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Quantum Symmetries of Graph C^*-algebras

Simon Schmidt and Moritz Weber

Abstract. The study of graph C^*-algebras has a long history in operator algebras. Surprisingly, their quantum symmetries have not yet been computed. We close this gap by proving that the quantum automorphism group of a finite, directed graph without multiple edges acts maximally on the corresponding graph C^*-algebra. This shows that the quantum symmetry of a graph coincides with the quantum symmetry of the graph C^*-algebra. In our result, we use the definition of quantum automorphism groups of graphs as given by Banica in 2005. Note that Bichon gave a different definition in 2003; our action is inspired from his work. We review and compare these two definitions and we give a complete table of quantum automorphism groups (with respect to either of the two definitions) for undirected graphs on four vertices.

Introduction

Symmetry constitutes one of the most important properties of a graph. It is captured by its automorphism group

$$\mathrm{Aut}(\Gamma) := \{\sigma \in S_n \mid \sigma\varepsilon = \varepsilon\sigma\} \subseteq S_n,$$

where $\Gamma = (V, E)$ is a finite graph with n vertices and no multiple edges, $\varepsilon \in M_n(\{0,1\})$ is its adjacency matrix, and S_n is the symmetric group. In modern mathematics, notably in operator algebras, symmetries are no longer described only by groups, but by quantum groups. In 2005, Banica [1] gave a definition of a quantum automorphism group of a finite graph within Woronowicz's theory of compact matrix quantum groups [20]. In our notation, $G^+_{\mathrm{aut}}(\Gamma)$ is based on the C^*-algebra

$$\begin{aligned} &C(G^+_{\mathrm{aut}}(\Gamma)) \\ &\quad := C(S_n^+)\langle u\varepsilon = \varepsilon u\rangle \\ &\quad = C^*\Big(u_{ij}, i, j = 1, \dots, n \;\Big|\; u_{ij} = u_{ij}^* = u_{ij}^2, \sum_l u_{il} = 1 = \sum_l u_{lj}, R_{\mathrm{Ban}}\Big), \end{aligned}$$

where S_n^+ is Wang's quantum symmetric group [18] and R_{Ban} are the relations

$$\sum_k u_{ik}\varepsilon_{kj} = \sum_k \varepsilon_{ik} u_{kj}.$$

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Earlier, in 2003, Bichon [5] defined a quantum automorphism group $G^*_{\mathrm{aut}}(\Gamma)$ via

$$C(G^*_{\mathrm{aut}}(\Gamma)) := C^*\Big(u_{ij}, i,j=1,\dots,n \;\Big|\; u_{ij} = u^*_{ij} = u^2_{ij}, \sum_l u_{il} = 1 = \sum_l u_{lj}, R_{\mathrm{Bic}}\Big),$$

where R_{Bic} are the relations

$$\sum_k u_{ik}\varepsilon_{kj} = \sum_k \varepsilon_{ik}u_{kj}, \quad u_{s(e)s(f)}u_{r(e)r(f)} = u_{r(e)r(f)}u_{s(e)s(f)} \text{ for } e,f \in E,$$

and $r\colon E \to V$ and $s\colon E \to V$ are range and source maps, respectively. We immediately see that

$$\mathrm{Aut}(\Gamma) \subseteq G^*_{\mathrm{aut}}(\Gamma) \subseteq G^+_{\mathrm{aut}}(\Gamma)$$

holds, in the sense that there are surjective *-homomorphisms:

$$\begin{array}{ccccc} C(G^+_{\mathrm{aut}}(\Gamma)) & \longrightarrow & C(G^*_{\mathrm{aut}}(\Gamma)) & \longrightarrow & C(\mathrm{Aut}(\Gamma)) \\ u_{ij} & \longmapsto & u_{ij} & \longmapsto & (\sigma \mapsto \sigma_{ij}) \end{array}$$

Relatively little is known about these two quantum automorphism groups of graphs, and we refer the reader to Section 3.4 for an overview on all published articles in this area.

Graph C^*-algebras in turn are well-established objects in operator algebras. They emerged from Cuntz and Krieger's work [8] in the 1980's and have become one of the most important classes of examples of C^*-algebras; see, for instance, Raeburn's book for an overview [15]. Given a finite graph $\Gamma = (V,E)$ the associated graph C^*-algebra $C^*(\Gamma)$ is defined as

$$C^*(\Gamma) := C^*\Big(p_v, v \in V, s_e, e \in E \;\Big|\; p_v = p^*_v = p^2_v,\ p_v p_w = 0 \text{ for } v \neq w,\ s^*_e s_e = p_{r(e)},\ \sum_{\substack{e\in E\\ s(e)=v}} s_e s^*_e = p_v, \text{ if } s^{-1}(v) \neq \varnothing\Big).$$

A natural question is then: what is the quantum symmetry group of the graph C^*-algebra, and is it one of the above two quantum automorphism groups of the underlying graphs? The answer is that it is given by the one defined by Banica. Note however, that Bichon's definition has its justification in other contexts, such as in [4, 6] or in the recent work by Speicher and the second author [16]. Moreover, Bichon's work [5] inspired the formulatation our main theorem; see also Remark 4.2.

1 Main Result

Intuitively speaking, our main result is that the quantum symmetry of a finite graph without multiple edges coincides with the quantum symmetry of the associated graph

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C^*-algebra. In other words, the following diagram is commutative.

$$\begin{tikzcd}
\text{finite graphs} \arrow[rr, "\Gamma\mapsto C^*(\Gamma)"] \arrow[dr, "\Gamma\mapsto G_{\mathrm{aut}}^+(\Gamma)"'] & & \text{graph } C^*\text{-algebras} \arrow[dl, "C^*(\Gamma)\mapsto \mathrm{QSym}(C^*(\Gamma))"] \\
& \text{quantum symmetry groups} &
\end{tikzcd}$$

More precisely, we have the following result.

Main Theorem *Let Γ be a finite graph with n vertices $V = \{1,\dots,n\}$ and m edges $E = \{e_1,\dots,e_m\}$ having no multiple edges. The maps*

$$\begin{aligned}
\alpha\colon C^*(\Gamma) &\longrightarrow C(G_{\mathrm{aut}}^+(\Gamma)) \otimes C^*(\Gamma),\\
p_i &\longmapsto \sum_{k=1}^{n} u_{ik}\otimes p_k, && 1\le i\le n,\\
s_{e_j} &\longmapsto \sum_{l=1}^{m} u_{s(e_j)s(e_l)}u_{r(e_j)r(e_l)}\otimes s_{e_l}, && 1\le j\le m,
\end{aligned}$$

and

$$\begin{aligned}
\beta\colon C^*(\Gamma) &\longrightarrow C(G_{\mathrm{aut}}^+(\Gamma)) \otimes C^*(\Gamma),\\
p_i &\longmapsto \sum_{k=1}^{n} u_{ki}\otimes p_k, && 1\le i\le n,\\
s_{e_j} &\longmapsto \sum_{l=1}^{m} u_{s(e_l)s(e_j)}u_{r(e_l)r(e_j)}\otimes s_{e_l}, && 1\le j\le m
\end{aligned}$$

define a left and a right action of $G_{\mathrm{aut}}^+(\Gamma)$ on $C^*(\Gamma)$, respectively. Moreover, whenever G is a compact matrix quantum group acting on $C^*(\Gamma)$ in the above way, we have $G\subseteq G_{\mathrm{aut}}^+(\Gamma)$. In this sense, the quantum automorphism group $G_{\mathrm{aut}}^+(\Gamma)$ of Γ is the quantum symmetry group of $C^*(\Gamma)$; see also Remark 4.1.

