6) and (5.8). We shall first show that, in each case, $\Theta_{\mathrm{gen}}(\chi(s))$ is a singleton set, by computing Whittaker vectors. We then proceed to check that it is really what we expect. This is accomplished for noncuspidal representations by computing Jacquet functors. The case of cuspidal generic representations is settled by using base change, or Shintani descent.

6. Restriction to Maximal Parabolics

Henceforth, we assume that $p \geqslant 5$, so that p is a good prime for $G = E_6$ or E_7, and the Killing form is nondegenerate. For the purposes of computing Whittaker vectors and Jacquet functors, we need to know the restriction of Π to various maximal parabolic subgroups of E_6 and E_7.

In each case, there is a maximal parabolic subgroup $P_0 = M_0 \ltimes N_0$, whose unipotent radical N_0 is abelian. In the case of E_6, the derived group of M_0 is of type D_5, and N_0 is a Spin representation of D_5 via the adjoint action of M_0. In the case of E_7, the derived group of M_0 is of type E_6 and N_0 is the 27-dimensional representation of E_6 via adjoint action, and we can identify N_0 with the exceptional Jordan algebra over $k = \mathbb{F}_q$. Note that this Jordan algebra is split.

Since N_0 is abelian, we can identify its character group $N_0^\vee$ with $\overline{N}_0$, the unipotent radical of the opposite parabolic $\overline{P}_0$, as follows. If

$$\langle , \rangle : N_0 \times \overline{N}_0 \to k \tag{6.1}$$

is the pairing induced by the Killing form, which is nondegenerate by assumption, and $\psi : k \to \mathbb{C}^*$ is a fixed nontrivial additive character, then

$$\psi_x = \psi(\langle -, x\rangle) : N_0 \to \mathbb{C}^* \tag{6.2}$$

is a character of N_0, and the map $x \mapsto \psi_x$ gives an identification of $N_0^\vee$ with $\overline{N}_0$. Note that the minimal nontrivial M_0-orbit ω in $\overline{N}_0$ is the orbit of the highest weight vector.

The following proposition, which describes the restriction of Π to P_0, is the finite field analogue of Theorem (1.1) in [MS].

PROPOSITION 6.3. *$\Pi \downarrow_{P_0} \cong \Pi^{N_0} \oplus V$ where $\Pi^{N_0} \cong 1 \oplus \Pi(M_0)$ and as a N_0-module $V \cong \bigoplus_{x\in\omega} \mathbb{C}\psi_x$ and M_0 acts on V by its permutation action on ω.*

Now consider the Heisenberg maximal parabolic $P = M \cdot N$, so-called because N is a Heisenberg group. P corresponds to the unique vertex α joined to the negative of the highest root in the extended Dynkin diagram. So for E_6, M has derived group of type A_5, and for E_7, it is of type D_6.

Let Z be the one-dimensional center of N, and $\overline{Z}$ that of $\overline{N}$. Then the Killing form induces a nondegenerate pairing

$$\langle\, , \,\rangle : N/Z \times \overline{N}/\overline{Z} \to k. \tag{6.4}$$

With the fixed character ψ, we can identify the character group of N/Z with $\overline{N}/\overline{Z}$ as before. Let Ω be the minimal nontrivial M-orbit in $\overline{N}/\overline{Z}$.

The following Proposition is the finite field analogue of Theorem (6.1) in [MS].

PROPOSITION 6.5. *$\Pi^Z \cong \Pi^N \oplus V$ where $\Pi^N \cong 1 \oplus \Pi(M)$ and, as a N/Z-module $V \cong \bigoplus_{x\in\Omega} \mathbb{C}\psi_x$ and M acts on V via its permutation action on Ω.*

7. Whittaker Vectors

In this section, let U be the unipotent radical of a Borel subgroup of G_2 (respectively PGSp_6), and let ψ be a generic character of U. We shall compute $\Pi_{U,\psi}$ for E_6 (respectively E_7). It is a pleasure to thank G. Savin for his suggestion to do this computation.

The main result of this section is

THEOREM 7.1. (1) *If Π is the reflection representation of E_6, and ψ a generic character of U, then $\Pi_{U,\psi}$ is the Gelfand–Graev representation of* PGL_3.

(2) *Similarly, if Π is the reflection representation of E_7, and ψ a generic character of U, then $\Pi_{U,\psi}$ is the Gelfand–Graev representation of G_2.*

Proof. Let us consider the case of E_6 first. Let $P_2 = L_2U_2$ be the Heisenberg parabolic of G_2. Write $U = U_2 \rtimes U'$, with $U' \cong k$. Then we can denote ψ by $\psi = (\phi, \varphi)$, with $\phi := \psi|_{U_2}$, and $\varphi := \psi|_{U'}$.

