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Dwork's conjecture on the logarithmic growth of solutions of p-adic differential equations

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Abstract

This is a study of the asymptotic behaviour of solutions of p-adic linear differential equations near the boundary of their convergence disks. We prove Dwork's conjecture on the logarithmic growth of solutions in generic versus special disks.

1. Introduction

1.1 We consider an ordinary linear differential equation

$$\frac{d^\mu y}{dx^\mu} + f_{\mu-1}(x)\frac{d^{\mu-1}y}{dx^{\mu-1}} + \cdots + f_0(x)y = 0, \qquad (*)$$

where the $f_i(x)$ are *bounded* analytic functions in the unit disk $|x| < 1$, with coefficients in a p-adic field (a finite extension of $\mathbf{Q}_p$). We assume that there is a full set of solutions y which are analytic in the unit disk.

Since his early studies on p-adic differential equations, Dwork drew attention to the growth of solutions near the boundary of the unit disk. He proved that every solution y has at most logarithmic growth of order $\mu - 1$, i.e.

$$|y|_0(r) = O((\log 1/r)^{1-\mu}) \quad \text{for } r < 1$$

(cf. § 2.1 below for the standard notation $|y|_0(r)$). In order to encode the precise logarithmic growth of the solutions, he introduced the (special) log-growth Newton polygon of $(*)$.

On the other hand, he considered the solutions of $(*)$ in the so-called generic unit disk $|x-t| < 1$, proved that there is a full set solutions y analytic in that disk which have at most logarithmic growth of order $\mu-1$, and introduced the generic log-growth Newton polygon of $(*)$, cf. [Dwo73a, Dwo73b].

In case $(*)$ has a strong Frobenius structure, he put forward some empirical relationship between the logarithmic growth rate of the solutions and the slopes of the Frobenius structure. By analogy with Grothendieck's specialization theorem [Gro74] for Frobenius–Newton polygons, he then proposed the following.

Conjecture 1.1.1 (Dwork [Dwo73b]). *The special log-growth Newton polygon lies above the generic log-growth Newton polygon.*

As an interesting example, he mentioned in the same paper the case of the hypergeometric differential equation with parameters $(1/2, 1/2, 1)$ which controls the variation of the p-adic cohomology of the Legendre family of elliptic curves. The log-growth Newton polygons then coincide with the (suitably normalized) Frobenius–Newton polygons; in particular, in a supersingular disk, the special log-growth Newton polygon lies strictly above the generic log-growth Newton polygon.

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Later on, Dwork computed the generic and special log-growth of p-adic solutions of classical hypergeometric equations[1] [Dwo82, §§ 9.6, 9.7 and 16.9]; see also the studies by Adolphson and Sperber of p-adic confluent hypergeometric differential equations.

The study of the generic log-growth filtration has given birth to factorization techniques for differential operators which have had a considerable impact on the development of p-adic analysis (works by Robba [Rob75], Christol [Chr83] and Dwork [Dwo73a, Dwo73b]).

1.2 In this paper, we *prove Dwork's conjecture, without assuming the existence of a Frobenius structure*: the special Newton polygon lies above the generic one (the right end-point being fixed); cf. Theorem 4.1.1 for a precise and more general statement. In this generality, irrational slopes may occur in both Newton polygons, cf. [CT07, § 5].

1.3 For p-adic differential equations which admit a Frobenius structure, the relationship between the Frobenius filtration (up to shift) and the log-growth filtration remains mysterious; it is not even known whether the log-growth slopes are rational in this situation. Some partial results have been obtained in [CT07] (where the rank 2 case is settled).

No straightforward connection can be expected since the two filtrations exhibit rather different qualitative features. Notably, in the generic situation, there is a horizontal filtration from which the Frobenius slope filtration can be seen on the *diagonal blocks* of a full solution matrix (of the associated linear system), whereas the log-growth filtration can only be detected by taking into account the *first row of blocks*.

Also, the log-growth filtration is not strictly compatible with the tensor product, which prevents us from using Grothendieck's well-known argument of exterior powers in the case of the Frobenius slope filtration (cf. e.g. [CT07, Theorem 6.12] in our context) in order to prove the specialization theorem.

Our method is somewhat inspired by another specialization theorem, which concerns the Newton polygon controlling the irregularity at zero of an ordinary linear differential equation with coefficients in $\mathbb{C}((x))$ (see [And07, Appendix]).

2. Logarithmic growth

2.1 Let K be a field of characteristic zero which is complete with respect to a discrete non-archimedean absolute value $|\cdot|$. Let $\mathcal{O}_K$ be its ring of integers and let k be its residue field. We assume that k is of characteristic $p > 0$.

For any $a \in \mathcal{O}_K$, let $\mathcal{A}(a, 1^-)$ be the K-algebra of analytic functions (with coefficients in K) in the residue class $D(a, 1^-)$ of a. It is endowed with the family of (multiplicative) norms $|\cdot|_a(r)$ (for $r \in [0, 1[$):

$$f = \sum b_n (x-a)^n \mapsto |f|_a(r) = \sup |b_n| r^n.$$

If $\bar{a} \in k$ denotes the residue class of a, $\mathcal{A}(a, 1^-)$ depends only (up to canonical isomorphism) on $\bar{a}$, and we sometimes write $\mathcal{A}(\bar{a}, 1^-)$ to emphasize this.

