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A WEAK LIMIT THEOREM FOR GENERALIZED JIŘINA PROCESSES

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Abstract

In this paper we prove that a sequence of scaled generalized Jiřina processes can converge weakly to a nonlinear diffusion process with Lévy jumps under certain conditions.

Keywords: Generalized branching processes; weak convergence; nonlinear diffusion processes

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1. Introduction

By a generalized Jiřina process we mean a continuous-state population-size-dependent branching process (continuous-state PSDBP) which is a modification of a classical Jiřina process, namely, a continuous-state branching process with discrete time [5], taking into account the fact that the reproductive behavior may depend on the size of the population. Here we recall its definition.

A time-homogeneous Markov process $\{Y(k),\ k = 0, 1, 2, \dots\}$ with state space $[0, \infty)$ is called a continuous-state PSDBP if its one-step transition function $P(x, \mathrm{d}y)$ satisfies

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} P(x, \mathrm{d}y) = \exp\biggl\{-x\biggl(\gamma(x)\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda u})\nu(x, \mathrm{d}u)\biggr)\biggr\}.$$

Here $\gamma(x)$ is a nonnegative Borel function and $(1 \wedge u)\nu(x, \mathrm{d}u)$ is a finite kernel from $[0, \infty)$ to $(0, \infty)$.

Obviously, a continuous-state PSDBP is determined by the pair of functions $\gamma(x)$ and $\nu(x, \cdot)$.

For any $x \geq 0$, define $m(x) := \gamma(x) + \int_{(0,\infty)} u\nu(x, \mathrm{d}u)$ and $\sigma^2(x) := \int_{(0,\infty)} u^2\nu(x, \mathrm{d}u)$. We call $m(x)$ and $\sigma^2(x)$ the offspring mean and the offspring variance (when the parent population is of size x), respectively, if the corresponding integral is finite. When $m(x)$ and $\sigma(x)$ are finite for all x, it is easy to obtain, for any $k > 0$,

$$\mathrm{E}[Y(k) \mid Y(k-1)] = m(Y(k-1))Y(k-1) \tag{1.1}$$

and

$$\mathrm{E}[(Y(k) - mY(k-1))^2 \mid Y(k-1)] = \sigma^2(Y(k-1))Y(k-1). \tag{1.2}$$

The continuous-state PSDBP was first introduced by Li [9], who showed that it can arise from the limit of a sequence of suitably scaled PSDBPs with discrete states [3], [8].

Diffusion approximation for branching processes was formulated by Feller [2] in 1951. He described a procedure for obtaining diffusions as limits of Galton–Watson processes.

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Jiřina [6] gave a more precise proof of Feller's assertion. Since then, many authors have done much work in this field. See, for example, [4], [11], and the references therein.

In 1977, Lipow [13] studied the diffusion approximation for state-dependent branching processes. He considered a sequence of continuous-time discrete-state branching processes $\{Z_n(t)\}$, $n = 1, 2, \dots$, whose reproductive behavior depends on the size of the population. Using Kurtz's theorem for the convergence of a semigroup, Lipow proved that, under certain conditions, the sequence $\{Z_n(nt)/n,\ t \geq 0\}$, $n = 1, 2, \dots$, converges weakly to a diffusion process with generator

$$\mathcal{A} f(x) = \lambda x[\beta f''(x) + \alpha(x) f'(x)],$$

where λ and β are positive constants, and $\alpha(x)$ is a bounded continuous function on $[0, \infty)$.

In addition, by means of a semigroup, Rosenkranz [14] showed that, under certain conditions, a sequence of density-dependent branching processes with random environment converges weakly to a diffusion process which can be obtained as a solution of a stochastic differential equation.

Motivated by their work, in this paper we discuss the relation between continuous-state PSDBPs and diffusion processes. More precisely, we are interested in the relation between the continuous-state PSDBPs and the diffusion processes with Lévy generators $\mathcal{L}$ satisfying

$$\mathcal{L} f(x) = x\alpha(x) f'(x) + x\beta(x) f''(x) + x \int_{(0,\infty)} (f(x+u) - f(x) - f'(x)u)\mu(x, \mathrm{d}u) \quad (1.3)$$

for any $f \in C_c^\infty[0, \infty)$, where $\alpha(x)$ and $\beta(x)$ are two functions, and $C_c^\infty[0, \infty)$ is the set of all infinite differentiable functions $f : [0, \infty) \to (-\infty, \infty)$ with compact support.