Now it was shown in [MS] that there is an embedding of the dual pair $G_2 \times \mathrm{PGL}_3$ such that $G_2 \cap P = P_2$. Then ϕ is a degenerate character of U_2 in the smallest L_2 orbit. We first compute $\Pi_{U_2,\phi}$. By Proposition (6.5), we need to compute $V_{U_2,\phi}$. The same considerations as in [GrS, Prop. 2.8, Sect. VI] show that $\Pi_{U_2,\phi} = \mathbb{C}[\Omega_\phi]$, as a representation of $\mathrm{PGL}_3 \times L_{2,\phi}$, where Ω_ϕ is the set of nilpotent 3×3 matrices, and $L_{2,\phi}$ is the stabilizer of ϕ in L_2. In particular, $U' \subset L_{2,\phi}$.

To finish the computation, we need to know the action of $U' \cong k$ on Ω_ϕ. If $\lambda \in k$ and $z \in \Omega_\phi$, then the action of λ on z is given by $\lambda\colon z \mapsto z + 2\lambda e \times (z \times z)$, where e is the identity matrix, and $z \times z$ is the adjoint matrix of z. Hence, we see that U' fixes each z of rank less than 2, and acts freely on the rank 2 elements.

Now the action of PGL_3 on Ω_ϕ is by conjugation, and so PGL_3 has 3 orbits on Ω_ϕ, characterized by rank. Since U' acts trivially on the elements of rank less than 2, we have $\Pi_{U,\psi} = \mathbb{C}[\{z \in \Omega_\phi\colon \mathrm{rank}(z) = 2\}]_{U',\varphi}$.

Let $z \in \Omega_\phi$ be given by

$$z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the U'-orbit of z are elements of the form

$$\begin{pmatrix} 0 & 1 & \lambda \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and the stabilizer of z in PGL_3 is

$$H = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : b, c \in k \right\}.$$

So as a representation of $\mathrm{PGL}_3 \times U'$,

$$\mathbb{C}[\{z \in \Omega_\phi : \mathrm{rank}(z) = 2\}] \cong \mathrm{Ind}_{U_{\mathrm{PGL}_3} \times k}^{\mathrm{PGL}_3 \times k} \mathbb{C}[k],$$

where the action of k is by right translation, and the action of U_{PGL_3}, the group of upper triangular unipotent matrices, factors through the quotient

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto a - b \in k.$$

Hence, we deduce that

$$\Pi_{U,\psi} = \mathrm{Ind}_{U_{\mathrm{PGL}_3}}^{\mathrm{PGL}_3} \mathbb{C}[k]_{U',\varphi} = \mathrm{Ind}_{U_{\mathrm{PGL}_3}}^{\mathrm{PGL}_3} \varphi_0,$$

where φ_0 is the character of U_{PGL_3} given by

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \varphi(b - a).$$

Hence, $\Pi_{U,\psi}$ is indeed the Gelfand–Graev representation of PGL_3.

Now we consider the case of E_7. Let P_0 be the maximal parabolic of E_7 considered in the previous section. Then $P_0 \cap \mathrm{PGSp}_6 = \mathrm{GL}_3 \ltimes U_0$ is the Siegel parabolic of PGSp_6, and U_0 can be identified with the set of all 6×6 matrices of the form

$$\begin{pmatrix} I_3 & B \\ 0 & I_3 \end{pmatrix}, \quad B^t = B.$$

As before, write $U = U_0 \rtimes U'$, where U' is the unipotent radical of GL_3. Also, let $\phi := \psi|_{U_0}$, and $\varphi := \psi|_{U'}$.

Note that

$$\phi\colon \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \mapsto \phi_0(B_{33}),$$

where ϕ_0 is a nontrivial character of k.

By Proposition (6.3), we see that $\Pi_{U,\psi} = \mathbb{C}[\omega]_{U,\psi}$.