We also provide some tools for comparing and dealing with the two definitions of quantum automorphism groups of graphs, $G_{\mathrm{aut}}^+(\Gamma)$ and $G_{\mathrm{aut}}^*(\Gamma)$, notably depending on the complement Γ^c of Γ; see Section 3.5. Moreover, we provide a list of all $\mathrm{Aut}(\Gamma)$, $G_{\mathrm{aut}}^+(\Gamma)$ and $G_{\mathrm{aut}}^*(\Gamma)$ for undirected graphs Γ on four vertices, having no multiple edges and no loops; see Section 3.6.

2 Preliminaries

2.1 Graphs

We fix some notation for graphs used throughout this article. A graph $\Gamma = (V, E)$ is *finite*, if the set V of *vertices* and the set E of edges are finite. We denote by $r\colon E\to V$ the *range map* and by $s\colon E\to V$ the *source map*. A graph is *undirected* if for every $e\in E$, there is a $f\in E$ with $s(f) = r(e)$ and $r(f) = s(e)$; it is *directed* otherwise. Elements $e\in E$ with $s(e) = r(e)$ are called *loops*. A graph *without multiple edges* is a directed graph where there are no $e, f\in E$, $e\neq f$, such that $s(e) = s(f)$ and $r(e) = r(f)$. For

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a finite graph $\Gamma = (V, E)$ with $V = \{1, \dots, n\}$, its *adjacency matrix* $\varepsilon \in M_n(\mathbb{N}_0)$ is defined as $\varepsilon_{ij} := \#\{e \in E \mid s(e) = i, r(e) = j\}$. Here $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Throughout this article we restrict to finite graphs having no multiple edges.

If $\Gamma = (V, E)$ is a directed graph without multiple edges, we denote by $\Gamma^c = (V, E')$ the *complement* of Γ, where $E' = (V \times V)\backslash E$. Within the category of graphs having no loops, the complement Γ^c is defined using $E' = (V \times V)\backslash(E \cup \{(i,i);\ i \in V\})$.

2.2 Automorphism Groups of Graphs

For a finite graph $\Gamma = (V, E)$ without multiple edges, a *graph automorphism* is a bijective map $\sigma\colon V \to V$ such that $(\sigma(i), \sigma(j)) \in E$ if and only if $(i, j) \in E$. In other words, $\varepsilon_{\sigma(i)\sigma(j)} = 1$ if and only if $\varepsilon_{ij} = 1$. The set of all graph automorphisms of Γ forms a group, the *automorphism group* $\mathrm{Aut}(\Gamma)$. We can view $\mathrm{Aut}(\Gamma)$ as a subgroup of the symmetric group S_n if Γ has n vertices:

$$\mathrm{Aut}(\Gamma) = \{\sigma \in S_n \mid \sigma\varepsilon = \varepsilon\sigma\} \subseteq S_n.$$

2.3 Graph C^*-algebras

The theory of Graph C^*-algebras has its roots in Cuntz and Krieger's work [8] in 1980. Nowadays, it forms a well-developed and very active part of the theory of C^*-algebras; see [15] for an overview or [9] for recent developments. For a finite, directed graph $\Gamma = (V, E)$ without multiple edges, the *graph C^*-algebra* $C^*(\Gamma)$ is the universal C^*-algebra generated by mutually orthogonal projections p_v, $v \in V$ and partial isometries s_e, $e \in E$ such that

(i) $s_e^* s_e = p_{r(e)}$ for all $e \in E$,
(ii) $p_v = \sum_{e \in E: s(e) = v} s_e s_e^*$ for every $v \in V$ with $s^{-1}(v) \neq \varnothing$.

It follows immediately that $s_e^* s_f = 0$ for $e \neq f$ and $\sum_{v \in V} p_v = 1$ hold true in $C^*(\Gamma)$.

2.4 Compact Matrix Quantum Groups

Compact matrix quantum groups were defined by Woronowicz [19, 20] in 1987. They form a special class of compact quantum groups; see [13, 17] for recent books. A *compact matrix quantum group G* is a pair $(C(G), u)$, where $C(G)$ is a unital (not necessarily commutative) C^*-algebra that is generated by u_{ij}, $1 \le i, j \le n$, the entries of a matrix $u \in M_n(C(G))$. Moreover, the $*$-homomorphism $\Delta\colon C(G) \to C(G) \otimes C(G)$, $u_{ij} \mapsto \sum_{k=1}^n u_{ik} \otimes u_{kj}$ must exist, and u and its transpose u^t must be invertible.

Example 2.1 Consider the *quantum symmetric group* $S_n^+ = (C(S_n^+), u)$, as defined by Wang [18] in 1998. It is the compact matrix quantum group given by

$$C(S_n^+) := C^*\Big(u_{ij} \;\Big|\; u_{ij} = u_{ij}^* = u_{ij}^2,\ \sum_{l=1}^n u_{il} = 1 = \sum_{l=1}^n u_{li} \text{ for all } 1 \le i, j \le n\Big).$$

One can show that the quotient of $C(S_n^+)$ by the relations that all u_{ij} commute is exactly $C(S_n)$. Moreover, the symmetric group S_n can be viewed as a compact matrix quantum group $S_n = (C(S_n), u)$, where $u_{ij}\colon S_n \to \mathbb{C}$ are the evaluation maps of the matrix entries. This justifies the name "quantum symmetric group".

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If $G = (C(G), u)$ and $H = (C(H), v)$ are compact matrix quantum groups with $u \in M_n(C(G))$ and $v \in M_n(C(H))$, we say that G is a *compact matrix quantum subgroup* of H if there is a surjective $*$-homomorphism from $C(H)$ to $C(G)$ mapping generators to generators. In this case we write $G \subseteq H$. For example, $S_n \subseteq S_n^+$. The compact matrix quantum groups G and H are equal as compact matrix quantum groups, writing $G = H$, if we have $G \subseteq H$ and $H \subseteq G$.

2.5 Actions of Quantum Groups

Let $G = (C(G), u)$ be a compact matrix quantum group and let B be a C^*-algebra. A *left action* of G on B is a unital $*$-homomorphism $\alpha: B \to C(G) \otimes B$ such that

(i) $(\Delta \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \alpha) \circ \alpha$,
(ii) $\alpha(B)(C(G) \otimes 1)$ is linearly dense in $C(G) \otimes B$.