Let $\mathcal{B}(a, 1^-) = \mathcal{O}_K[[x-a]] \otimes_{\mathcal{O}_K} K$ be the subalgebra of bounded analytic functions. This is a Banach algebra with respect to the (multiplicative) norm

$$f \mapsto |f| := |f|_a(1) = \sup |b_n|.$$

[1] I am indebted to F. Baldassarri for these references.

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Similarly, $\mathcal{B}(a, 1^-)$ depends only (up to canonical isomorphism) on $\bar{a}$, and we sometimes write $\mathcal{B}(\bar{a}, 1^-)$ to emphasize this independence.

Remarks 2.1.1. (1) The ring $\mathcal{B}(a, 1^-)$ is *principal*. Indeed, let $\mathcal{I}_K$ be an ideal of $\mathcal{B}(a, 1^-)$, and let $\mathcal{I}$ be the ideal $\mathcal{I}_K \cap \mathcal{O}_K[[x-a]]$ of $\mathcal{O}_K[[x-a]]$. Then $\mathcal{I}$ is either principal or of the form (π, f), $f \in (x-a)\mathcal{O}_K[[x-a]]$, and we see that, in both cases, $\mathcal{I}_K = \mathcal{I}[\frac{1}{\pi}]$ is principal in $\mathcal{B}(a, 1^-) = \mathcal{O}_K[[x-a]][\frac{1}{\pi}]$.

On the other hand, $\mathcal{B}(a, 1^-)$ is *differentially simple*: there is no non-trivial ideal stable under d/dx.

(2) In contrast, $\mathcal{A}(a, 1^-)$ is not noetherian and *a fortiori* not principal. However, according to a result of Lazard, it is *Bézout* (every finitely generated ideal is principal), hence integrally closed and coherent.

On the other hand, it is not differentially simple (cf. the differential ideal generated by an analytic function with zeroes of unbounded multiplicities), but there is *no non-trivial finitely generated differential ideal* (see e.g. [And02, § 2] for a discussion of this matter).

2.2 For any $\sigma \in \mathbb{R}_{\geqslant 0}$, let $\mathcal{W}_{a,\sigma}$ (or $\mathcal{W}_{K,a,\sigma}$) be the K-space formed by all series $f = \sum b_n(x-a)^n \in K[[x-a]]$ such that

$$|f|_{a,\sigma} := \sup_n (n+1)^{-\sigma}|b_n| < \infty,$$

and let $\mathcal{W}_{a,\sigma^+}$ (or $\mathcal{W}_{K,a,\sigma^+}$) be the K-vector space formed by all series $f = \sum b_n(x-a)^n \in K[[x-a]]$ such that

$$\overline{\lim_n} \frac{\log |b_n|}{\log(n+1)} \leqslant \sigma.$$

Lemma 2.2.1. *We have the following.*

(i) *Here $\sigma \mapsto \mathcal{W}_{a,\sigma}$ is an increasing sequence of $\mathcal{B}(a, 1^-)$-submodules of $\mathcal{A}(a, 1^-)$.*

(ii) *We have $(\mathcal{W}_{a,0}, |\cdot|_{a,0}) = (\mathcal{B}(a, 1^-), |\cdot|)$.*

(iii) *We have $\mathcal{W}_{a,\sigma^+} = \bigcap_{\sigma'>\sigma} \mathcal{W}_{a,\sigma'}$.*

(iv) *Each $\mathcal{W}_{a,\sigma}$ is a K-Banach space with respect to $|\cdot|_{a,\sigma}$. It is the space of analytic functions f of* logarithmic growth *at most σ in $D(a, 1^-)$, i.e. such that*

$$|f|_a(r) = O((\log 1/r)^{-\sigma}) \quad \text{for } r < 1.$$

(v) *For a, a' in the same residue class, $\mathcal{W}_{a,\sigma} = \mathcal{W}_{a',\sigma}$ and $|\cdot|_{a,\sigma}, |\cdot|_{a',\sigma}$ are equivalent norms. A fortiori $\mathcal{W}_{a,\sigma^+} = \mathcal{W}_{a',\sigma^+}$. (We sometimes write $\mathcal{W}_{\bar{a},\sigma}$ and $\mathcal{W}_{\bar{a},\sigma^+}$ instead of $\mathcal{W}_{a,\sigma}$ and $\mathcal{W}_{a,\sigma^+}$, respectively, to emphasize this independence.)*

(vi) *We have $\mathcal{W}_{a,\sigma}.\mathcal{W}_{a,\sigma'} \subset \mathcal{W}_{a,\sigma+\sigma'}$.*

(vii) *Here $\mathcal{W}_{a,\sigma}$ is stable under derivation.*

(viii) *For any $f \in \mathcal{A}(a, 1^-)$, $df/dx \in \mathcal{W}_{a,\sigma}$ implies $f \in \mathcal{W}_{a,\sigma+1}$.*

Proof. These properties are standard and left to the reader. We just indicate that property (iv) follows from the inequality

$$\sup_n (n+1)^{\sigma} r^n \leqslant \frac{1}{r}\left(\frac{\sigma}{e}\right)^{\sigma} (\log 1/r)^{-\sigma}. \qquad \square$$

Remark 2.2.2. Strictly speaking, the classical notion of logarithmic growth corresponds to the case when the absolute value is normalized by $|p| = p^{-1}$.