We first compute $\mathbb{C}[\omega]_{U_0,\phi}$. As before, we have $\Pi_{U_0,\phi} = \mathbb{C}[\omega_\phi]$, where ω_ϕ is the set of all X in ω of the form

$$X = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 1 \end{pmatrix},$$

with x, y, z in the octonion algebra Θ satisfying

$$\mathrm{tr}(x) = \mathrm{tr}(y) = \mathrm{tr}(z) = 0, \quad x^2 = y^2 = z^2 = 0,$$

$$xz = yz = 0, \quad xy = -z.$$

Now we need to consider the action of U' on ω_ϕ. Note that U' is a Heisenberg group of dimension 3. The U'-orbit of an X as above consists of all elements of ω_ϕ of the form

$$\begin{pmatrix} 1 & z & -(y+cz) \\ & 0 & x+ay+bz \\ & & 0 \end{pmatrix}, \quad a, b, c \in k.$$

Now, $G_2 \times U'$ has various orbits in ω_ϕ. The generic orbit is the one consisting of all those X's such that the (x, y, z) of X span a three-dimensional space. Call this orbit $\mathcal{O}$. Then U' acts freely on $\mathcal{O}$. One checks that only $\mathbb{C}[\mathcal{O}]$ will contribute to the space of Whittaker vectors, i.e. $\Pi_{U,\psi} = \mathbb{C}[\mathcal{O}]_{U',\varphi}$. Now as a representation of $G_2 \times U'$, $\mathbb{C}[\mathcal{O}] = \mathrm{Ind}_{U_{G_2}\times U'}^{G_2\times U'} \mathbb{C}[U']$. The action of U_{G_2} on $\mathbb{C}[\mathcal{O}]$ is as follows. Note that U_{G_2} is a 4-step nilpotent group

$$U_{G_2} \supset U_{G_2}^{(1)} \supset U_{G_2}^{(2)} \supset U_{G_2}^{(3)} \supset \{1\}$$

and $U_{G_2}/U_{G_2}^{(2)} \cong U'$. Hence the action of U_{G_2} is via the quotient: $U_{G_2} \to U_{G_2}/U_{G_2}^{(2)}$. Hence, we see that $\mathbb{C}[\mathcal{O}]_{U',\varphi} \cong \mathrm{Ind}_{U_{G_2}}^{G_2} \varphi^{-1}$, which is the Gelfand–Graev representation of G_2. □

COROLLARY 7.2. (1) *For any generic representation* π *of* PGL_3, $\Theta_{\mathrm{gen}}(\pi)$ *is a singleton set.*

(2) *For any generic representation* π *of* G_2, $\Theta_{\mathrm{gen}}(\pi)$ *is a singleton set.*

COROLLARY 7.3. (1) *If* σ *is a nongeneric representation of* PGL_3, *then* $\Theta_{\mathrm{gen}}(\sigma) = \emptyset$.

(2) *If* σ *is a nongeneric representation of* G_2, *then* $\Theta_{\mathrm{gen}}(\sigma) = \emptyset$.

8. Jacquet Functors

By the previous corollaries, it remains to check that the correspondence of the parameters of generic representations is given by the natural inclusions of dual groups. For noncuspidal representations, this can be checked by computing Jacquet functors. Most of what we need have been computed in [MS]; so we begin by transferring their results to the finite field situation.

Let us denote our two dual pairs by $G_2 \times H$, and let P_0 be the maximal parabolic introduced in Section 6. Recall that the unipotent radical of P_0 is abelian. Also, we have

$$(G_2 \times H) \cap P_0 = G_2 \times Q_0, \tag{8.1}$$

where $Q_0 = L_0 \cdot U_0$ is a maximal parabolic of H. In the case of E_6, $L_0 = \mathrm{GL}_2$, and for E_7, Q_0 is the Siegel parabolic of PGSp_6, so that $L_0 = \mathrm{GL}_3$. We also denote by $P_1 = L_1 \cdot U_1$ (respectively $P_2 = L_2 \cdot U_2$) the maximal parabolic of G_2 whose Levi factor L_1 (respectively L_2) is spanned by the long (respectively short) simple root.

The following Propositions give the structure of the $G_2 \times L_0$-module Π^{U_0}, and are the finite field analogues of Theorem (4.3) and Theorem (5.3) in [MS].

PROPOSITION 8.2. *For* E_6

$$\Pi^{U_0} \cong \Pi^{N_0} \oplus \mathrm{Ind}_{P_2\times\mathrm{GL}_2}^{G_2\times\mathrm{GL}_2}\mathbb{C}[\mathrm{GL}_2] \oplus \mathrm{Ind}_{P_1\times B}^{G_2\times\mathrm{GL}_2}\mathbb{C}[\mathrm{GL}_1].$$

Here, B *is the Borel subgroup of* GL_2, *and the actions of* P_1, P_2 *and* B *on the appropriate spaces are via the quotients*

$$P_2 \twoheadrightarrow L_2 \cong \mathrm{GL}_2,$$

$$P_1 \twoheadrightarrow L_1 \cong \mathrm{GL}_2 \to \mathrm{GL}_1(\text{determinant map}),$$

$$B \twoheadrightarrow \mathrm{GL}_1 \times \mathrm{GL}_1 \twoheadrightarrow \mathrm{GL}_1(\text{projection onto first factor}).$$