A *right action* is a unital $*$-homomorphism $\beta: B \to C(G) \otimes B$ with

(i) $((F \circ \Delta) \otimes \mathrm{id})) \circ \beta = (\mathrm{id} \otimes \beta) \circ \beta$,
(ii) $\beta(B)(C(G) \otimes 1)$ is linearly dense in $C(G) \otimes B$,

where F is the flip map:

$$F: C(G) \otimes C(G) \to C(G) \otimes C(G), \quad a \otimes b \mapsto b \otimes a.$$

Note that in some articles (for instance in [18]), the property (ii) is replaced by

(ii′) $(\varepsilon \otimes \mathrm{id}) \circ \alpha = \mathrm{id}$,
(iii′) there is a dense $*$-subalgebra of B such that α restricts to a right coaction of the Hopf $*$-algebra on the $*$-subalgebra.

One can show that (ii′) and (iii′) are equivalent to (ii); see [14].

2.6 Quantum Symmetry Group of n Points

According to Wang's work [18], we know that S_n^+ (from Example 2.1) is the quantum symmetry group of n points in the sense that

(i) S_n^+ acts from the left and right on

$$C^*\Big(p_1, \dots, p_n \mid p_i = p_i^* = p_i^2, \sum_l p_l = 1\Big)$$

by $\alpha(p_i) := \sum_{k=1}^n u_{ik} \otimes p_k$ and $\beta(p_i) := \sum_{k=1}^n u_{ki} \otimes p_k$, respectively,

(ii) S_n^+ is maximal with these actions; *i.e.,* any other compact matrix quantum group with actions defined as α and β is a compact matrix quantum subgroup of S_n^+.

See also [11] for similar questions around quantum symmetries.

3 Quantum Automorphism Groups of Graphs

Wang's work in the 1990's was the starting point of the investigations of quantum symmetry phenomena for discrete structures (within Woronowicz's framework). Note that n points can be viewed as the totally disconnected graph on n vertices. A decade later, Banica and Bichon extended Wang's approach to a theory of quantum automorphism groups of finite graphs. In the sequel, we restrict our attention to finite graphs having no multiple edges.

3.1 Bichon's Quantum Automorphism Group of a Graph

In 2003, Bichon [5] defined a quantum automorphism group as follows.

Definition 3.1 Let $\Gamma = (V, E)$ be a finite graph with n vertices $V = \{1, \dots, n\}$ and m edges $E = \{e_1, \dots, e_m\}$. The *quantum automorphism group* $G^*_{\mathrm{aut}}(\Gamma)$ is the compact matrix quantum group $(C(G^*_{\mathrm{aut}}(\Gamma)), u)$, where $C(G^*_{\mathrm{aut}}(\Gamma))$ is the universal C^*-algebra with generators u_{ij}, $1 \le i, j \le n$ and relations

$$u_{ij} = u_{ij}^*, \quad u_{ij}u_{ik} = \delta_{jk}u_{ij}, \quad u_{ji}u_{ki} = \delta_{jk}u_{ji}, \qquad 1 \le i, j, k \le n, \tag{3.1}$$

$$\sum_{l=1}^{n} u_{il} = 1 = \sum_{l=1}^{n} u_{li}, \qquad 1 \le i \le n, \tag{3.2}$$

$$u_{s(e_j)i}u_{r(e_j)k} = u_{r(e_j)k}u_{s(e_j)i} = 0, \qquad e_j \in E, (i,k) \notin E, \tag{3.3}$$

$$u_{is(e_j)}u_{kr(e_j)} = u_{kr(e_j)}u_{is(e_j)} = 0, \qquad e_j \in E, (i,k) \notin E, \tag{3.4}$$

$$u_{s(e_j)s(e_l)}u_{r(e_j)r(e_l)} = u_{r(e_j)r(e_l)}u_{s(e_j)s(e_l)}, \qquad e_j, e_l \in E. \tag{3.5}$$

In Bichon's original definition, there is actually another relation that is implied by the others:

$$\sum_{l=1}^{m} u_{s(e_l)s(e_j)}u_{r(e_l)r(e_j)} = 1 = \sum_{l=1}^{m} u_{s(e_j)s(e_l)}u_{r(e_j)r(e_l)}, \qquad e_j \in E \tag{3.6}$$

Indeed, relations (3.6) are implied by relations (3.2), (3.3), and (3.4):

$$\sum_{l=1}^{m} u_{s(e_l)s(e_j)}u_{r(e_l)r(e_j)} = \sum_{i,k=1}^{n} u_{is(e_j)}u_{kr(e_j)} = \Big(\sum_{i=1}^{n} u_{is(e_j)}\Big)\Big(\sum_{k=1}^{n} u_{kr(e_j)}\Big) = 1.$$

3.2 Banica's Quantum Automorphism Group of a Graph

Two years later, Banica [1] gave the following definition.

Definition 3.2 Let $\Gamma = (V, E)$ be a finite graph with n vertices and adjacency matrix $\varepsilon \in M_n(\{0,1\})$. The *quantum automorphism group* $G^+_{\mathrm{aut}}(\Gamma)$ is the compact matrix quantum group $(C(G^+_{\mathrm{aut}}(\Gamma)), u)$, where $C(G^+_{\mathrm{aut}}(\Gamma))$ is the universal C^*-algebra with generators u_{ij}, $1 \le i, j \le n$ and relations (3.1), (3.2) together with $u\varepsilon = \varepsilon u$, which is nothing but $\sum_k u_{ik}\varepsilon_{kj} = \sum_k \varepsilon_{ik}u_{kj}$.

3.3 Link Between the Two Definitions

It is easy to see ([10, Lemma 3.1.1] or [16, Lemma 6.7]) that Banica's definition can be expressed as

$$C(G^+_{\mathrm{aut}}(\Gamma)) = C^*(u_{ij} \mid \text{relations (3.1)–(3.4) hold}).$$

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We thus have

$$\mathrm{Aut}(\Gamma) \subseteq G^*_{\mathrm{aut}}(\Gamma) \subseteq G^+_{\mathrm{aut}}(\Gamma)$$

in the sense of compact matrix quantum subgroups; see Section 2.4. Equality holds if $C(G^*_{\mathrm{aut}}(\Gamma))$ and $C(G^+_{\mathrm{aut}}(\Gamma))$ are commutative. Moreover, note that (see Example 2.1):

$$C(S_n^+) = C^*(u_{ij} \mid \text{relations (3.1) and (3.2) hold}).$$

Example 3.3 Let Γ be the complete graph (*i.e.*, $E = V \times V$). Then

$$\mathrm{Aut}(\Gamma) = G^*_{\mathrm{aut}}(\Gamma) = S_n, \qquad G^+_{\mathrm{aut}}(\Gamma) = S_n^+.$$

For its complement Γ^c (*i.e.*, $E = \varnothing$), we have

$$\mathrm{Aut}(\Gamma^c) = S_n, \qquad G^*_{\mathrm{aut}}(\Gamma^c) = G^+_{\mathrm{aut}}(\Gamma^c) = S_n^+.$$

