SIMULTANEOUS DUAL INTEGRAL EQUATIONS

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1. Lowengrub [1] has considered simultaneous dual integral equations of the form

$$S_{\frac{1}{2}\mu_i-\alpha,2\alpha}\sum_{j=1}^{n}c_{ij}\phi_j(x)=f_i(x) \qquad (x\in I_1), \tag{1}$$

$$S_{\frac{1}{2}\nu_i-\beta,2\beta}\phi_i(x)=0 \qquad (x\in I_2), \tag{2}$$

where $i=1, 2, \ldots, n$, $I_1=\{x:0\leqq x<1\}$, $I_2=\{x:x>1\}$, the c_{ij} are constants, the $f_i(x)$ are known functions and the functions $\phi_i(x)$ are to be determined.

$$S_{\eta,\alpha}f(x)=2^{\alpha}x^{-\alpha}\int_0^{\infty}t^{1-\alpha}J_{2\eta+\alpha}(xt)f(t)\,dt \tag{3}$$

denotes the modified operator of the Hankel transform with the inversion formula

$$S_{\eta,\alpha}^{-1}=S_{\eta+\alpha,-\alpha}. \tag{4}$$

Assuming integral representations for the functions $\phi_i(x)$, Lowengrub applied the Erdélyi–Kober operators of fractional integration to obtain detailed solutions of the equations when $n=2$ or $n=3$ and he indicated an extension of the method for solving the equations in the general case. His solutions, however, are not correct. This is due to the fact that in the course of the analysis assumptions are made about the functions $\phi_i(x)$ which are equivalent to assuming *at least one* extra equation in addition to the equations (1) and (2) (cf. equations (3.5), (4.6) and (6.4) of [1]).

In this note we use the Erdélyi–Kober operators to reduce the equations (1) and (2) to a set of simultaneous integral equations for the determination of the functions $\phi_i(x)$. The solution to a simple example of the equations when $n=2$ is then given.

2. A comprehensive account of the Erdélyi–Kober operators can be found in [2]. They are defined by the formulae

$$I_{\eta,\alpha}f(x)=\begin{cases}\dfrac{2x^{-2(\alpha+\eta)}}{\Gamma(\alpha)}\displaystyle\int_0^x(x^2-u^2)^{\alpha-1}u^{2\eta+1}f(u)\,du & (0<\alpha), \qquad (5)\\[2ex] x^{-2(\alpha+\eta)-1}\mathscr{D}_x^m\{x^{2(\alpha+\eta+m)+1}I_{\eta,\alpha+m}f(x)\} & (-m<\alpha<0), \qquad (6)\end{cases}$$

$$K_{\eta,\alpha}f(x)=\begin{cases}\dfrac{2x^{2\eta}}{\Gamma(\alpha)}\displaystyle\int_x^{\infty}(u^2-x^2)^{\alpha-1}u^{-2(\alpha+\eta)+1}f(u)\,du & (0<\alpha), \qquad (7)\\[2ex] (-1)^m x^{2\eta-1}\mathscr{D}_x^m\{x^{2(m-\eta)+1}K_{\eta-m,\alpha+m}f(x)\} & (-m<\alpha<0), \qquad (8)\end{cases}$$

where m is a positive integer and $\mathscr{D}_x=\frac{1}{2}(d/dx)x^{-1}$.

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We shall also require the relations

$$I_{\eta+\alpha,\beta}S_{\eta,\alpha}=S_{\eta,\alpha+\beta}, \qquad K_{\eta,\alpha}S_{\eta+\alpha,\beta}=S_{\eta,\alpha+\beta}. \tag{9}$$

3. If we write $\lambda_i=\frac{1}{2}(\mu_i+\nu_i)-(\alpha-\beta)$ and apply the operators $I_{\frac{1}{2}\mu_i+\alpha,\lambda_i-\mu_i}$, $K_{\frac{1}{2}\mu_i-\alpha,\nu_i-\lambda_i}$ $(i=1,2,\ldots,n)$, to equations (1) and (2) respectively, we find that they become

$$S_{\frac{1}{2}\mu_i-\alpha,\lambda_i-\mu_i+2\alpha}\sum_{j=1}^{n}c_{ij}\phi_j(x)=I_{\frac{1}{2}\mu_i+\alpha,\lambda_i-\mu_i}f_i(x) \qquad (x\in I_1), \tag{10}$$

$$S_{\frac{1}{2}\mu_i-\alpha,\lambda_i-\mu_i+2\alpha}\phi_i(x)=0 \qquad (x\in I_2), \tag{11}$$

where we have used the results (9).

If the constants $c_{ii}\neq 0$, we can write the above equations in the form

$$S_{\frac{1}{2}\mu_i-\alpha,\lambda_i-\mu_i+2\alpha}\phi_i(x)$$
$$=\begin{Bmatrix} c_{ii}^{-1}I_{\frac{1}{2}\mu_i+\alpha,\lambda_i-\mu_i}f_i(x)-S_{\frac{1}{2}\mu_i-\alpha,\lambda_i-\mu_i+2\alpha}\Phi_i(x) & (x\in I_1), \\ 0 \quad (x\in I_2), & \end{Bmatrix} \quad (i=1,2,\ldots,n), \tag{12}$$

where

$$\Phi_i(x)=c_{ii}^{-1}{\sum_{j=1}^{n}}' c_{ij}\phi_j(x) \tag{13}$$

and the dash in the summation sign denotes that the term for which $j=i$ is omitted.