PROPOSITION 8.3. *For* E_7

$$\Pi^{U_0} \cong \Pi^{N_0} \oplus \mathrm{Ind}_{P_2\times Q_2}^{G_2\times\mathrm{GL}_3}\mathbb{C}[\mathrm{GL}_2] \oplus \mathrm{Ind}_{P_1\times Q_1}^{G_2\times\mathrm{GL}_3}\mathbb{C}[\mathrm{GL}_1].$$

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Here, $Q_1 = M_1 \cdot V_1$ (respectively $Q_2 = M_2 \cdot V_2$) is the maximal parabolic of GL_3 *which stabilizes a line (respectively a plane) in k^3, and the actions of P_i and Q_i on the appropriate spaces are via the quotients*

$$P_2 \twoheadrightarrow L_2,$$

$$P_1 \twoheadrightarrow L_1 \longrightarrow \mathrm{GL}_1(\text{determinant map}),$$

$$Q_2 \twoheadrightarrow M_2 \cong \mathrm{GL}_2 \times \mathrm{GL}_1 \longrightarrow \mathrm{GL}_2(\text{projection onto first factor}),$$

$$Q_1 \twoheadrightarrow M_1 \cong \mathrm{GL}_1 \times \mathrm{GL}_2 \longrightarrow \mathrm{GL}_1(\text{projection onto first factor}).$$

Now we consider the Heisenberg parabolic P. It was shown in [MS] that there is an embedding of the dual pairs such that

$$(G_2 \times H) \cap P = P_2 \times H, \tag{8.4}$$

where P_2 is as defined before and is the Heisenberg parabolic of G_2. The following Proposition gives the structure of the $L_2 \times H$-module Π^{U_2} and is the finite field analogue of Theorem (7.6) in [MS].

PROPOSITION 8.5. *For E_7*

$$\Pi^{U_2} \cong \Pi^N \oplus \mathrm{Ind}_{L_2 \times Q_2'}^{L_2 \times \mathrm{PGSp}_6} \mathbb{C}[\mathrm{GL}_2] \oplus \mathrm{Ind}_{B \times Q}^{L_2 \times \mathrm{PGSp}_6} \mathbb{C}[\mathrm{GL}_1].$$

Here, Q is the maximal parabolic subgroup of PGSp_6 *corresponding to the middle vertex in the Dynkin diagram, and Q_2' is the minimal parabolic of* PGSp_6 *which intersects the Levi factor of the Siegel parabolic in Q_2.*

For our purposes, we also need to compute the Jacquet functor of $\Pi(E_7)$ with respect to the maximal parabolic $P_1 = L_1 U_1$ of G_2. Over a p-adic field, this is computed in Proposition 6.8 in [SG]. The result over finite fields can be checked along similar lines, but to state it, we need to introduce some more notations.

Let P' be the maximal parabolic of E_7 corresponding to the unique vertex β joined to α in the Dynkin diagram. Then

$$(G_2 \times \mathrm{PGSp}_6) \cap P' = P_1 \times \mathrm{PGSp}_6.$$

Let $Q = L \cdot U$ be the maximal parabolic subgroup of PGSp_6 corresponding to the middle vertex in the Dynkin diagram of type C_3. Hence, its Levi factor is $L \cong (\mathrm{GL}_2 \times \mathrm{GL}_2)/k^\times$, with $k^\times$ embedded via: $a \mapsto (a, a^{-1})$. Let

$$R_0 = \{(g_1, g_2, g_3) \in L_1 \times \mathrm{GL}_2 \times \mathrm{GL}_2 \colon \det(g_1 g_2 g_3) = 1\}.$$

Then R_0 has a natural Weil representation W which can be realized on $\mathbb{C}[M_2(k)]$, where $M_2(k)$ is the space of 2×2 matrices over k, which we describe below. Let

$\{\cdot,\cdot\}$ be the standard nondegenerate symplectic form on k^2, and let $V = k^2 \otimes k^2 \otimes k^2$ with symplectic form

$$\{u_1 \otimes u_2 \otimes u_3, v_1 \otimes v_2 \otimes v_3\} = \prod_{i=1}^{3} \{u_i, v_i\}.$$

Then $R_0 \to \mathrm{Sp}(V)$ under the natural action of GL_2 on k^2. Now, it is well-known that $\mathrm{Sp}(V)$ has a natural Weil representation [Ge], which can be realized on the space of $\mathbb{C}$-valued functions on the maximal isotropic subspace $ke_1 \otimes k^2 \otimes k^2$. Then the representation W is the pullback of the Weil representation of $\mathrm{Sp}(V)$ to R_0. In particular, the action of $\mathrm{SL}_2 \subset L_1$ is via the usual Weil representation formulas [Ge], whereas the action of the subgroup

$$R_1 = \{(g,h): \det(gh) = 1\} \subset \mathrm{GL}_2 \times \mathrm{GL}_2$$

is geometric, and we see that W is actually a representation of the quotient

$$R = \{(g,h) \in L_1 \times L: \det(gh) = 1\}$$

of R_0. Let

$$\tilde{W} = \mathrm{Ind}_R^{L_1 \times L} W. \tag{8.6}$$

Then the following proposition is the finite field analogue of Proposition 6.8 of [SG].