3.4 Review of the Literature on Quantum Automorphism Groups of Graphs

At the moment there are only few articles about quantum automorphism groups of graphs. Some results are the following. In [6], Bichon defined the hyperoctahedral quantum group and showed that this group is the quantum automorphism group of some graph. Banica computed the Poincaré series of $G^+_{\mathrm{aut}}(\Gamma)$ for homogenous graphs with less than eight vertices in [1]. Banica, Bichon, and Chenevier considered circulant graphs having p vertices for p prime in [3]. They proved that $G^+_{\mathrm{aut}}(\Gamma) = \mathrm{Aut}(\Gamma)$ if the graph Γ does fulfill certain properties. Banica and Bichon investigated $G^+_{\mathrm{aut}}(\Gamma)$ for vertex-transitive graphs of order less or equal to eleven in [2]. They also computed $G^+_{\mathrm{aut}}(\Gamma)$ for the direct product, the Cartesian product, and the lexicographic product of specific graphs. Chassaniol also studied the lexicographic product of graphs in [7]. In her Ph.D. thesis [10], Fulton studied undirected trees Γ such that $\mathrm{Aut}(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, where we have k kopies of the cyclic group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. She proved that $\mathrm{Aut}(\Gamma) = G^*_{\mathrm{aut}}(\Gamma) = G^+_{\mathrm{aut}}(\Gamma)$ for $k = 1$ and $\mathrm{Aut}(\Gamma) \neq G^*_{\mathrm{aut}}(\Gamma) = G^+_{\mathrm{aut}}(\Gamma)$ for $k \geq 2$. See also [4,12] for links to quantum isometry groups.

3.5 Comparing with the Complement of the Graph

As can be seen from Section 3.4, the theory of quantum automorphism groups of graphs is still in its infancy. We now provide some basic results on the link between $G^*_{\mathrm{aut}}(\Gamma)$ and $G^*_{\mathrm{aut}}(\Gamma^c)$. Note that while we have

$$\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma^c) \quad \text{and} \quad G^+_{\mathrm{aut}}(\Gamma) = G^+_{\mathrm{aut}}(\Gamma^c)$$

for all graphs Γ (using $\varepsilon_{\Gamma^c} = A - \varepsilon_\Gamma$ for the adjacency matrices, with $A \in M_n(\{1\})$ the matrix filled with units, and $uA = A = Au$ by relation (3.2)), we may have

$$G^*_{\mathrm{aut}}(\Gamma) \neq G^*_{\mathrm{aut}}(\Gamma^c),$$

for instance, when Γ is the complete graph; see Example 3.3.

Lemma 3.4 *If* $G^*_{\mathrm{aut}}(\Gamma) \subseteq G^*_{\mathrm{aut}}(\Gamma^c)$, *then* $G^*_{\mathrm{aut}}(\Gamma) = \mathrm{Aut}(\Gamma)$.

Proof Relation (3.5) in $C(G^*_{\mathrm{aut}}(\Gamma^c))$ implies that u_{ik} and u_{jl} commute in $C(G^*_{\mathrm{aut}}(\Gamma))$ whenever $(i,j) \notin E$ and $(k,l) \notin E$. Together with relations (3.3), (3.4), and (3.5) in $C(G^*_{\mathrm{aut}}(\Gamma))$, this yields commutativity of all generators. ∎

Lemma 3.5 *If $G^*_{\mathrm{aut}}(\Gamma^c) = G^+_{\mathrm{aut}}(\Gamma^c)$, then $G^*_{\mathrm{aut}}(\Gamma) = \mathrm{Aut}(\Gamma)$.*

Proof We have $G^*_{\mathrm{aut}}(\Gamma) \subseteq G^+_{\mathrm{aut}}(\Gamma) = G^+_{\mathrm{aut}}(\Gamma^c) = G^*_{\mathrm{aut}}(\Gamma^c)$ and apply Lemma 3.4. ∎

The next lemma shows that the quantum automorphism groups of a graph without loops does not change if we add those.

Lemma 3.6 *Let $\Gamma = (V,E)$ be a finite graph without loops. Consider $\Gamma' = (V,E')$ with $E' = E \cup \{(i,i), i \in V\}$. The following hold:*

(i) $G^+_{\mathrm{aut}}(\Gamma) = G^+_{\mathrm{aut}}(\Gamma')$,
(ii) $G^*_{\mathrm{aut}}(\Gamma) = G^*_{\mathrm{aut}}(\Gamma')$.

Proof For (i), we use $\varepsilon_{\Gamma'} = 1 + \varepsilon_\Gamma$, where 1 is the identity matrix in $M_n(\{0,1\})$. Thus, $u\varepsilon_\Gamma = \varepsilon_\Gamma u$ is equivalent to $u\varepsilon_{\Gamma'} = \varepsilon_{\Gamma'} u$.

For (ii), all we need to check is that $u_{is(e_j)}u_{ir(e_j)} = u_{ir(e_j)}u_{is(e_j)}$ is fulfilled in $C(G^*_{\mathrm{aut}}(\Gamma))$ for all $i \in V$, $e_j \in E$, which is true due to relation (3.1). ∎

3.6 Quantum Automorphism Groups on Four Vertices

For a small number of vertices of undirected graphs, a complete classification of $G^*_{\mathrm{aut}}(\Gamma)$ and $G^+_{\mathrm{aut}}(\Gamma)$ is possible. For $n \in \{1,2,3\}$, we have $C(S_n^+) = C(S_n)$, hence $\mathrm{Aut}(\Gamma) = G^*_{\mathrm{aut}}(\Gamma) = G^+_{\mathrm{aut}}(\Gamma)$. For $n = 4$, we now provide a complete table for graphs having no loops. We restrict to undirected graphs in order to keep it simple. We need the following lemma to compute the quantum automorphism groups.

Lemma 3.7 *Let $\Gamma = (V,E)$ be a finite graph with $V = \{1,\dots,n\}$ and let $e_j \in E$. Let $q \in V$ with $s^{-1}(q) = \varnothing$. For the generators of $C(G^+_{\mathrm{aut}}(\Gamma))$, we have*

$$u_{qs(e_j)} = 0 = u_{s(e_j)q}.$$

Proof By relations (3.2) and (3.4), we get

$$u_{qs(e_j)} = u_{qs(e_j)}\Big(\sum_{i=1}^n u_{ir(e_j)}\Big) = \sum_{i=1}^n u_{qs(e_j)}u_{ir(e_j)} = 0,$$

because $(q,i) \notin E$ for all $i \in V$. Likewise, we get $u_{s(e_j)q} = 0$. ∎

In the following, D_4 denotes the dihedral group defined as

$$D_4 := \langle x, y \mid x^2 = y^2 = (xy)^4 = e\rangle,$$

H_2^+ denotes the hyperoctahedral quantum group defined by Bichon in [6], and $\mathbb{Z}_2$ denotes the cyclic group $\mathbb{Z}/2\mathbb{Z}$. The quantum group $\widehat{\mathbb{Z}_2 * \mathbb{Z}_2} = (C^*(\mathbb{Z}_2 * \mathbb{Z}_2), u)$ is