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3. Log-growth filtration and Newton polygon

3.1 Let M be a *differential module* of rank μ over $\mathcal{B}(a,1^-)$, i.e. a $\mathcal{B}(a,1^-)[d/dx]$-module whose underlying $\mathcal{B}(a,1^-)$-module is free of rank μ. We denote by $\nabla(d/dx)$ or simply ∇ the action of d/dx on M.

For any $\mathcal{B}(a,1^-)[d/dx]$-submodule $\mathcal{W} \subset K[[x-a]]$ which contains $\mathcal{B}(a,1^-)$, we endow

$$M_{\mathcal{W}} := M \otimes_{\mathcal{B}(a,1^-)} \mathcal{W}$$

with the action of d/dx defined by $\nabla(d/dx) \otimes 1 + 1 \otimes d/dx$.

We say that M is *solvable in* $\mathcal{W}$ if and only if the following equivalent conditions are fulfilled:

- the natural morphism $(M_{\mathcal{W}})^{\nabla} \otimes_K K[[x-a]] \to M_{K[[x-a]]}$ is surjective;
- $(M_{\mathcal{W}})^{\nabla} \otimes_K K[[x-a]] \to M_{K[[x-a]]}$ is bijective;
- $\dim_K (M_{\mathcal{W}})^{\nabla} = \mu$.

It is well-known that M is always solvable in $K[[x-a]]$: in fact, the K-linear endomorphism

$$\Pi_a = \sum_{n \geqslant 0} \frac{(a-x)^n}{n!} \nabla\left(\frac{d}{dx}\right)^n$$

of $M_{K[[x-a]]}$ is a projector which induces an isomorphism of K-vector spaces of dimension μ:

$$\bar{\Pi}_a : M/(x-a)M \xrightarrow{\cong} (M_{K[[x-a]]})^{\nabla},$$

cf. [Kat70, 8.9]. Therefore, M is solvable in $\mathcal{W}$ if and only if every formal solution belongs to $\mathcal{W}$.

If M is solvable in $\mathcal{A}(a,1^-)$, then $\bar{\Pi}_a : M/(x-a)M \xrightarrow{\cong} (M_{\mathcal{A}(\bar{a},1^-)})^{\nabla}$, and $M/(x-a)M$ becomes canonically isomorphic to $M/(x-a')M$ for any a' in the same residue class as a.

3.2 The importance of logarithmic growth in the context of p-adic differential equations lies the following result.

Theorem 3.2.1 (Dwork). *If M is solvable in $\mathcal{A}(a,1^-)$, then it is also solvable in $\mathcal{W}_{a,\mu-1}$.*

See [Dwo73a, Dwo73b], and also [Chr83, § 5]; the proof uses a transfer from the generic disk to the special disk $D(a,1^-)$. The theorem is also a straightforward consequence of the Dwork–Robba effective bounds for the Taylor coefficients of solutions [DR80].

3.3 We assume henceforth that M is solvable in $\mathcal{A}(a,1^-)$.

It follows from Dwork's theorem that for any non-zero $m \in (M_{\mathcal{A}(\bar{a},1^-)})^{\nabla}$, there exists a maximal value $\sigma = \sigma(m) \in \mathbb{R}_{\geqslant 0}$ for which $m \in (M_{\mathcal{W}_{\bar{a},\sigma+}})^{\nabla}$. We call it the *log-growth slope of* m (this is the infimum of all σ' for which $m \in (M_{\mathcal{W}_{\bar{a},\sigma'}})^{\nabla}$).

Let us consider the corresponding exhaustive increasing filtration of $(M_{\mathcal{A}(\bar{a},1^-)})^{\nabla}$ given by

$$\mathcal{W}_{\sigma+}(M_{\mathcal{A}(\bar{a},1^-)})^{\nabla} = (M_{\mathcal{W}_{\bar{a},\sigma+}})^{\nabla}.$$

We set

$$gr_{\sigma}(M_{\mathcal{A}(\bar{a},1^-)})^{\nabla} = (M_{\mathcal{W}_{\bar{a},\sigma+}})^{\nabla} \Big/ \bigcup_{\tau<\sigma} (M_{\mathcal{W}_{\bar{a},\tau+}})^{\nabla}.$$

We denote by $\Sigma_{\bar{a}}(M) \subset \mathbb{R}_{\geqslant 0}$ the finite set of σ (the *breaks*) for which $gr_{\sigma}(M_{\mathcal{A}(\bar{a},1^-)})^{\nabla} \neq 0$.

The *log-growth Newton polygon* $NP_{\log,\bar{a}}(M)$ (at $\bar{a} \in k$), which we place by convention in the fourth quadrant, is defined as follows: its last vertex on the right is $(\mu, 0)$, and for each break σ,

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it has an edge of slope σ and horizontal length

$$\mu_{\bar{a},\sigma}(M) := \dim gr_\sigma(M_{\mathcal{A}(\bar{a},1^-)})^\nabla.$$

Example 3.3.1. Let M be $\mathcal{B}(0,1^-)e_1 \oplus \mathcal{B}(0,1^-)e_2$, with

$$\nabla\left(\frac{d}{dx}\right)e_1 = 0, \quad \nabla\left(\frac{d}{dx}\right)e_2 = \frac{1}{x-1}e_1.$$

Let S^nM be the nth symmetric power of M (a basis of solutions being given by $\log^j(1-x), j \leqslant n$). Then $NP_{\log,0}(S^nM)$ has slopes $0, 1, \ldots, n$.