PROPOSITION 8.7. *In the case E_7*

$$\Pi^{U_1} \cong \Pi^{N'} \oplus \mathrm{Ind}_{B \times Q}^{\mathrm{GL}_2 \times \mathrm{PGSp}_6} \mathbb{C}[\mathrm{GL}_1] \oplus \mathrm{Ind}_{\mathrm{GL}_2 \times Q}^{\mathrm{GL}_2 \times \mathrm{PGSp}_6} \tilde{W}.$$

9. Shintani Descent

The results of the last section will allow us to determine the correspondence for non-cuspidal representations. Before doing that, we review the results about Shintani descent that we need for the correspondence of cuspidal representations. We refer the reader to the article of Digne [D] for a quick introduction.

Let $k_m := \mathbb{F}_{q^m}$ and $G(q^m) := G^{F^m}$. There is a norm map $\mathbb{N}_m: G(q^m) \to G(q)$, which induces a bijection of F-conjugacy classes in $G(q^m)$ and conjugacy classes in $G(q)$. Hence, given any F-class function ψ of $G(q^m)$, $\mathrm{Sh}_m(\psi) := \psi \circ \mathbb{N}_m^{-1}$ is a class function on $G(q)$, and Sh_m is an isomorphism of the vector space of F-class function on $G(q^m)$ and the vector space of class functions on $G(q)$. Moreover, it is an isometry for the natural inner products on the two vector spaces

$$\langle \psi_1, \psi_2 \rangle_{G(q^m)} = \langle \mathrm{Sh}_m(\psi_1), \mathrm{Sh}_m(\psi_2) \rangle_{G(q)}.$$

We will call ψ the base change of $\mathrm{Sh}_m(\psi)$ or equivalently, $\mathrm{Sh}_m(\psi)$ the Shintani descent of ψ. Note that our Sh_m is equal to the map $i \circ \mathrm{Sh}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$ in [D].

If χ is (the character of) an F-stable representation of $G(q^m)$, then χ extends (nonuniquely) to a representation of $G(q^m) \rtimes \langle F \rangle$, denoted $\widetilde{\chi}$. If χ is irreducible, then $\widetilde{\chi}$ is well-determined up to an m-th root of unity. In any case, the function $g \mapsto \widetilde{\chi}(Fg)$ is an F-class function on $G(q^m)$, and so we can consider $\mathrm{Sh}_m(\widetilde{\chi}(F\cdot))$. For ease of notation, we shall denote this class function on $G(q)$ simply by $\mathrm{Sh}_m(\widetilde{\chi})$.

Now, the results we need are summarized in the the following Propositions, which are special cases of results of Digne–Michel [D] and Digne [D2] respectively.

PROPOSITION 9.1. *There is an extension $\widetilde{\Pi_m}$ of the reflection representation Π_m of $G(q^m)$ (for G of type E_n) such that* $\mathrm{Sh}_m(\widetilde{\Pi_m}) = \Pi$.

Remark. We note that Π_m is F-stable by the characterization of the reflection representation in Section 3.

PROPOSITION 9.2. *Assume that p is a good prime for G, and $(m, p) = 1$. Let θ be a character of $T(q)$, where T is an F-stable maximal torus of G. Let $R(\theta)_m := R^{G(q^m)}_{T(q^m)}(\theta \circ \mathbb{N}_m)$, a virtual character. We shall write $R(\theta)$ for $R(\theta)_1$. Then there is an extension of $R(\theta)_m$ to a virtual character $\widetilde{R(\theta)_m}$ of $G(q^m) \rtimes \langle F \rangle$ such that* $\mathrm{Sh}_m\left(\widetilde{R(\theta)_m}\right) = R(\theta)$.

Remark. In our applications, we shall only need to consider $m = 2$ or 3. Hence $p \geqslant 5$ will suffice.