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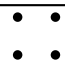
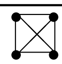
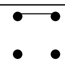
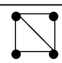
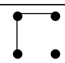
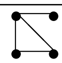
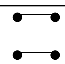
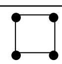
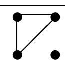
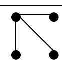
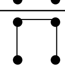
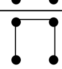
understood as the compact matrix quantum group with matrix

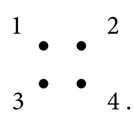
$$\begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix},$$

where $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ is seen as the universal unital C^*-algebra generated by two projections p and q. Recall that $\operatorname{Aut}(\Gamma) = \operatorname{Aut}(\Gamma^c)$ and $G^+_{\mathrm{aut}}(\Gamma) = G^+_{\mathrm{aut}}(\Gamma^c)$, where Γ^c is the complement of Γ within the category of graphs having no loops. Parts of the following table were also computed in [2, 6].

Theorem 3.8 *Let Γ be an undirected graph on four vertices having no loops and no multiple edges. Then we have the following table:*

	Γ	Γ^c	$\operatorname{Aut}(\Gamma)$	$G^*_{\mathrm{aut}}(\Gamma^c)$	$G^*_{\mathrm{aut}}(\Gamma)$	$G^+_{\mathrm{aut}}(\Gamma)$
(1)			S_4	S_4	S_4^+	S_4^+
(2)			$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\widehat{\mathbb{Z}_2 * \mathbb{Z}_2}$	$\widehat{\mathbb{Z}_2 * \mathbb{Z}_2}$
(3)			$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
(4)			D_4	D_4	H_2^+	H_2^+
(5)			S_3	S_3	S_3	S_3
(6)			$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Proof For every row of the table, we compute $G^+_{\mathrm{aut}}(\Gamma)$, and we show that $G^+_{\mathrm{aut}}(\Gamma) = G^*_{\mathrm{aut}}(\Gamma)$. We then obtain $G^*_{\mathrm{aut}}(\Gamma^c)$ by using Lemma 3.5. We label the points of the graphs as follows:



(1) Obvious, see Example 3.3.
(2) Let $(u_{ij})_{1\le i,j\le 4}$ be the generators of $C(G^+_{\mathrm{aut}}(\Gamma))$. Lemma 3.7 yields

$$u_{31} = u_{32} = u_{41} = u_{42} = u_{13} = u_{23} = u_{14} = u_{24} = 0.$$

Using relations (3.2), we deduce that

$$u = \begin{pmatrix} u_{11} & 1-u_{11} & 0 & 0 \\ 1-u_{11} & u_{11} & 0 & 0 \\ 0 & 0 & u_{33} & 1-u_{33} \\ 0 & 0 & 1-u_{33} & u_{33} \end{pmatrix}.$$

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Thus, $G^+_{\text{aut}}(\Gamma) = \widehat{\mathbb{Z}_2 * \mathbb{Z}_2}$. Since $u_{ij}u_{kl} = u_{kl}u_{ij}$ holds for $(i,k),(j,l) \in \{(1,2),(2,1)\}$ in $C(G^+_{\text{aut}}(\Gamma))$, we get $G^*_{\text{aut}}(\Gamma) = G^+_{\text{aut}}(\Gamma)$.

(3) Lemma 3.7 yields

$$u_{14} = u_{24} = u_{34} = u_{41} = u_{42} = u_{43} = 0.$$

This implies that $G^+_{\text{aut}}(\Gamma) \subseteq S^+_3 = S_3$; thus, $G^+_{\text{aut}}(\Gamma)$ is commutative, and hence $G^+_{\text{aut}}(\Gamma) = G^*_{\text{aut}}(\Gamma) = \text{Aut}(\Gamma) = \mathbb{Z}_2$.

(4) Let Δ and Δ' be the comultiplication maps of $G^+_{\text{aut}}(\Gamma)$ and H^+_2, respectively. We first show that these two quantum groups coincide as compact quantum groups, *i.e.,* there is a *-isomorphism

$$\varphi\colon C(H^+_2) \longrightarrow C(G^+_{\text{aut}}(\Gamma))$$

such that $\Delta' \circ \varphi = (\varphi \otimes \varphi) \circ \Delta$.

Step 1: The map φ exists and we have $\Delta' \circ \varphi = (\varphi \otimes \varphi) \circ \Delta$.
From $\varepsilon u = u\varepsilon$ we get

$$u = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{12} & u_{11} & u_{14} & u_{13} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{32} & u_{31} & u_{34} & u_{33} \end{pmatrix}.$$

Define $v_{11} := u_{11} - u_{12}$, $v_{12} := u_{13} - u_{14}$, $v_{21} := u_{31} - u_{32}$, and $v_{22} := u_{33} - u_{34}$. One can compute that v_{ij}, $i,j = 1,2$ fulfill the relations of $C(H^+_2)$, and with the universal property we get a *-homomorphism $\varphi\colon C(H^+_2) \to C(G^+_{\text{aut}}(\Gamma))$. Since $\Delta' \circ \varphi = (\varphi \otimes \varphi) \circ \Delta$ also holds, we get that $G^+_{\text{aut}}(\Gamma)$ is a quantum subgroup of H^+_2.

Step 2: The map φ is a *-isomorphism.
Let $(v_{ij})_{i,j=1,2}$ be the generators of $C(H^+_2)$. Define

$$\begin{aligned} u_{11} &:= u_{22} := \frac{v_{11}^2 + v_{11}}{2}, & u_{12} &:= u_{21} := \frac{v_{11}^2 - v_{11}}{2}, \\ u_{13} &:= u_{24} := \frac{v_{12}^2 + v_{12}}{2}, & u_{14} &:= u_{23} := \frac{v_{12}^2 - v_{12}}{2}, \\ u_{31} &:= u_{42} := \frac{v_{21}^2 + v_{21}}{2}, & u_{41} &:= u_{32} := \frac{v_{21}^2 - v_{21}}{2}, \\ u_{33} &:= u_{44} := \frac{v_{22}^2 + v_{22}}{2}, & u_{34} &:= u_{43} := \frac{v_{22}^2 - v_{22}}{2}. \end{aligned}$$

One can show that the $(u_{ij})_{1\le i,j\le 4}$ fulfill the relations of $C(G^+_{\text{aut}}(\Gamma))$. The universal property now gives us a *-homomorphism $\varphi'\colon C(G^+_{\text{aut}}(\Gamma)) \to C(H^+_2)$ and φ' is the inverse of φ and vice versa.