Remarks 3.3.2. (1) For any finite extension K'/K, $\mathcal{W}_{K',\bar{a},\sigma^+} = \mathcal{W}_{K,\bar{a},\sigma^+} \otimes_K K'$, and $(M_{\mathcal{W}_{\bar{a},\sigma^+}\otimes_K K'})^\nabla = (M_{\mathcal{W}_{\bar{a},\sigma^+}})^\nabla \otimes_K K'$ (cf. [CT07, 1.9]). *A fortiori*, $NP_{\log,\bar{a}}(M)$ does not change if one replaces K by K'.

(2) If N is another differential module solvable in $\mathcal{A}(a,1^-)$, then the tensor product $M \otimes_{\mathcal{B}(a,1^-)} N$ is also solvable in $\mathcal{A}(a,1^-)$, and

$$((M \otimes N)_{\mathcal{A}(a,1^-)})^\nabla = (M_{\mathcal{A}(\bar{a},1^-)})^\nabla \otimes_K (N_{\mathcal{A}(\bar{a},1^-)})^\nabla.$$

One has

$$(M_{\mathcal{W}_{\bar{a},\sigma^+}})^\nabla \otimes (N_{\mathcal{W}_{\bar{a},\tau^+}})^\nabla \subset ((M \otimes N)_{\mathcal{W}_{\bar{a},(\sigma+\tau)^+}})^\nabla.$$

3.4 Let π be a uniformizer of $\mathcal{O}_K$. The π-adic completion of

$$\mathcal{O}_K((x-a)) := \mathcal{O}_K[[x-a]]\left[\frac{1}{x-a}\right]$$

coincides after inverting π with the completion of the fraction field of $\mathcal{B}(a,1^-)$: this is the field

$$\mathcal{E}_{x-a} = \left\{\sum_{n\in\mathbf{Z}} b_n(x-a)^n, (b_n) \text{ bounded}, \lim_{n\to-\infty} b_n = 0\right\}.$$

Let t be another indeterminate, which we view as a π-adic unit (Dwork's generic point) in the complete field $K_t := \mathcal{E}_{t-a}$ (whose residue field is $k_t := k((\bar{t}-\bar{a}))$). One has isometric embeddings of normed rings

$$\mathcal{B}(a,1^-) \subset \mathcal{E}_{x-a} \subset \mathcal{B}(t,1^-),$$

(where the latter is the ring of bounded analytic functions, with coefficients in K_t, in the generic disk), and, correspondingly, isometric embeddings of complete π-adic subrings:

$$\mathcal{O}_K[[x-a]] \subset \mathcal{O}_{\mathcal{E}_{x-a}} \subset \mathcal{O}_{K_t}[[x-t]].$$

The embedding $\mathcal{O}_K[[x-a]] \subset \mathcal{O}_K[[t-a, x-t]]$ has a retraction given by the specialization map $t \mapsto a$.

We can apply the constructions of § 3 to $M_{\bar{t}} = M \otimes_{\mathcal{B}(\bar{a},1^-)} \mathcal{B}(\bar{t},1^-)$, replacing (a,K,k) by (t,K_t,k_t) (and denoting by $\bar{t}$ the image of t in k_t). In this way, we get the *generic* filtration $\mathcal{W}_{\sigma^+}(M_{\mathcal{A}(\bar{t},1^-)})^\nabla$, and the *generic log-growth Newton polygon* $NP_{\log,\bar{t}}(M_{\bar{t}})$ at $\bar{t}$.

When compared with $NP_{\log,\bar{t}}(M_{\bar{t}})$, $NP_{\log,\bar{a}}(M)$ is usually referred to as the *special* log-growth Newton polygon (in the residue class $D(\bar{a},1^-)$).

Remark 3.4.1. It is known that the generic filtration $\mathcal{W}_{\sigma^+}(M_{\mathcal{A}(\bar{t},1^-)})^\nabla$ comes from a horizontal filtration of $M_{\mathcal{E}_x}$ (see Robba [Rob75] and Christol [Chr83, 4.3]; see also [CT07, 3.2]).

4. Main theorem

4.1 Let M be a differential module over $\mathcal{B}(\bar{a},1^-)$, which is solvable in $\mathcal{A}(\bar{a},1^-)$.

THEOREM 4.1.1. *The special log-growth Newton polygon $NP_{\log,\bar{a}}(M)$ lies above the generic log-growth Newton polygon $NP_{\log,\bar{t}}(M_{\bar{t}})$.*

It is important here to keep in mind that the *right end-point* of the log-growth Newton polygon is fixed: it is $(\mu, 0)$ by convention. The theorem implies in particular that the left end-point of the special log-growth Newton polygon lies above the left end-point of the generic log-growth Newton polygon. Whether or not these points coincide remains an open question.

Remark 4.1.2. The main difficulty in comparing the two Newton polygons is the following: one has

$$(M \otimes \mathcal{A}(t, 1^-))^\nabla = (M \otimes (\mathcal{O}_K[[t-a]] \otimes K)[[x-t]])^\nabla \otimes_{(\mathcal{O}_K[[t-a]]\otimes K)} \mathcal{E}_{t-a}$$

but the log-growth filtration does not come from a filtration of

$$(M \otimes (\mathcal{O}_K[[t-a]] \otimes K)[[x-t]])^\nabla$$

in general, and thus cannot be specialized for $t \mapsto a$.

This difficulty is settled below by using appropriate approximations of the log-growth slope function by continuous numerical functions, combined with a compacity argument.