10. The Dual Pair $G_2 \times \mathrm{PGL}_3$

In this section, we consider the restriction of Π to $G_2 \times \mathrm{PGL}_3 \subset E_6$. We shall first show that (5.4) holds for noncuspidal generic characters of PGL_3. Suppose $\chi(s)_{\mathrm{PGL}_3}$ is a noncuspidal generic character. Then we can assume that $s \in L_0^* \cong \mathrm{GL}_2$, so that

$$\chi(s)_{\mathrm{PGL}_3} \hookrightarrow \mathrm{Ind}_{Q_0}^{\mathrm{PGL}_3} \chi(s)_{\mathrm{GL}_2}.$$

By Frobenius reciprocity,

$$\langle \Pi, \chi(i(s))_{G_2} \otimes \mathrm{Ind}_{Q_0}^{\mathrm{PGL}_3} \chi(s)_{\mathrm{GL}_2} \rangle = \langle \Pi^{U_0}, \chi(i(s))_{G_2} \otimes \chi(s)_{\mathrm{GL}_2} \rangle.$$

This is nonzero, since by Proposition 8.2, $\chi(i(s))_{G_2} \otimes \chi(s)_{\mathrm{GL}_2}$ occurs in the second summand of Π^{U_0}. By Corollaries 7.2 and 7.3, we have

PROPOSITION 10.1. *For any noncuspidal generic representation $\chi(s)_{\mathrm{PGL}_3}$*

$$\Theta_{\mathrm{gen}}(\chi(s)_{\mathrm{PGL}_3}) = \{\chi(i(s))_{G_2}\}.$$

EXAMPLE. Since the Steinberg character has parameter the trivial class 1 $\Theta_{\mathrm{gen}}(\mathrm{St}_{\mathrm{PGL}_3}) = \{\mathrm{St}_{G_2}\}$.

Now we have

THEOREM 10.2. *For* $p \geqslant 5$, $\Theta_{\mathrm{gen}}(\chi(s)_{\mathrm{PGL}_3}) = \{\chi(i(s))_{G_2}\}$.

Proof. After Proposition 10.1, it remains to prove this for cuspidal representations. Let $G = \mathrm{PGL}_3 \times G_2$. Note that if $\chi(s)_{\mathrm{PGL}_3}$ is cuspidal, then s must be regular in SL_3, the dual group of PGL_3. So any cuspidal representation of PGL_3 is of the form $R_T(\theta)$, where $T(q) = k_3^\times / k^\times$, and θ is a regular character of $T(q)$. This has parameter

$$s = \begin{pmatrix} a & & \\ & a^q & \\ & & a^{q^2} \end{pmatrix} \in \mathrm{SL}_3(q^3),$$

with all elements on the diagonal distinct. Now $i(s) \in G_2$ is also regular and is contained in an elliptic torus. So it corresponds to a cuspidal generic representation $R_{T'}(\theta')$ of G_2. Hence, we need to show that $\langle \Pi, R(\theta) \otimes R(\theta') \rangle = 1$.

Now, by Propositions 9.1 and 9.2, we can find extensions such that

$$\mathrm{Sh}_3(\widetilde{\Pi_3}) = \Pi, \qquad \mathrm{Sh}_3(\widetilde{R(\theta)_3}) = R(\theta), \qquad \mathrm{Sh}_3(\widetilde{R(\theta')_3}) = R(\theta').$$

Note that since T and T' both split over k_3, $R(\theta)_3$ and $R(\theta')_3$ are irreducible principal series representations of $\mathrm{PGL}_3(q^3)$ and $G_2(q^3)$, respectively, with parameters s and $i(s)$ in $\mathrm{SL}_3(q^3)$ and $G_2(q^3)$ respectively. Hence, by Proposition 10.1, their tensor product occurs in Π_3 with multiplicity 1. Let us denote $R(\theta)_3 \otimes R(\theta')_3$ by $\Pi(\theta)_3$, for simplicity. Note that the extension of Π_3 above induces an extension of $\Pi(\theta)_3$, but this may or may not be the same as the extension of $\Pi(\theta)_3$ chosen above.