This kind of problem was also considered by Kawazu and Watanabe [7]. In [7], a continuous-state branching process (CSBP) and a CSBP with immigration, when immigration components exist, were defined. They pointed out that a conservative CSBP with immigration has the following generator:

$$\begin{aligned}\mathcal{A} f(x) = {} & axf''(x) + (bx + d) f'(x) + \int_{(0,\infty)} (f(x+y) - f(x))\nu_1(\mathrm{d}y) \\ & + x \int_{(0,\infty)} \left(f(x+y) - f(x) - \frac{y}{1+y^2} f'(x) \right) \nu(\mathrm{d}y),\end{aligned}$$

and that a Markov process with the above generator is a CSBP with immigration, where $a \geq 0$, b, and $d \geq 0$ are constants, and ν_1 and ν are two nonnegative measures on $(0, \infty)$ such that

$$\int_{(0,\infty)} \frac{y^2}{1+y^2} \nu(\mathrm{d}y) + \int_{(0,\infty)} \frac{y}{1+y} \nu_1(\mathrm{d}y) < \infty.$$

Then, Kawazu and Watanabe [7] proved that, under mild conditions, a sequence of scaled Galton–Watson processes with immigrations converges in finite-dimensional distributions to a CSBP with immigration. In addition, Li [12] extended this result from the convergence of finite-dimensional distributions to weak convergence in the Skorokhod space $D_{[0,\infty)}[0, \infty)$, namely, the space of càdlàg functions (i.e. those that are right continuous with left limits) from $[0, \infty)$ to $[0, \infty)$ with Skorokhod topology.

From [7] and [12] we can readily find that if $\int_{(0,\infty)} (y \wedge y^2)\nu(\mathrm{d}y) < \infty$ then the generator of the conservative CSBP can be written as

$$\mathcal{A} f(x) = axf''(x) + \bar{b}xf'(x) + x \int_{(0,\infty)} (f(x+y) - f(x) - yf'(x))\nu(\mathrm{d}y)$$

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for some $\bar{b} \geq b$, and that, under mild conditions, a sequence of scaled Galton–Watson processes converges weakly in $D_{[0,\infty)}[0, \infty)$ to a CSBP. For the similarity between $\mathcal{L}$ and $\mathcal{A}$, we naturally regard the diffusion process X with Lévy generator $\mathcal{L}$ as a generalized CSBP. Hence, the work of the present paper can be seen as a generalization of that of [7] and [12] in some sense. It is proved in the present paper that, under certain conditions, a sequence of scaled continuous-state PSDBPs converges weakly in the Skorokhod space $D_{[0,\infty)}[0, \infty)$ to a generalized CSBP.

The main tool used in this paper is the convergence theory of martingale problems; see [1, Chapter 4, Corollary 8.17]). Below, we briefly introduce some basic definitions on martingale problems. For further details, we refer the reader to [1, Chapter 3].

For a metric space E, $D_E[0, \infty)$ denotes the Skorokhod space of càdlàg functions from $[0, \infty)$ to E. Let $\mathcal{T}$ be an operator from $D(\mathcal{T}) \subset B(E)$ to $B(E)$, where $B(E)$ is the collection of bounded measurable functions on E and $D(\mathcal{T})$ is the domain of $\mathcal{T}$. By a solution of the martingale problem for $\mathcal{T}$ we mean a measurable stochastic process X with value in E defined on some probability space $(\Omega, \mathcal{F}, \mathrm{P})$ such that, for each $f \in D(\mathcal{T})$,

$$f(X(t)) - \int_0^t \mathcal{T} f(X(s))\, \mathrm{d}s$$

is a martingale with respect to the filtration

$$^*\mathcal{F}_t^X = \mathcal{F}_t^X \vee \sigma\bigg(\int_0^s h(X(u))\, \mathrm{d}u \colon s \leq t,\ h(\cdot) \in B(E)\bigg),$$

where $\mathcal{F}_t^X = \sigma(X(s),\ 0 \leq s \leq t)$. We say that a solution X of the martingale problem for $\mathcal{T}$ is a solution of the martingale problem for $(\mathcal{T}, x)$ in $D_E[0, \infty)$ if $X(0) = x$ almost surely and X is right continuous with left limits in path. We say that the uniqueness holds for solutions of the martingale problem for $(\mathcal{T}, x)$ in $D_E[0, \infty)$ if any two solutions in $D_E[0, \infty)$ have the same finite-dimensional distributions. If there exists a solution of the martingale problem for $(\mathcal{T}, x)$ in $D_E[0, \infty)$ and the uniqueness holds, we say that the martingale problem for $(\mathcal{T}, x)$ is well posed in $D_E[0, \infty)$.