As was mentioned in the introduction, another difficulty is that the log-growth filtration is neither exact (which prevents us from arguing by devissage and induction on μ), nor strictly compatible with the tensor product (which prevents us from applying Grothendieck's well-known argument of exterior powers in the case of the Frobenius slope filtration, cf. e.g. [CT07, 6.12] in our context).

4.2 For any $\xi \in \mathbb{R}_{\geqslant 0}$, we write

$$\sigma_{\bar{a},\xi} = \sum_{\sigma\in\Sigma_{\bar{a}},\sigma>\xi} \mu_{\bar{a},\sigma}(M).(\sigma - \xi),$$

$$\sigma_{\bar{t},\xi} = \sum_{\tau\in\Sigma_{\bar{t}},\tau>\xi} \mu_{\bar{t},\tau}(M_{\bar{t}}).(\tau - \xi).$$

Let us first explain how to derive the theorem from the following lemma.

LEMMA 4.2.1. *One has $\sigma_{\bar{a},\xi} \leqslant \sigma_{\bar{t},\xi}$.*

For $x \in [0, \mu]$, let us denote by $g_{\bar{a}}(x)$ (respectively $g_{\bar{t}}(x)$) the convex, piecewise affine function (with non-positive values) whose graph is the boundary of $NP_{\log,\bar{a}}(M)$ (respectively $NP_{\log,\bar{t}}(M_{\bar{t}})$). For $\xi \in [0, \infty[$, let us denote by

$$g^*_{\bar{a}}(\xi) = \sup_x (x\xi - g_{\bar{a}}(x)) \quad \Big(\text{respectively } g^*_{\bar{t}}(\xi) = \sup_x (x\xi - g_{\bar{t}}(x))\Big)$$

the *Legendre transform* of $g_{\bar{a}}$ (respectively $g_{\bar{t}}$), which is given by

$$g^*_{\bar{a}}(\xi) = \mu\xi + \sigma_{\bar{a},\xi} \quad (\text{respectively } g^*_{\bar{t}}(\xi) = \mu\xi + \sigma_{\bar{t},\xi}).$$

On the other hand, for any $x \in [0, \mu], \xi \in [0, \infty[$, one has the usual Young inequality

$$x\xi \leqslant g_{\bar{a}}(x) + g^*_{\bar{a}}(\xi) \quad (\text{respectively } x\xi \leqslant g_{\bar{t}}(x) + g^*_{\bar{t}}(\xi)),$$

that is

$$\sigma_{\bar{a},\xi} \geqslant -g_{\bar{a}}(x) - \xi(\mu - x), \quad (\text{respectively } \sigma_{\bar{t},\xi} \geqslant -g_{\bar{t}}(x) - \xi(\mu - x)).$$

For x equal to the abscissa of a vertex of $NP_{\log,\bar{a}}(M)$, and for ξ equal to the slope of the edge on the right of this vertex, the Young inequality for $(g_{\bar{a}}, g^*_{\bar{a}})$ becomes an equality: $x\xi = g_{\bar{a}}(x) + g^*_{\bar{a}}(\xi)$. It then follows from the Young inequality for $(g_{\bar{t}}, g^*_{\bar{t}})$ and from the inequality of Lemma 4.2.1 that

$$g_{\bar{a}}(x) \geqslant g_{\bar{t}}(x)$$

for any break x of the piecewise affine function $g_{\bar{a}}$. Since $g_{\bar{t}}$ is convex, this implies that $g_{\bar{a}}(x) \geqslant g_{\bar{t}}(x)$ for every $x \in [0,\mu]$, i.e. $NP_{\log,\bar{a}}(M)$ lies above $NP_{\log,\bar{t}}(M_{\bar{t}})$. □

5. Proof of Lemma 4.2.1

To simplify notation, we assume that $a = 0$.

5.1 Some numerical functions of bases of M

We fix once for all a free $\mathcal{O}_K$-submodule V of M, of rank μ, which generates M over $\mathcal{B}(0,1^-)$. For any sub-$\mathcal{O}_K$-algebra R of $K_t = \mathcal{E}_t$, we set

$$V_R = V \otimes_{\mathcal{O}_K} R,$$

endowed with its natural norm $|\cdot|_{V_R}$.

We denote by V_0 (respectively V_t) the (isomorphic) image of V (respectively $V_{\mathcal{O}_{K_t}}$) in M/xM (respectively $M_t/(x-t)M_t$). This is an $\mathcal{O}_K$-lattice of the K-space M/xM (respectively an $\mathcal{O}_{K_t}$-lattice of the K_t-space $M_t/(x-t)M_t$). We denote by $B_0 \subset V_0^\mu$ (respectively $B_t \subset V_t^\mu$) the metric space of all bases of V_0 (respectively V_t).

To any $v \in V_0$, we attach a sequence of elements $(v_n)_{n\geqslant 0} \in V_K\colon v_n$ is the image modulo x of the coefficient of x^n in the Taylor expansion of $\bar{\Pi}_0(v) \in V_K \otimes_K \mathcal{A}(0,1^-) = M_{\mathcal{A}(0,1^-)}$.

In the same way, we attach to any $w \in V_t$ a sequence of elements $(w_n)_{n\geqslant 0} \in V_{K_t}$.