Step 3: We have $G^+_{\text{aut}}(\Gamma) = G^*_{\text{aut}}(\Gamma)$.
We have seen in Step 1 that

$$\begin{aligned} &u_{11} = u_{22}, \quad u_{12} = u_{21}, \quad u_{13} = u_{24}, \quad u_{14} = u_{23}, \\ &u_{31} = u_{42}, \quad u_{32} = u_{41}, \quad u_{33} = u_{44}, \quad u_{34} = u_{43} \end{aligned}$$

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and therefore we get

$$u_{ij}u_{kl} = u_{kl}^2 = u_{kl}u_{ij}$$

for all $(i,k),(j,l) \in E$. Thus, $G^+_{\mathrm{aut}}(\Gamma) = G^*_{\mathrm{aut}}(\Gamma)$.

(5) We conclude as in (3).

(6) Some direct computations using $\varepsilon u = u\varepsilon$ and relations (3.2) show

$$u = \begin{pmatrix} u_{33} & 1-u_{33} & 0 & 0 \\ 1-u_{33} & u_{33} & 0 & 0 \\ 0 & 0 & u_{33} & 1-u_{33} \\ 0 & 0 & 1-u_{33} & u_{33} \end{pmatrix}.$$

Thus $G^+_{\mathrm{aut}}(\Gamma)$ is commutative. ∎

4 Proof of the Main Result

We now prove the main result of this article (see Section 1) for a finite graph Γ with vertices $V = \{1,\dots,n\}$ and edges $E = \{e_1,\dots,e_m\}$ having no multiple edges.

Remark 4.1 We define the quantum symmetry group $\mathrm{QSym}(C^*(\Gamma))$ of $C^*(\Gamma)$ to be the maximal compact matrix quantum group G acting on $C^*(\Gamma)$ by $\alpha: C^*(\Gamma) \to C(G) \otimes C^*(\Gamma)$ and $\beta: C^*(\Gamma) \to C(G) \otimes C^*(\Gamma)$ as defined in the statement of our Main Theorem. We thus have to show that $G^+_{\mathrm{aut}}(\Gamma)$ acts on $C^*(\Gamma)$ via α and β (see Sections 4.1 and 4.2) and that it is maximal with these actions (see Section 4.3).

4.1 Existence of the Maps α and β

In order to prove that

$$\begin{aligned} \alpha: C^*(\Gamma) &\longrightarrow C(G^+_{\mathrm{aut}}(\Gamma)) \otimes C^*(\Gamma) \\ p_i &\longmapsto p_i' := \sum_{k=1}^n u_{ik} \otimes p_k, && 1 \le i \le n \\ s_{e_j} &\longmapsto s_{e_j}' := \sum_{l=1}^m u_{s(e_j)s(e_l)} u_{r(e_j)r(e_l)} \otimes s_{e_l}, && 1 \le j \le m \end{aligned}$$

defines a $*$-homomorphism, all we have to show is that the relations of $C^*(\Gamma)$ hold for p_i' and s_{e_j}'. We can then use the universal property of $C^*(\Gamma)$. The proof for the existence of β is analogous.

4.1.1 The p_i' are Mutually Orthogonal Projections.

Obviously, $p_i' = (p_i')^*$ holds. Moreover, using $p_k p_l = \delta_{kl} p_k$ and relations (3.1), we have

$$p_i' p_j' = \sum_{k,l=1}^n u_{ik}u_{jl} \otimes p_k p_l = \sum_{k=1}^n u_{ik}u_{jk} \otimes p_k = \delta_{ij} p_i'.$$

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4.1.2 The s'_{e_j} are Partial Isometries with $(s'_{e_j})^* s'_{e_j} = p'_{r(e_j)}$.

Using $s^*_{e_l} s_{e\,i} = \delta_{il} p_{r(e_i)}$ (see Section 2.3) and relations (3.1), we have

$$\begin{aligned}(s'_{e_j})^* s'_{e_j} &= \sum_{l,i=1}^{m} u_{r(e_j)r(e_l)} u_{s(e_j)s(e_l)} u_{s(e_j)s(e_i)} u_{r(e_j)r(e_i)} \otimes s^*_{e_l} s_{e_i}\\ &= \sum_{i=1}^{m} u_{r(e_j)r(e_i)} u_{s(e_j)s(e_i)} u_{r(e_j)r(e_i)} \otimes p_{r(e_i)}.\end{aligned}$$

By relations (3.3), we have $u_{r(e_j)j'} u_{s(e_j)i'} u_{r(e_j)j'} = 0$ for $(i', j') \notin E$. This yields

$$\sum_{i=1}^{m} u_{r(e_j)r(e_i)} u_{s(e_j)s(e_i)} u_{r(e_j)r(e_i)} \otimes p_{r(e_i)} = \sum_{i',j'=1}^{n} u_{r(e_j)j'} u_{s(e_j)i'} u_{r(e_j)j'} \otimes p_{j'}.$$

Using relations (3.2), we obtain $\sum_{i=1}^{n} u_{s(e_j)i'} = 1$, and thus

$$(s'_{e_j})^* s'_{e_j} = \sum_{i',j'=1}^{n} u_{r(e_j)j'} u_{s(e_j)i'} u_{r(e_j)j'} \otimes p_{j'} = \sum_{j'=1}^{n} u_{r(e_j)j'} \otimes p_{j'} = p'_{r(e_j)}.$$

4.1.3 We Have $\sum_{j:s(e_j)=v} s'_{e_j}(s'_{e_j})^* = p'_v$ for $s^{-1}(v) \neq \varnothing$.

Using relations (3.1), we get for $v \in V$ with $s^{-1}(v) \neq \varnothing$:

$$\begin{aligned}\sum_{\substack{j\in\{1,\dots,m\}\\ s(e_j)=v}} s'_{e_j}(s'_{e_j})^* &= \sum_{\substack{j\in\{1,\dots,m\}\\ s(e_j)=v}} \sum_{i,l=1}^{m} u_{vs(e_l)} u_{r(e_j)r(e_l)} u_{r(e_j)r(e_i)} u_{vs(e_i)} \otimes s_{e_l} s^*_{e_i}\\ &= \sum_{l=1}^{m} \sum_{\substack{i\in\{1,\dots,m\}\\ r(e_i)=r(e_l)}} u_{vs(e_l)} \Bigg(\sum_{\substack{j\in\{1,\dots,m\}\\ s(e_j)=v}} u_{r(e_j)r(e_l)} \Bigg) u_{vs(e_i)} \otimes s_{e_l} s^*_{e_i}.\end{aligned}$$

Now,

$$\sum_{\substack{j\in\{1,\dots,m\}\\ s(e_j)=v}} u_{r(e_j)r(e_l)} = \sum_{\substack{q\in V\\ (v,q)\in E}} u_{qr(e_l)},$$

and for $q \in V$ with $(v, q) \notin E$, we have $u_{vs(e_l)} u_{qr(e_l)} = 0$ by relations (3.4). Thus, for any $l \in \{1, \dots, m\}$, we have using relations (3.2)

$$u_{vs(e_l)} \sum_{\substack{q\in V\\ (v,q)\in E}} u_{qr(e_l)} = u_{vs(e_l)} \sum_{q\in V} u_{qr(e_l)} = u_{vs(e_l)},$$

and hence

$$\sum_{\substack{j\in\{1,\dots,m\}\\ s(e_j)=v}} s'_{e_j}(s'_{e_j})^* = \sum_{l=1}^{m} \sum_{\substack{i\in\{1,\dots,m\}\\ r(e_i)=r(e_l)}} u_{vs(e_l)} u_{vs(e_i)} \otimes s_{e_l} s^*_{e_i}.$$