This paper is organized as follows. In Section 2 we introduce the main assumptions and results of this paper. The proofs of the results are given in Section 3.

2. The main results

In the sequel, unless otherwise stated, let $\mathcal{L}$ be the operator defined in (1.3). Furthermore, we suppose that the following conditions hold.

(H1) $x\beta(x)$ and $\beta(x)$ are bounded and continuous and $\beta(x) > 0$ for $x > 0$.

(H2) $x\alpha(x)$ and $\alpha(x)$ are bounded and continuous.

(H3) For any Borel measurable set $\Gamma \subset (0, \infty)$, $\int_\Gamma u^2 x\mu(x, \mathrm{d}u)$ is bounded and continuous.

(H4) $b(x) := \int_{(0,\infty)} u^2 \mu(x, \mathrm{d}u)$ is bounded and continuous.

Let $\{Y_n(k),\ k \geq 0\}_n$ be a sequence of continuous-state PSDBPs given by a sequence of parameters $\gamma_n(x)$ and $\nu_n(x, \cdot)$. The corresponding offspring mean and offspring variance are respectively denoted by $m_n(x)$ and $\sigma_n^2(x)$. We assume the following conditions.

(E1) For $x \geq 0$, $m_n(x) = 1 + \alpha_n(x)/n > 0$ and $\sigma_n^2(x) = \beta_n(x)/n > 0$, where $\alpha_n(x)$ and $\beta_n(x)$ are uniformly bounded.

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(E2) $\alpha_n(x)$ and $\beta_n(x)$ converge locally uniformly to continuous functions $\alpha(x)$ and $2\beta(x) + b(x)$, respectively.

(E3) For $x \in [0, \infty)$, let $\mu_n(x, \cdot) = \nu_n(x, \cdot) - n^{-1}\mu(x, \cdot)$ and $\tau_n(x) = \int_{(0,\infty)} u^3 |\mu_n|(x, \mathrm{d}u)$, where $|\mu_n|(x, \cdot)$ is the total variation measure of $\mu_n(x, \cdot)$. Then $n\tau_n(x)$ converges locally uniformly to 0.

The fact that a sequence of functions $T_n(x)$ converges locally uniformly to a function $T(x)$ means that, for any bounded set $I \subset [0, \infty)$, $\lim_{n\to+\infty} \sup_{x\in I} |T_n(x) - T(x)| = 0$.

We have the following result for

$$X_n(t) = Y_n([nt]), \tag{2.1}$$

where $[nt]$ is the largest integer bounded by nt.

Theorem 2.1. *Suppose that (H1)–(H4) and (E1)–(E3) hold. Let $Y_n(0) \equiv x_0 \geq 0$. Then there exists a solution X of the martingale problem for $(\mathcal{L}, x_0)$ such that X_n converges weakly to X in the Skorokhod space $D_{[0,\infty)}[0, \infty)$.*

The converse of Theorem 2.1 holds in some sense.

Theorem 2.2. *Suppose that (H1)–(H4) hold and that $X(t)$ is the unique solution of the martingale problem for $(\mathcal{L}, x_0)$, where $x_0 \geq 0$. Furthermore, we assume that $a(x) := \int_{(0,\infty)} u\mu(x, \mathrm{d}u)$ is a bounded continuous function. Then there exists a sequence of continuous-state PSDBPs Y_n satisfying assumptions (E1)–(E3) and $Y_n(0) = x_0$. Hence, there exists a version of $X(t)$ such that $X_n(t) = Y_n([nt])$ converges weakly in $D_{[0,\infty)}[0, \infty)$ to this version.*

Remark 2.1. Theorem 2.2 still shows that conditions (E1)–(E3) are meaningful.