Remarks 5.1.1. (1) Let $\mathbf{v}$ be a basis of V and $\mathbf{v}^*$ be the dual basis. Let $G_{\mathbf{v},1}$ (respectively $G_{\mathbf{v}^*,1} = -(G_{\mathbf{v},1})^t$) be the matrix of $\nabla(d/dx)$ in $\mathbf{v}$ (respectively $\mathbf{v}^*$). Let $G_{\mathbf{v}^*,n} \in M_\mu(\mathcal{B}(0,1^-))$ be the sequence of matrices defined inductively by

$$G_{\mathbf{v}^*,n+1} = \frac{d}{dx}G_{\mathbf{v}^*,n} + G_{\mathbf{v}^*,n}.G_{\mathbf{v}^*,1},$$

so that $G_{\mathbf{v}^*,n}$ is the matrix of $\nabla(d/dx)^n$ in $\mathbf{v}^*$. Then if v (respectively v_n) is identified with the column vector of its coordinates in $\mathbf{v}$, one has $v_n = (1/n!)(G_{\mathbf{v}^*,n}(0))^t.v$.

(2) The choice of a basis $\mathbf{v}$ identifies V_0 with the set of K-points of the unit polydisk, and B_0 with the set of K-points of an affinoid.

For $i \leqslant j$ and for any basis $\mathbf{v}$ of V_0, we set

$$s_{0,\xi}(\mathbf{v}) = \sum_{v\in\mathbf{v}} \max(0, \sigma(\bar{\Pi}_0(v)) - \xi),$$

$$s_{0,\xi,i}(\mathbf{v}) = \sum_{v\in\mathbf{v}} \max\left(0, \max_{n\geqslant i} \frac{\log|v_n|_{V_K}}{\log(n+1)} - \xi\right).$$

$$s_{0,\xi,i,j}(\mathbf{v}) = \sum_{v\in\mathbf{v}} \max\left(0, \max_{i\leqslant n\leqslant j} \frac{\log|v_n|_{V_K}}{\log(n+1)} - \xi\right).$$

We use similar notation, with t instead of 0, for any basis $\mathbf{w}$ of V_t.

Lemma 5.1.2. *We have the following:*

(i) $s_{0,\xi,i,j}$ *is continuous on* B_0;

(ii) $s_{0,\xi,i}$ *is lower semi-continuous on* B_0;

(iii) $s_{0,\xi} = \inf_i s_{0,\xi,i} = \inf_i \sup_j s_{0,\xi,i,j}$;

(iv) $\sigma_{\bar{0},\xi} = \inf_{\mathbf{v}\in B_0} s_{0,\xi}(\mathbf{v})$ *(cf. § 4.2)*;

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(v) *for any* $\epsilon > 0$, *there exist* i *such that for any* $j \geqslant i$,

$$\sigma_{\bar{0},\xi} \geqslant \inf_{\mathbf{v}\in B_0} s_{0,\xi,i,j}(\mathbf{v}) - \epsilon.$$

The analogous statements with t *instead of* 0 *hold.*

Proof. Part (i) follows from the previous remark, and the continuity of $|\cdot|_{V_K}$ on V_K.

Part (ii) follows from part (i) since $s_{0,\xi,i}$ is the upper hull of the functions $s_{0,\xi,i,j}$.

Part (iii) follows from the definition of the log-slope $\sigma(\bar{\Pi}_0(v))$.

We now prove part (iv). Via $\bar{\Pi}_0$, the log-growth filtration of $M_{\mathcal{A}(0,1^-)}$ induces a filtration of V_0 by saturated $\mathcal{O}_K$-submodules. It is clear that $\sigma_{\bar{0},\xi} \leqslant s_{0,\xi}(\mathbf{v})$, with equality if and only if $\mathbf{v}$ induces a basis of $\oplus gr_\sigma V_0$.

By (iii) and (iv), one has

$$\sigma_{\bar{0},\xi} = \inf_i \inf_{\mathbf{v}\in B_0} \sup_j s_{0,\xi,i,j}(\mathbf{v}).$$

Hence, for any $\epsilon > 0$, there exists i for which

$$\sigma_{\bar{0},\xi} \geqslant \inf_{\mathbf{v}\in B_0} \sup_j s_{0,\xi,i,j}(\mathbf{v}) - \epsilon,$$

which gives part (v). □

5.2 Descent: from $\mathcal{O}_{\mathcal{E}_t}$ to $\mathcal{O}_K[[t]]$

Let $B'_t \subset B_t$ be the (metric) space of all bases of $V_{\mathcal{O}_K[[t]]}$.

Lemma 5.2.1. *We have the following.*

(i) *We have* $\inf_{\mathbf{w}\in B_t} s_{t,\xi,i,j}(\mathbf{w}) = \inf_{\mathbf{w}\in B'_t} s_{t,\xi,i,j}(\mathbf{w})$. *The infimum is attained.*

(ii) *For any* $\mathbf{w} \in B'_t$, *its specialization* $\mathbf{v} \in B_0$ *(for* $t \mapsto 0$*) satisfies*

$$s_{t,\xi,i,j}(\mathbf{w}) \geqslant s_{0,\xi,i,j}(\mathbf{v}).$$

Proof. For any $w \in V_{K_t}$, let us write

$$S_{t,\xi,i,j}(w) = \max\left(0, \max_{i\leqslant n\leqslant j} \frac{\log|w_n|_{V_{K_t}}}{\log(n+1)} - \xi\right),$$

so that

$$s_{t,\xi,i,j}(\mathbf{w}) = \sum_{w\in\mathbf{w}} S_{t,\xi,i,j}(w).$$

Note that $S_{t,\xi,i,j}$ is a continuous function on V_{K_t}.