Now, we have

$$\begin{aligned} \sum_{i=0}^{2} \tfrac{1}{3} \langle \widetilde{\Pi}_3(F^i \cdot), \widetilde{\Pi(\theta)_3}(F^i \cdot) \rangle_{G(q^3)} &= \langle \widetilde{\Pi}, \widetilde{\Pi(\theta)} \rangle_{G(q^3) \rtimes \langle F \rangle} \\ &= 0 \text{ or } 1. \end{aligned}$$

But we know, by Proposition 10.1, that $\langle \widetilde{\Pi}_3, \widetilde{\Pi(\theta)_3} \rangle_{G(q^3)} = 1$. Moreover, since Sh_m is an isometry, for $i = 1$ or 2:

$$\langle \widetilde{\Pi}_3(F^i \cdot), \widetilde{\Pi(\theta)_3}(F^i \cdot) \rangle_{G(q^3)} = \langle \Pi, \Pi(\theta) \rangle_{G(q)} \in \mathbb{N}.$$

Thus we see that the only possibility is that $\langle \Pi, \Pi(\theta) \rangle = 1$. This completes the proof of the theorem. □

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11. The Dual Pair $G_2 \times \mathrm{PGSp}_6$

In this section, we consider the dual pair $G_2 \times \mathrm{PGSp}_6$ in E_7. As before, we have the set $\Theta_{\mathrm{gen}}(\chi_{G_2})$, and we shall prove (5.8). Before that, we prove a lemma concerning the Weil representation.

LEMMA 11.1. *Let $\tilde{W}$ denote the representation $\mathrm{Ind}_{R_0}^{\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2} W$ as in* (8.6), *with $R_0 = \{(g_1, g_2, g_3)\colon \det(g_1g_2g_3) = 1\}$. Then for any generic representation π of* GL_2 $\langle \tilde{W}, \pi \otimes \pi \otimes \pi\rangle = 1$.

Proof. First, we claim that to prove the lemma, it suffices to show that as a representation of $\mathrm{GL}_2 \times \mathrm{GL}_2$ (the last two copies),

$$\langle \tilde{W}, \pi \otimes \sigma\rangle_{\mathrm{GL}_2\times\mathrm{GL}_2} = \begin{cases} \dim(\pi), & \text{if } \sigma = \pi, \\ 0, & \text{if } \sigma \neq \pi, \end{cases}$$

where σ and π are generic representations of GL_2. But this is clear, since we could use the maximal isotropic subspace $k^2 \otimes ke_1 \otimes k^2$, or $k^2 \otimes k^2 \otimes ke_1$ to define W.

Now, one sees that, as a representation of $\mathrm{GL}_2 \times \mathrm{GL}_2$ (the last two copies)

$$\begin{aligned} \tilde{W} &\cong \mathrm{Ind}_{R_1}^{\mathrm{GL}_2\times\mathrm{GL}_2}\mathbb{C}[M_2(k)] \\ &\cong (\mathrm{Ind}_{R_1}^{\mathrm{GL}_2\times\mathrm{GL}_2}1) \otimes \mathbb{C}[M_2(k)] \\ &\cong (\oplus_{\psi\in k^\vee}\mathbb{C}\psi) \otimes \mathbb{C}[M_2(k)]. \end{aligned}$$

From this, the lemma follows easily. □

Now we can determine the correspondence for noncuspidal representations

PROPOSITION 11.2. *If $\chi(s)_{G_2}$ is a noncuspidal generic representation of G_2, then $\Theta_{\mathrm{gen}}(\chi(s)_{G_2}) = \{\chi(i(s))_{\mathrm{PGSp}_6}\}$.*

Proof. The argument is essentially the same as in the proof of Proposition 10.1. For generic representations of G_2 which are induced from P_2, we use Proposition 8.5. For generic representations induced from P_1, we use Proposition 8.7, and the previous lemma. □

Now we have

THEOREM 11.3. *If $p \geqslant 5$, then $\Theta_{\mathrm{gen}}(\chi(s)_{G_2}) = \{\chi(i(s))_{\mathrm{PGSp}_6}\}$.*

Proof. Parts of the proof involve the same sort of considerations as in the previous sections and so we shall be brief on those parts. Also, after the previous proposition, it remains to prove this for cuspidal generic representations only.

Now, if the parameter s is contained in $\mathrm{SL}_3 \subset G_2$, then since the induced representation $\mathrm{Ind}_{Q_0}^{\mathrm{PGSp}_6}\chi(s)_{\mathrm{PGL}_3}$ is always irreducible for s in the elliptic torus of $\mathrm{SL}_3(q)$, we have the required result, using Proposition 8.3. Hence we are left with those

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cuspidal generic representations which do not come from PGL_3. Again, these have parameters s which are regular in $G_2^* = G_2$. Moreover, s lies in a unique maximal torus T_w^*, where w is either the nontrivial element in the center of the Weyl group, or the class of elements of order 6. Let us consider the latter case first. Then the regular parameter s looks like

$$\begin{pmatrix} a & & \\ & a^{-q} & \\ & & a^{q^2} \end{pmatrix},$$

with $a \in k_6 = \mathbb{F}_{q^6}$, and $a^{q^2-q+1} = 1$. It is straightforward to check that in Spin_7, the parameter $i(s)$ is still regular. Hence, we have $\chi(s) = R_T(\theta)$, $\chi(i(s)) = R_{T'}(\theta')$, with θ, θ' regular characters of $T(q)$ and $T'(q)$ respectively.