We will prove Theorem 2.1 via Corollary 8.17 of [1, Chapter 4], which requires two preconditions. One is the uniqueness of the solutions to the martingale problem for $(\mathcal{L}, x_0)$ in $D_{[0,\infty)}[0, \infty)$. This work was done in [10] by a standard method; see [10, Theorem 2.1]. In fact, based on Theorem 4.3 of [15], by the stopping time arguments, we can readily conceive that, under conditions (H1)–(H4), the martingale problem for $(\mathcal{L}, x_0)$ is well posed in $D_{[0,\infty)}[0, \infty)$. The other requirement is that the sequence X_n satisfies the compact containment condition in $D_{[0,\infty)}[0, \infty)$. We have the following lemma.

Lemma 2.1. *Let $\{Y_n\}$ be a sequence of continuous-state PSDBPs with $0 < m_n(x) = 1 + \alpha_n(x)/n$. Suppose that there exists a constant $\xi > 0$ such that $\alpha_n(x) < \xi$ for all n and x. Then $\{X_n\}$ defined by (2.1) satisfies the compact containment condition.*

From this lemma we immediately obtain the compact containment condition for X_n.

Corollary 2.1. *Under the assumptions of Theorem 2.1, $\{X_n\}$ satisfies the compact containment condition.*

The proofs of Theorem 2.1, Theorem 2.2, and Lemma 2.1 are given in the next section. For convenience, let $Z_n(x) := Y_n(1) - x$ and $\mathrm{E}_x[f(Y_n(1))] := \mathrm{E}[f(Y_n(1)) \mid Y_n(0) = x]$.

3. The proofs of the main results

Proof of Lemma 2.1. The proof is equal to checking that, for any $\eta > 0$ and $T \geq 0$, there exists a compact set $\Gamma_{\eta,T} \subset [0, \infty)$ such that

$$\liminf_{n\to\infty} \mathrm{P}\{X_n(t) \in \Gamma_{\eta,T} \text{ for } 0 \leq t \leq T\} \geq 1 - \eta. \tag{3.1}$$

Define a compact set $\Gamma_{\eta,T} := [0, 1/\eta e^{\xi T} x_0]$. Let $M_n(t) := X_n(t)/\sum_{i=0}^{[nt]-1} m_n(Y_n(i))$. Then, for any $n = 1, 2, \ldots$, M_n is a martingale. Observe that

$$\begin{aligned}
&\mathrm{P}\{X_n(t) \in \Gamma_{\eta,T} \text{ for } 0 \le t \le T\} \\
&\quad = 1 - \mathrm{P}\bigg\{M_n(t) > \frac{1}{\eta} e^{\xi T} x_0 \bigg/ \sum_{i=0}^{[nt]-1} m_n(Y_n(i)) \text{ for some } 0 \le t \le T\bigg\} \\
&\quad \ge 1 - \mathrm{P}\bigg\{\sup_{0 \le t \le T} M_n(t) > \frac{e^{\xi T} x_0}{\eta(1+\xi/n)^{[nT]}}\bigg\}.
\end{aligned} \tag{3.2}$$

Doob's inequality indicates that

$$\mathrm{P}\bigg\{\sup_{0 \le t \le T} M_n(t) > \frac{e^{\xi T} x_0}{\eta(1+\xi/n)^{[nT]}}\bigg\} \le \frac{\eta(1+\xi/n)^{[nT]}}{e^{\xi T} x_0} \sup_{0 \le t \le T} \mathrm{E}[M_n(t)] = \frac{\eta(1+\xi/n)^{[nT]}}{e^{\xi T}}.$$

Hence, (3.2) implies that

$$\mathrm{P}\{X_n(t) \in \Gamma_{\eta,T} \text{ for } 0 \le t \le T\} \ge 1 - \frac{\eta(1+\xi/n)^{[nT]}}{e^{\xi T}}. \tag{3.3}$$

Then (3.1) follows from (3.3) as $n \to \infty$. This completes the proof.