Let R be a noetherian sub-$\mathcal{O}_K$-algebra of $\mathcal{O}_{K_t}$. Let us define inductively a (finite) decreasing filtration $F^\bullet_{t,\xi,i,j}V_R$ of V_R by sub-R-modules as follows:

$$F^0_{t,\xi,i,j}V_R = V_R, \tag{5.1}$$

$$F^{\ell+1}_{t,\xi,i,j}V_R = \left\{w \in F^\ell_{t,\xi,i,j}V_R, S_{t,\xi,i,j}(w) < \sup_{w'\in F^\ell_{t,\xi,i,j}V_R} S_{t,\xi,i,j}(w')\right\}. \tag{5.2}$$

We denote by $\bar{F}^\bullet_{t,\xi,i,j}V_R$ the image of the filtration $F^\bullet_{t,\xi,i,j}V_R$ in $V_R/\pi V_R$.

The cases of interest are $R = \mathcal{O}_K[[t]], \mathcal{O}_K((t))$ and $\mathcal{O}_{K_t}$. Since t is a π-adic unit and $(tv)_n = t.v_n$, it is clear that

$$F^\ell_{t,\xi,i,j}V_{\mathcal{O}_K((t))} = (F^\ell_{t,\xi,i,j}V_{\mathcal{O}_K[[t]]})\left[\frac{1}{t}\right],$$

and that $\bar{F}^\ell_{t,\xi,i,j}V_{\mathcal{O}_K[[t]]}/\bar{F}^{\ell+1}_{t,\xi,i,j}V_{\mathcal{O}_K[[t]]}$ has no t-torsion, hence is a free $k[[t]]$-module of rank at most μ.

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On the other hand, since $S_{t,\xi,i,j}$ is continuous and since $\mathcal{O}_K((t))$ is dense in $\mathcal{O}_{K_t}$, it follows (by induction on ℓ) that $F^{\ell}_{t,\xi,i,j}V_{\mathcal{O}_K((t))}$ is dense in $F^{\ell}_{t,\xi,i,j}V_{\mathcal{O}_{K_t}}$. In other words, $F^{\ell}_{t,\xi,i,j}V_{\mathcal{O}_{K_t}}$ is the topological closure of $F^{\ell}_{t,\xi,i,j}V_{\mathcal{O}_K((t))}$ in $V_{\mathcal{O}_{K_t}}$, which is nothing but $F^{\ell}_{t,\xi,i,j}V_{\mathcal{O}_K((t))}\otimes_{\mathcal{O}_K((t))}\mathcal{O}_{K_t}$ according to [Bou98, III.3.4, Theorem 3]. Therefore

$$F^{\ell}_{t,\xi,i,j}V_{\mathcal{O}_{K_t}} = F^{\ell}_{t,\xi,i,j}V_{\mathcal{O}_K[[t]]}\otimes_{\mathcal{O}_K[[t]]}\mathcal{O}_{K_t}, \tag{5.3}$$

$$\bar{F}^{\ell}_{t,\xi,i,j}V_{\mathcal{O}_{K_t}} = \bar{F}^{\ell}_{t,\xi,i,j}V_{\mathcal{O}_K[[t]]}\otimes_{k[[t]]}k((t)). \tag{5.4}$$

Let us now characterize those bases $\mathbf{w}$ of V_R for which the infimum of $s_{t,\xi,i,j}(\mathbf{w})$ is attained. For this purpose, we define inductively a (finite) decreasing sequence $\ell_\bullet$ of natural integers as follows:

- ℓ_0 is the last index for which $\bar{F}^{\ell_0}_{t,\xi,i,j}V_R\neq 0$;
- ℓ_{h+1} is the last index less than ℓ_h for which $\bar{F}^{\ell_{h+1}}_{t,\xi,i,j}V_R\neq \bar{F}^{\ell_h}_{t,\xi,i,j}V_R$.

If $\bar{F}^{\ell_h}_{t,\xi,i,j}V_R/\bar{F}^{\ell_h+1}_{t,\xi,i,j}V_R$ is a free $R/\pi R$-module, we denote its rank by ν_h, so that $\sum\nu_h=\mu$.

Then $\inf_{\mathbf{w}\text{ basis of }V_R}s_{t,\xi,i,j}(\mathbf{w})$ is attained at any basis $\mathbf{w}$ which induces a basis of

$$\bigoplus_h \bar{F}^{\ell_h}_{t,\xi,i,j}V_R/\bar{F}^{\ell_h+1}_{t,\xi,i,j}V_R$$

(i.e. such that for any h, $(w_{\nu_0+\cdots+\nu_{h-1}+1},\dots,w_{\nu_0+\cdots+\nu_h})$ is a ν_h-uple of $F^{\ell_h}_{t,\xi,i,j}V_R$ which induces a basis of $\bar{F}^{\ell_h}_{t,\xi,i,j}V_R/\bar{F}^{\ell_h+1}_{t,\xi,i,j}V_R$).