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Since Γ has no multiple edges by assumption, $r(e_i) = r(e_l)$ and $s(e_i) = s(e_l)$ imply that $e_i = e_l$. We thus infer, using relations (3.1) that

$$\sum_{\substack{j\in\{1,\dots,m\}\\ s(e_j)=v}} s'_{e_j}(s'_{e_j})^* = \sum_{l=1}^{m} u_{vs(e_l)} \otimes s_{e_l}s_{e_l}^*$$

Now, for $V' := \{q \in V \mid s^{-1}(q) \neq \varnothing\}$, we have, using the relations in $C^*(\Gamma)$,

$$\sum_{l=1}^{m} u_{vs(e_l)} \otimes s_{e_l}s_{e_l}^* = \sum_{q\in V'} \sum_{\substack{l\in\{1,\dots,m\}\\ s(e_l)=q}} u_{vq} \otimes s_{e_l}s_{e_l}^* = \sum_{q\in V'} u_{vq} \otimes p_q$$

Since we know that $u_{vq} = 0$ for $q \notin V'$ by Lemma 3.7, we finally conclude that

$$\sum_{\substack{j\in\{1,\dots,m\}\\ s(e_j)=v}} s'_{e_j}(s'_{e_j})^* = \sum_{q=1}^{n} u_{vq} \otimes p_q = p'_v.$$

This settles the existence of α.

4.2 The Map α is a Left Action and β is a Right Action.

We only prove this claim for α, the proof for β being analogous.

4.2.1 $(\Delta \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \alpha) \circ \alpha$ Holds and α is Unital.

Using relations (3.3), this is straightforward to check.

It remains to show that

$$\mathcal{S} := \operatorname{span} \alpha(C^*(\Gamma))\big(C(G_{\mathrm{aut}}^+(\Gamma)) \otimes 1\big)$$

is dense in $C(G_{\mathrm{aut}}^+(\Gamma)) \otimes C^*(\Gamma)$, which we will do in the sequel.

4.2.2 The Elements $1 \otimes p_l, 1 \otimes s_{e_l}$ and $1 \otimes s_{e_l}^*$ are in $\mathcal{S}$.

Using relations (3.1) and (3.2), we infer

$$\mathcal{S} \ni \sum_{i=1}^{n} \alpha(p_i)(u_{il} \otimes 1) = \sum_{i=1}^{n}\sum_{j=1}^{n} u_{ij}u_{il} \otimes p_j = \sum_{i=1}^{n} u_{il} \otimes p_l = 1 \otimes p_l.$$

Moreover, for $e_l \in E$ we get, using relations (3.1) and $V' := \{v \in V \mid s^{-1}(v) \neq \varnothing\}$,

$$\begin{aligned}
&\sum_{v\in V'} \sum_{\substack{j\in\{1,\dots,m\}\\ s(e_j)=v}} \alpha(s_{e_j})(u_{r(e_j)r(e_l)}u_{vs(e_l)} \otimes 1)\\
&= \sum_{v\in V'} \sum_{\substack{j\in\{1,\dots,m\}\\ s(e_j)=v}} \Big(\sum_{k=1}^{m} u_{vs(e_k)}u_{r(e_j)r(e_k)}u_{r(e_j)r(e_l)}u_{vs(e_l)} \otimes s_{e_k}\Big)\\
&= \sum_{v\in V'} \left(\sum_{\substack{k\in\{1,\dots,m\}\\ r(e_k)=r(e_l)}} u_{vs(e_k)} \left(\sum_{\substack{j\in\{1,\dots,m\}\\ s(e_j)=v}} u_{r(e_j)r(e_l)} \right) u_{vs(e_l)} \otimes s_{e_k} \right).
\end{aligned}$$

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We proceed in a similar way as in Step 4.1.3. By relations (3.4), we know that $u_{qr(e_l)}u_{vs(e_l)} = 0$ for $(v,q) \notin E$. Thus, by relations (3.1) and (3.2) and using that Γ has no multiple edges, we obtain

$$\begin{aligned}
&\sum_{v\in V'}\sum_{\substack{j\in\{1,\dots,m\}\\ s(e_j)=v}} \alpha(s_{e_j})(u_{r(e_j)r(e_l)}u_{vs(e_l)}\otimes 1)\\
&= \sum_{v\in V'}\Bigg(\sum_{\substack{k\in\{1,\dots,m\}\\ r(e_k)=r(e_l)}} u_{vs(e_k)}\Big(\sum_{q=1}^{n} u_{qr(e_l)}\Big)u_{vs(e_l)}\otimes s_{e_k}\Bigg)\\
&= \sum_{v\in V'}\Bigg(\sum_{\substack{k\in\{1,\dots,m\}\\ r(e_k)=r(e_l)}} u_{vs(e_k)}u_{vs(e_l)}\otimes s_{e_k}\Bigg)\\
&= \sum_{v\in V'} u_{vs(e_l)}\otimes s_{e_l}.
\end{aligned}$$

Finally, Lemma 3.7 yields $u_{vs(e_l)} = 0$ for $v \notin V'$. Hence, using relations (3.2),

$$\mathcal{S} \ni \sum_{v\in V'}\sum_{\substack{j\in\{1,\dots,m\}\\ s(e_j)=v}} \alpha(s_{e_j})(u_{r(e_j)r(e_l)}u_{s(e_j)s(e_l)}\otimes 1) = \sum_{i=1}^{n} u_{is(e_l)}\otimes s_{e_l} = 1\otimes s_{e_l}.$$

Define $V'' := \{v\in V \mid r^{-1}(v)\neq\varnothing\}$. Similar to the computations above, we get

$$\mathcal{S} \ni \sum_{v\in V''}\sum_{\substack{j\in\{1,\dots,m\}\\ s(e_j)=v}} \alpha(s_{e_j}^*)(u_{s(e_j)s(e_l)}u_{r(e_j)r(e_l)}\otimes 1) = 1\otimes s_{e_l}^*.$$

4.2.3 If $1\otimes x, 1\otimes y\in\mathcal{S}$, then also $1\otimes xy\in\mathcal{S}$.

The remainder of the proof of Step 4.2 consists in general facts for actions of compact matrix quantum groups.