Proof of Theorem 2.1. According to Corollary 8.17 of [1, Chapter 4], it suffices to prove that

$$\lim_{n \to +\infty} \sup_{x \in [0,\infty)} |\mathcal{A}_n f(x) - \mathcal{L} f(x)| = 0$$

for any $f \in C_c^\infty[0, \infty)$, where $\mathcal{A}_n f(x) = n(\mathrm{E}_x[f(Y_n(1))] - f(x))$. To this end, it is enough to prove that, for any $x_n \in [0, \infty)$, if $x_n \to x \in [0, \infty]$ then

$$\lim_{n \to +\infty} (\mathcal{A}_n f(x_n) - \mathcal{L} f(x_n)) = 0 \tag{3.4}$$

for any given $f \in C_c^\infty[0, \infty)$.

From (1.1), it follows that

$$n\, \mathrm{E}_{x_n}[f'(x_n) Z_n(x_n)] = n x_n f'(x_n)(m_n(x_n) - 1) = x_n \alpha_n(x_n) f'(x_n).$$

By Taylor's expansion we obtain

$$\begin{aligned}
\mathcal{A}_n f(x_n) &= n\, \mathrm{E}_{x_n}[f(Y_n(1)) - f(x_n)] \\
&= n\, \mathrm{E}_{x_n}[f'(x_n) Z_n(x_n)] + \Delta_n(x_n) \\
&= x_n \alpha_n(x_n) f'(x_n) + \Delta_n(x_n),
\end{aligned}$$

where

$$\Delta_n(x_n) = \mathrm{E}_{x_n}\bigg[n \int_0^1 (1-w) f''(x_n + w Z_n(x_n)) Z_n^2(x_n)\, \mathrm{d}w\bigg].$$

In the remainder of the proof, we consider two cases.

Case 1: $x_n \to x = +\infty$. In this case, for sufficiently large n, we have $x_n > \delta$, where δ is an upper bound of the support set of $f \in C_c^\infty[0, \infty)$. Therefore, for sufficiently large n, $\mathcal{L} f(x_n) = 0$ and $x_n \alpha_n(x_n) f'(x_n) = 0$. Consequently, (3.4) is equivalent to

$$\lim_{n \to \infty} \Delta_n(x_n) = 0. \tag{3.5}$$

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Observe that

$$x_n + wZ_n(x_n) = x_n + w(Y_n(1) - x_n) \geq (1-w)x_n.$$

The integrand in $\Delta_n(x_n)$ is 0 if $w < 1 - \delta/x_n$. Hence, from (1.2), it follows that

$$\begin{aligned}\Delta_n(x_n) &\leq \mathrm{E}_{x_n}\biggl[\int_{1-\delta/x_n}^{1} (1-w)\|f''\| n Z_n^2(x_n)\,\mathrm{d}w\biggr] \\ &= \frac{\delta^2}{2x_n^2} n\,\mathrm{E}_{x_n}[Z_n^2(x_n)] \\ &= \frac{\delta^2}{2x_n^2}(n(m_n(x_n)-1)^2 x_n^2 + n\sigma_n^2(x_n)x_n) \\ &= \frac{\delta^2}{2}\biggl(\frac{\alpha_n^2(x_n)}{n} + \frac{\beta_n(x_n)}{x_n}\biggr). \end{aligned} \tag{3.6}$$

From assumption (E1) we have

$$\lim_{n\to\infty}\biggl(\frac{\alpha_n^2(x_n)}{n} + \frac{\beta_n(x_n)}{x_n}\biggr) = 0. \tag{3.7}$$

Then (3.5) follows from (3.6) and (3.7).

Case 2: $x_n \to x < +\infty$. In this case, by (H2) and (E2), we can readily obtain

$$x_n f'(x_n)\alpha_n(x_n) \to x\alpha(x) f'(x) \quad \text{as } n \to \infty.$$

At the same time, (H3) and (H4) imply that

$$\begin{aligned} &\biggl|\int_{(0,\infty)} (f(x_n+u) - f(x_n) - f'(x_n)u)x_n\mu(x_n,\mathrm{d}u) \\ &\quad - \int_{(0,\infty)} (f(x+u) - f(x) - f'(x)u)x_n\mu(x_n,\mathrm{d}u)\biggr| \\ &\qquad \leq \int_{(0,\infty)} |f'(z_n+u) - f'(z_n) - f''(z_n)u||x_n - x|x_n\mu(x_n,\mathrm{d}u) \\ &\qquad \leq \|f'''\||x_n - x|\int_{(0,\infty)} u^2 x_n\mu(x_n,\mathrm{d}u) \\ &\qquad \to 0, \end{aligned} \tag{3.8}$$