In view of (5.4), the sequence $\ell_\bullet$ is the same for $R=\mathcal{O}_K[[t]]$ and for $R=\mathcal{O}_{K_t}$, and we conclude that any $\mathbf{w}\in B'_t$ which induces a basis of $\bigoplus_h\bar{F}^{\ell_h}_{t,\xi,i,j}V_{\mathcal{O}_K[[t]]}/\bar{F}^{\ell_h+1}_{t,\xi,i,j}V_{\mathcal{O}_K[[t]]}$ realizes the infimum $\inf_{\mathbf{w}\in B_t}s_{t,\xi,i,j}(\mathbf{w})$. This establishes point (i) of the lemma.

Point (ii) is immediate. □

5.3 A compacity argument

Let us fix an arbitrary positive real number ϵ and choose an index i as in point (v) of Lemma 5.1.2. For any $j\geqslant i$, let $\mathbf{w}_{i,j}\in B'_t$ realize the infimum in Lemma 5.2.1(i). Let $\mathbf{v}_{i,j}\in B_0$ be the specialization of $\mathbf{w}_{i,j}$ for $t\mapsto 0$. By Lemmas 5.1.2(v) and 5.2.1(ii), we have

$$\sigma_{\bar{t},\xi}\geqslant s_{t,\xi,i,j}(\mathbf{w}_{i,j})-\epsilon\geqslant s_{0,\xi,i,j}(\mathbf{v}_{i,j})-\epsilon. \tag{5.5}$$

Let us first assume that K is locally compact, i.e. a finite extension of $\mathbf{Q}_p$. Then B_0 is compact. Let $(\mathbf{v}_{i,j_k})_k$ be a convergent subsequence, and let $\mathbf{v}_i\in B_0$ be its limit. Since $s_{0,\xi,i}$ is lower semi-continuous (Lemma 5.1.2(ii)), there is a compact neighborhood $K(\mathbf{v}_i)$ of $\mathbf{v}_i$ in B_0 on which $s_{0,\xi,i}$ takes values of at least $s_{0,\xi,i}(\mathbf{v}_i)-\epsilon$. It follows that, when k grows, the functions

$$\min(s_{0,\xi,i,j_k}+\epsilon, s_{0,\xi,i}(\mathbf{v}_i))$$

form an increasing family of continuous functions on $K(\mathbf{v}_i)$ which converges to the constant function $s_{0,\xi,i}(\mathbf{v}_i)$.

By Dini's lemma, the convergence is uniform. Thus, for $k\gg 0$, we have

$$\mathbf{v}_{i,j_k}\in K(\mathbf{v}_i)$$

and

$$s_{0,\xi,i}(\mathbf{v}_i)-\min(s_{0,\xi,i,j_k}(\mathbf{v}_{i,j_k})+\epsilon, s_{0,\xi,i}(\mathbf{v}_i))\leqslant\epsilon,$$

that is,

$$s_{0,\xi,i,j_k}(\mathbf{v}_{i,j_k})\geqslant s_{0,\xi,i}(\mathbf{v}_i)-2\epsilon. \tag{5.6}$$

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Combining (5.5) and (5.6), we get $\sigma_{\bar{t},\xi} \geqslant s_{0,\xi,i}(\mathbf{v}_i) - 3\epsilon$, whence

$$\sigma_{\bar{t},\xi} \geqslant \sigma_{\bar{0},\xi} - 3\epsilon. \tag{5.7}$$

Since (5.7) holds for any $\epsilon > 0$, this ends the proof of Lemma 4.2.1 when K is locally compact.

If K is not locally compact, we take advantage of Remark 5.1.1(2). We replace $V_0 \cong \mathcal{O}_K^{\mu}$ by the Berkovich affinoid space V_0^{an} (isomorphic to the unit polydisk), and B_0 by the corresponding Berkovich affinoid space B_0^{an}, which is *compact* [Ber90]. More precisely, the choice of a basis $\mathbf{v}$ of V_0 identifies V_0^{an} with the Berkovich spectrum $\mathcal{M}(K\{\{x_1, \ldots, x_\mu\}\})$. The linear map $v \mapsto v_n$ corresponds to the analytic (hence continuous) morphism of polydisks

$$\mathcal{M}(K\{\{x_1, \ldots, x_\mu\}\}) \to \mathcal{M}(K\{\{x_1/r, \ldots, x_\mu/r\}\})$$

(for r big enough) induced by

$$x_i \mapsto \sum_j \frac{1}{n!}(G_{\mathbf{v}^*,n})_{ji}(0)x_j$$

(see Remark 5.1.1(1)). On the other hand, a point u of $\mathcal{M}(K\{\{x_1/r, \ldots, x_\mu/r\}\})$ corresponds to a certain seminorm $|\cdot|_u$ on the Tate algebra $K\{\{x_1/r, \ldots, x_\mu/r\}\}$, and

$$u \mapsto |u|_{V_K} := \max(|x_1|_u, \ldots, |x_\mu|_u)$$

is continuous by definition of the Berkovich topology. Moreover, if u is a classical K-point of $\mathcal{M}(K\{\{x_1/r, \ldots, x_\mu/r\}\})$ (corresponding to an element of V_K of norm $\leqslant r$), $|u|_{V_K}$ coincides with the usual norm of u.

Therefore $s_{0,\xi,i,j}$ defines a continuous function on the compact (metric) space B_0^{an}, and one can apply the same compacity argument as before.

Remark 5.3.1. Analogous results hold in the context of p-adic q-difference modules, for $|q-1| < |p|^{-1/p-1}$. The analog of Theorem 3.2.1 was proven by Di Vizio [Div04, 5.1]. The analog of Theorem 4.1.1 can be proven along the same lines as above, *mutatis mutandis*.

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