By going to k_2, the torus T_w^* becomes conjugate to the elliptic torus in $\mathrm{SL}_3(q^2)$. Thus both $\chi(s)$ and $\chi(i(s))$ lift to cuspidal representations with parameters in the elliptic torus in $\mathrm{SL}_3(q^2)$, and since we already know the result for representations associated to such tori, a base change argument as in the proof of Theorem 10.2 gives the result in this case.

Finally, if w is the nontrivial element in the center of the Weyl group of G_2, then the regular parameter s looks like

$$\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix},$$

with $a, b, c \in k_2^\times$, $a^{q+1} = b^{q+1} = c^{q+1} = 1$, and $abc = 1$. Now, if none of a, b or c equals -1, then $i(s)$ is still regular, and by base change to k_2, we obtain the required result. If, say, $a = -1$, then we find that the centralizer of $i(s)$ in Spin_7 has derived group SL_2. Hence, the geometric conjugacy class of PGSp_6 corresponding to $i(s)$ has 2 elements, denoted $\chi(i(s))$ and $\pi(s)$, with $\pi(s)$ a degenerate representation. In this case, if (T', θ') corresponds to the nonsplit torus in SL_2, and (T'', θ'') the split torus, we have

$$R_{T'}(\theta') = \pi(s) - \chi(i(s)), \qquad R_{T''}(\theta'') = \pi(s) + \chi(i(s)).$$

Over k_2, both T' and T'' split. Moreover, $R(\theta')_2$ and $R(\theta'')_2$ are both equal to the same (reducible) principal series representation, which has two irreducible components π' and $\chi'(i(s))$ (the generic component). One sees that $\chi'(i(s))$ is the base change of $\chi(i(s))$, and hence the same base change argument gives $\langle \Pi, \chi(s) \otimes \chi(i(s)) \rangle = 1$, which establishes the theorem. □

12. The Outer Form of E_6

We have seen that about half the generic characters of G_2 can be obtained as lifts from PGL_3. These are the generic representations whose parameter lies in SL_3, i.e. is of the form $i(s)$ for some $s \in \mathrm{SL}_3$. For $i(s) \in G_2$ regular, these can be characterized by their dimensions, which has the form $(q^3+1)\dim(\chi(s)_{\mathrm{PGL}_3}) = (q^3+1)P_{w(s)}(q)$, where $P_{w(s)}$ is a polynomial with integer coefficients, and depends only on the class of $w(s)$ in the Weyl group. Here, $w(s)$ is such that $G_s^* = T_{w(s)}$.

In this section, we see that the other generic representations are obtained as lifts from PU_3, the outer form of PGL_3, by using the dual pair $G_2 \times \mathrm{PU}_3 \subset {}^2E_6$, and the reflection representation Π of ${}^2E_6$. Before proceeding, we have to say what we mean by the reflection representation of $G^F := {}^2E_6$. As before, Π is a unipotent principal series representation. The Weyl group W^F of G^F is of type F_4, and so the irreducible components of $\mathrm{Ind}_{B^F}^{G^F} 1$ are parametrized by the irreducible characters of the Weyl group of F_4 (here, B^F is the Borel subgroup of G^F). In this case, however, Π does not correspond to the reflection representation of the Weyl group. Instead it corresponds to a two-dimensional representation which is denoted $\phi'_{2,4}$ in the notation of [C]. Hence, in particular, the space of B^F-fixed vectors in Π has dimension 2. Note, however, that the dimension of Π is $q^{11} - q^8 + q^7 + q^5 - q^4 + q$, which is the smallest among the irreducible representations of G^F of dimension greater than 1. Hence it might be more appropriate to call Π the minimal representation in this case. Now we can state the result

THEOREM 12.1. *Assume that* $p \geqslant 5$, *as before. Let* $i\colon \mathrm{SU}_3 \to G_2$ *be the natural embedding of dual groups (by regarding* SU_3 *as the Galois group of* $\mathbb{F}_{q^2}$ *in* $\mathbb{O}$, *the octonion algebra). Then* $\Theta_{\mathrm{gen}}(\chi(s)) = \{\chi(i(s))\}$.

The proof of this is similar to the other dual pairs, and so will be omitted. Notice that if $i(s) \in G_2$ is regular, then the dimension of $\chi(i(s))$ is given by $(q^3-1)|P_{w(s)}(-q)|$. This concludes our study of the Howe correspondence in the finite field situation.

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