where $z_n \in (x, x_n)$. Let $\phi_x(u) = (f(x+u) - f(x) - f'(x)u)/u^2$ for $u \in (0,\infty)$. From $f \in C_c^\infty[0,\infty)$ we know that $\phi_x(u)$ is bounded and continuous for any $x \geq 0$. Therefore, (H3) indicates that, as $n \to \infty$,

$$\begin{aligned} &\int_{(0,\infty)} (f(x+u) - f(x) - f'(x)u)x_n\mu(x_n,\mathrm{d}u) \\ &\quad - \int_{(0,\infty)} (f(x+u) - f(x) - f'(x)u)x\mu(x,\mathrm{d}u), \\ &\qquad = \int_{(0,\infty)} \phi_x(u)x_n u^2\mu(x_n,\mathrm{d}u) - \int_{(0,\infty)} \phi_x(u)xu^2\mu(x,\mathrm{d}u) \\ &\qquad \to 0. \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we obtain

$$\int_{(0,\infty)} (f(x_n+u) - f(x_n) - f'(x_n)u)x_n\mu(x_n, \mathrm{d}u) \\ \to \int_{(0,\infty)} (f(x+u) - f(x) - f'(x)u)x\mu(x, \mathrm{d}u),$$

which implies that $\mathcal{L}f(x_n) \to \mathcal{L}f(x)$ as $x_n \to x < \infty$. Consequently, to prove (3.4), it is enough to prove that

$$\Delta_n(x_n) \to \beta(x)xf''(x) + \int_{(0,\infty)} (f(x+u) - f(x) - f'(x)u)x\mu(x, \mathrm{d}u), \tag{3.10}$$

which is proved in the following lemma.

Lemma 3.1. *Using the same assumptions and notation as in Theorem 2.1 and its proof, if $x_n \to x < \infty$ then (3.10) holds.*

Proof. By Fubini's theorem we have

$$\begin{aligned}\Delta_n(x_n) = &\int_0^1 (1-w)\,\mathrm{E}_{x_n}[nf''(x_n + wZ_n(x_n))Z_n^2(x_n)]\,\mathrm{d}w \\ &- \int_0^1 (1-w)\,\mathrm{E}_{x_n}[nf''(x + wZ_n(x_n))Z_n^2(x_n)]\,\mathrm{d}w \\ &+ \int_0^1 (1-w)\,\mathrm{E}_{x_n}[nf''(x + wZ_n(x_n))Z_n^2(x_n)]\,\mathrm{d}w.\end{aligned}$$

Since f'' is uniformly continuous, the dominated convergence theorem implies that, as $x_n \to x$,

$$\int_0^1 (1-w)\,\mathrm{E}_{x_n}[nf''(x_n + wZ_n(x_n))Z_n^2(x_n)]\,\mathrm{d}w \\ - \int_0^1 (1-w)\,\mathrm{E}_{x_n}[nf''(x + wZ_n(x_n))Z_n^2(x_n)]\,\mathrm{d}w \to 0.$$

Hence, we only need to prove that

$$\int_0^1 (1-w)\,\mathrm{E}_{x_n}[nf''(x + wZ_n(x_n))Z_n^2(x_n)]\,\mathrm{d}w \\ \to \beta(x)xf''(x) + \int_{(0,\infty)} (f(x+u) - f(x) - f'(x)u)x\mu(x, \mathrm{d}u). \tag{3.11}$$

It suffices to prove that, as $x_n \to x$,

$$\mathrm{E}_{x_n}[nf''(x + wZ_n(x_n))Z_n^2(x_n)] \\ \to 2x\beta(x)f''(x) + x\int_{(0,\infty)} f''(x+wu)u^2\mu(x, \mathrm{d}u). \tag{3.12}$$

In fact, if (3.12) holds then (3.11) follows from Taylor's expansion.