Using the result (4), we can invert these equations to find that the functions $\phi_i(x)$ satisfy the simultaneous integral equations

$$\phi_i(x)=F_i(x)-S_{\frac{1}{2}\nu_i+\beta,\lambda_i-\nu_i-2\beta}H(1-x)S_{\frac{1}{2}\mu_i-\alpha,\lambda_i-\mu_i+2\alpha}\Phi_i(x), \tag{14}$$

where $i=1,2,\ldots,n$, $H(x)$ is the Heaviside unit function and the

$$F_i(x)=c_{ii}^{-1}S_{\frac{1}{2}\nu_i+\beta,\lambda_i-\nu_i-2\beta}H(1-x)I_{\frac{1}{2}\mu_i+\alpha,\lambda_i-\mu_i}f_i(x) \tag{15}$$

are known functions.

Lowengrub was able to obtain exact solutions to his equations because he made assumptions which are equivalent to taking equations (1) and (2) together with the equations

$$S_{\frac{1}{2}\nu_j-\beta,2\beta}\phi_i(x)=0 \qquad (x\in I_2;\ j=1,2,\ldots,i-1;\ i=2,3,\ldots,n), \tag{16}$$

that is, equations (10) and (11) and the additional equations

$$S_{\frac{1}{2}\mu_j-\alpha,\lambda_j-\mu_j+2\alpha}\phi_i(x)=0 \qquad (x\in I_2;\ j=1,2,\ldots,i-1;\ i=2,3,\ldots,n). \tag{17}$$

After a little manipulation it can easily be shown that we can write equations (14) as

$$\phi_i(x)=F_i(x)-x^{\nu_i-\lambda_i+2\beta}\int_0^\infty t^{1+\mu_i-\lambda_i-2\alpha}\Phi_i(t)L_{\lambda_i}(t,x)\frac{dt}{t^2-x^2}, \tag{18}$$

where

$$L_\lambda(t,x)=(t^2-x^2)\int_0^1 uJ_\lambda(ut)J_\lambda(ux)\,du=tJ_{\lambda+1}(t)J_\lambda(x)-xJ_\lambda(t)J_{\lambda+1}(x). \tag{19}$$

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4. As an example, Lowengrub considered the following special case of the equations which occur in the mathematical theory of elasticity when determining the stress field in the neighbourhood of a penny shaped crack in a solid under shear [3, p. 157].

$$\left.\begin{aligned} S_{0,0}[c_{11}A(x)+c_{12}B(x)] &= 1,\\ S_{1,0}[c_{21}A(x)+c_{22}B(x)] &= 0,\end{aligned}\right\} \quad (x\in I_1), \tag{20}$$

$$S_{-\frac{1}{2},1}A(x)=0, \quad S_{\frac{1}{2},1}B(x)=0 \qquad (x\in I_2). \tag{21}$$

These are equations (1) and (2) with $n=2$, $\mu_1=\nu_1=\alpha=0$, $\mu_2=\nu_2=4\beta=2$, $f_1(x)=1$ and $f_2(x)=0$.

Putting $\lambda_1=\frac{1}{2}$, $\lambda_2=\frac{5}{2}$ in equations (18), we find that the functions $A(x)$ and $B(x)$ can be determined from the pair of simultaneous integral equations

$$A(x)=F_1(x)-\frac{c_{12}}{c_{11}}x^{\frac{1}{2}}\int_0^\infty t^{\frac{1}{2}}B(t)L_{\frac{1}{2}}(t,x)\frac{dt}{t^2-x^2}, \tag{22}$$

$$B(x)=-\frac{c_{21}}{c_{22}}x^{\frac{1}{2}}\int_0^\infty t^{\frac{1}{2}}A(t)L_{\frac{3}{2}}(t,x)\frac{dt}{t^2-x^2}, \tag{23}$$

where

$$F_1(x)=c_{11}^{-1}S_{\frac{1}{2},-\frac{1}{2}}H(1-x)I_{0,\frac{1}{2}}[1]=\frac{1}{c_{11}}\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}J_{\frac{1}{2}}(x). \tag{24}$$

Lowengrub's (corrected) solution is

$$A(x)=\frac{1}{c_{11}}\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}J_{\frac{1}{2}}(x), \qquad B(x)=0, \tag{25}$$

which was arrived at by solving equations (20) and (21) together with the additional equation

$$S_{-\frac{1}{2},1}B(x)=0 \qquad (x\in I_2). \tag{26}$$

It can easily be shown that the expression for $A(x)$ given by equation (25) is the solution of the simple pair of dual integral equations

$$S_{0,0}A(x)=\frac{1}{c_{11}}\ (x\in I_1) \quad \text{and} \quad S_{-\frac{1}{2},1}A(x)=0\ (x\in I_2). \tag{27}$$

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