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Let $M_x = -\sup_n x_n$. For any $n \geq 0$, define a measure $Q_n(\cdot)$ on $[M_x, \infty)$ such that, for any Borel measurable set $A \subset [M_x, \infty)$,

$$Q_n(A) := \mathrm{E}_{x_n}[n\,\mathbf{1}_A(Z_n(x_n))Z_n^2(x_n)] = \mathrm{E}_{x_n}[n\,\mathbf{1}_A(Y_n(1)-x_n)(Y_n(1)-x_n)^2]. \tag{3.13}$$

Then, for any $\lambda \geq 0$,

$$\int_{[M_x,\infty)} \mathrm{e}^{-\lambda u} Q_n(\mathrm{d}u) = \mathrm{E}_{x_n}[n \exp\{-\lambda(Y_n(1)-x_n)\}(Y_n(1)-x_n)^2]. \tag{3.14}$$

Note that Y_n is the continuous-state PSDBP with parameters $\gamma_n(x)$ and $\nu_n(x, \cdot)$. We have

$$\begin{aligned}\mathrm{E}_{x_n}&[\exp\{-\lambda(Y_n(1)-x_n)\}]\\ &= \exp\bigg\{-x_n\bigg((\gamma_n(x_n)-1)\lambda + \int_{(0,\infty)}(1-\mathrm{e}^{-\lambda u})\nu_n(x_n,\mathrm{d}u)\bigg)\bigg\}\\ &= \exp\bigg\{-x_n\bigg((m_n(x_n)-1)\lambda + \int_{(0,\infty)}(1-\mathrm{e}^{-\lambda u}-\lambda u)\nu_n(x_n,\mathrm{d}u)\bigg)\bigg\}.\end{aligned}$$

Using this formula, via some simple calculation, we obtain

$$\begin{aligned}\mathrm{E}_{x_n}&[n\exp\{-\lambda(Y_n(1)-x_n)\}(Y_n(1)-x_n)^2]\\ &= \psi(n,x_n,\lambda)\phi(n,x_n,\lambda) + n\varphi(n,x_n,\lambda)^2\phi(n,x_n,\lambda),\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}\phi(n,x_n,\lambda) &= \exp\bigg\{-x_n\bigg((m_n(x_n)-1)\lambda + \int_{(0,\infty)}(1-\mathrm{e}^{-\lambda u}-\lambda u)\nu_n(x_n,\mathrm{d}u)\bigg)\bigg\},\\ \varphi(n,x_n,\lambda) &= -x_n\bigg(m_n(x_n)-1+\int_{(0,\infty)}u(\mathrm{e}^{-\lambda u}-1)\nu_n(x_n,\mathrm{d}u)\bigg),\\ \psi(n,x_n,\lambda) &= x_n\int_{(0,\infty)}u^2\mathrm{e}^{-\lambda u}\mu(x_n,\mathrm{d}u) + x_n\int_{(0,\infty)}u^2\mathrm{e}^{-\lambda u}n\mu_n(x_n,\mathrm{d}u).\end{aligned}$$

Note that

$$\bigg|-x_n\bigg((m_n(x_n)-1)\lambda+\int_{(0,\infty)}(1-\mathrm{e}^{-\lambda u}-\lambda u)\nu_n(x_n,\mathrm{d}u)\bigg)\bigg| \leq \lambda\frac{x_n|\alpha_n(x_n)|}{n} + \lambda^2\frac{\beta_n(x_n)}{n}$$

and that

$$|\varphi(n,x_n,\lambda)| \leq x_n\frac{|\alpha_n(x_n)|+\lambda\beta_n(x_n)}{n}.$$

Then by condition (A1) we have, as $n \to +\infty$,

$$\phi(n,x_n,\lambda) \to 1, \qquad n\varphi(n,x_n,\lambda)^2 \to 0. \tag{3.16}$$

Since

$$\int_{(0,\infty)} u^2\mu_n(x,\mathrm{d}u) = \frac{\beta_n(x)-b(x)}{n},$$

(E3) implies that

$$\bigg|\int_{(0,\infty)}u^2\mathrm{e}^{-\lambda u}n\mu_n(x_n,\mathrm{d}u) - (\beta_n(x_n)-b(x_n))\bigg| \leq \lambda\int_{(0,\infty)}u^3 n|\mu_n|(x_n,\mathrm{d}u) \to 0.$$

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Furthermore, from (E2)–(E3) and (H1)–(H2), it follows that $\beta_n(x_n) - b(x_n) \to 2\beta(x)$. Hence,

$$\psi(n, x_n, \lambda) \to x \int_{(0,\infty)} u^2 e^{-\lambda s} \mu(x, du) + 2x\beta(x). \tag{3.17}$$

Combining (3.15) with (3.16) and (3.17), we obtain

$$E_{x_n}[n \exp\{-\lambda(Y_n(1) - x_n)\}(Y_n(1) - x_n)^2] \to x \int_{(0,\infty)} u^2 e^{-\lambda s} \mu(x, du) + 2x\beta(x). \tag{3.18}$$

Define a measure $\bar{\mu}(x, \cdot)$ on $[M_x, \infty)$ as follows:

$$\bar{\mu}(x, A) = \begin{cases} \int_{(0,\infty)\cap A} xu^2 \mu(x, du) + 2x\beta(x), & 0 \in A, \\ \int_{(0,\infty)\cap A} xu^2 \mu(x, du), & \text{otherwise}, \end{cases}$$

for any Borel measurable set $A \subset [M_x, \infty)$. Equations (3.13), (3.14) and (3.18) imply that

$$\int_{[M_x,\infty)} e^{-\lambda u} Q_n(du) \to \int_{[M_x,\infty)} e^{-\lambda u} \bar{\mu}(du) \quad \text{as } n \to +\infty.$$

Hence, for any bounded continuous function $h(u)$ on $[M_x, \infty)$,

$$\begin{aligned} E_{x_n}[nh(Y_n(1) - x_n)(Y_n(1) - x_n)^2] &= \int_{[M_x,\infty)} h(u) Q_n(du) \\ &\to \int_{[M_x,\infty)} h(u) \bar{\mu}(du) \\ &= 2x\beta(x)h(0) + x \int_{(0,\infty)} h(u) u^2 \mu(x, du). \end{aligned}$$

Let $h(u) = f''(x + wu)$. Then (3.12) holds.

Proof of Theorem 2.2. From the assumptions of Theorem 2.2, for any $n \geq 1$, we can construct a continuous-state PSDBP Y_n which satisfies

$$\begin{aligned} &E[e^{-\lambda Y_n(k+1)} \mid Y_n(k) = x] \\ &\quad = \exp\left\{-x\left(\frac{e^{(\alpha(x)-a(x))/n}\lambda}{1 + n^{-1}\lambda\beta(x)} + \frac{1}{n}\int_{(0,\infty)} (1 - e^{-\lambda u})\mu(x, du)\right)\right\} \\ &\quad = \exp\left\{-x\left(\gamma_n(x)\lambda + \int_{(0,\infty)} (1 - e^{-\lambda u})\mu_n(x, du) + \frac{1}{n}\int_{(0,\infty)} (1 - e^{-\lambda u})\mu(x, du)\right)\right\} \end{aligned}$$

for any $\lambda \geq 0$, where

$$\gamma_n(x) = \begin{cases} 0, & \beta(x) > 0, \\ e^{(\alpha(x)-a(x))/n}, & \beta(x) = 0, \end{cases} \tag{3.19}$$

$$\mu_n(x, du) = \begin{cases} e^{(\alpha(x)-a(x))/n}\left(\dfrac{n}{\beta(x)}\right)^2 e^{-nu/\beta(x)}\, du, & \beta(x) > 0, \\ 0, & \beta(x) = 0. \end{cases} \tag{3.20}$$

By some simple calculation, we obtain

$$m_n(x) = \mathrm{e}^{(\alpha(x)-a(x))/n} + \frac{a(x)}{n}, \qquad \sigma_n^2(x) = \mathrm{e}^{(\alpha(x)-a(x))/n}\frac{2\beta(x)}{n} + \frac{b(x)}{n},$$

and

$$\tau_n(x) := \int_{(0,\infty)} u^3 \mu_n(x, \mathrm{d}u) = 6\mathrm{e}^{(\alpha(x)-a(x))/n}\frac{\beta^2(x)}{n^2}.$$

Since $\alpha(x)$, $\beta(x)$, $a(x)$, and $b(x)$ are bounded and continuous, the sequence $\{Y_n\}$ satisfies all the conditions of (E1)–(E3). Therefore, Theorem 2.2 follows from Theorem 2.1.

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