We can write $1\otimes x\in\mathcal{S}$ and $1\otimes y\in\mathcal{S}$ as

$$1\otimes x = \sum_{i=1}^{l}\alpha(z_i)(w_i\otimes 1), \qquad 1\otimes y = \sum_{j=1}^{k}\alpha(t_j)(v_j\otimes 1)$$

for some $z_i, t_j\in C^*(\Gamma)$ and $w_i, v_j\in C(G_{\mathrm{aut}}^+(\Gamma))$. Therefore,

$$\begin{aligned}
1\otimes xy &= \sum_{i=1}^{l}\alpha(z_i)(w_i\otimes 1)(1\otimes y) = \sum_{i=1}^{l}\alpha(z_i)(1\otimes y)(w_i\otimes 1)\\
&= \sum_{i=1}^{l}\sum_{j=1}^{k}\alpha(z_it_j)(v_jw_i\otimes 1)\in\mathcal{S}.
\end{aligned}$$

4.2.4 $\mathcal{S}$ is dense in $C(G_{\mathrm{aut}}^+(\Gamma))\otimes C^*(\Gamma)$.

Summarizing, we get that $1\otimes w\in\mathcal{S}$ for all monomials w in $p_i, s_{e_j}, s_{e_j}^*$, $1\le i\le n$, $1\le j\le m$. Since α is unital, we also have

$$C\big(G_{\mathrm{aut}}^+(\Gamma)\big)\otimes 1 \subseteq \alpha\big(C^*(\Gamma)\big)\big(C(G_{\mathrm{aut}}^+(\Gamma))\otimes 1\big)\subseteq\mathcal{S}.$$

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We conclude that $\mathcal{S}$ is dense in $C(G^+_{\mathrm{aut}}(\Gamma)) \otimes C^*(\Gamma)$, which settles Step 4.2.

4.3 The Quantum Group $G^+_{\mathrm{aut}}(\Gamma)$ Acts Maximally on $C^*(\Gamma)$

For proving the maximality, let $G = (C(G), u)$ be another compact matrix quantum group acting on $C^*(\Gamma)$ by $\alpha' : C^*(\Gamma) \to C(G) \otimes C^*(\Gamma)$ and $\beta' : C^*(\Gamma) \to C(G) \otimes C^*(\Gamma)$ in the way $G^+_{\mathrm{aut}}(\Gamma)$ acts on $C^*(\Gamma)$ via α and β. We want to show that there is a $*$-homomorphism $C(G^+_{\mathrm{aut}}(\Gamma)) \to C(G)$ sending generators to generators. Thus, we need to compute that the generators u_{ij} of $C(G)$ fulfill the relations of $C(G^+_{\mathrm{aut}}(\Gamma))$.

4.3.1 The Relations (3.1) Hold in $C(G)$.

The equation

$$\sum_{k=1}^{n} u_{ik} \otimes p_k = \alpha'(p_i) = \alpha'(p_i)^* = \sum_{k=1}^{n} u^*_{ik} \otimes p_k$$

yields $u_{ij} = u^*_{ij}$ after multiplying from the left with $1 \otimes p_j$. We also have

$$\sum_{i=1}^{n} u_{ji} u_{ki} \otimes p_i = \sum_{i,l=1}^{n} u_{ji} u_{kl} \otimes p_i p_l = \alpha'(p_j)\alpha'(p_k) = \delta_{jk}\alpha'(p_j) = \sum_{i=1}^{n} \delta_{jk} u_{ji} \otimes p_i,$$

from which we infer $u_{ji} u_{ki} = \delta_{jk} u_{ji}$. Using β', we also obtain $u_{ij} u_{ik} = \delta_{jk} u_{ij}$.

4.3.2 The Relations (3.2) Hold in $C(G)$.

From

$$\sum_{k=1}^{n} 1 \otimes p_k = 1 \otimes 1 = \alpha'(1) = \sum_{i=1}^{n} \alpha'(p_i) = \sum_{k=1}^{n} \Big(\sum_{i=1}^{n} u_{ik} \Big) \otimes p_k$$

we deduce $\sum_{i=1}^{n} u_{ik} = 1$, and likewise $\sum_{i=1}^{n} u_{ki} = 1$ using β'.

4.3.3 The Relations (3.3) Hold in $C(G)$.

Using $s^*_{e_l} s_{e_t} = \delta_{lt} p_{r(e_l)}$ (see Section 2.3) and relations (3.1) in $C(G)$, we obtain for any j,

$$\begin{aligned}
\sum_{q=1}^{n} u_{r(e_j)q} \otimes p_q = \alpha'(p_{r(e_j)}) &= \alpha'(s^*_{e_j} s_{e_j}) \\
&= \sum_{l,t=1}^{m} u_{r(e_j)r(e_l)} u_{s(e_j)s(e_l)} u_{s(e_j)s(e_t)} u_{r(e_j)r(e_t)} \otimes s^*_{e_l} s_{e_t} \\
&= \sum_{l=1}^{m} u_{r(e_j)r(e_l)} u_{s(e_j)s(e_l)} u_{r(e_j)r(e_l)} \otimes p_{r(e_l)}.
\end{aligned}$$

Multiplication by $1 \otimes p_k$ yields

$$u_{r(e_j)k} = \sum_{\substack{l \in \{1,\dots,m\} \\ r(e_l)=k}} u_{r(e_j)k} u_{s(e_j)s(e_l)} u_{r(e_j)k}.$$

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If $r^{-1}(k) = \varnothing$, then $u_{r(e_j)k} = 0$, and hence $u_{s(e_j)i}u_{r(e_j)k} = u_{r(e_j)k}u_{s(e_j)i} = 0$ for all $i \in V$.

Otherwise, if $r^{-1}(k) \neq \varnothing$, we use relations (3.1) and (3.2) in $C(G)$ and get

$$\sum_{\substack{l\in\{1,\dots,m\}\\ r(e_l)=k}} u_{r(e_j)k}u_{s(e_j)s(e_l)}u_{r(e_j)k} = u_{r(e_j)k} = u^2_{r(e_j)k} = \sum_{i=1}^{n} u_{r(e_j)k}u_{s(e_j)i}u_{r(e_j)k},$$

and therefore

$$\sum_{\substack{i\in V\\ (i,k)\notin E}} u_{r(e_j)k}u_{s(e_j)i}u_{r(e_j)k} = 0.$$

Since

$$u_{r(e_j)k}u_{s(e_j)i}u_{r(e_j)k} = \big(u_{s(e_j)i}u_{r(e_j)k}\big)^* u_{s(e_j)i}u_{r(e_j)k}$$

holds, the above is a vanishing sum of positive elements, and hence each summand vanishes. This yields $u_{s(e_j)i}u_{r(e_j)k} = 0$ for all $(i,k) \notin E$.

4.3.4 The Relations (3.4) Hold in $C(G)$.

The argument is analogous to the one for proving relations (3.3) when replacing α' by β'.

The proof of the main theorem is complete. ■

Remark 4.2 Let Γ be a finite graph with n vertices $V = \{1,\dots,n\}$ and m edges $E = \{e_1,\dots,e_m\}$. In [5], Bichon showed that $G^*_{\mathrm{aut}}(\Gamma)$ is the quantum symmetry group of Γ in his sense, where

$$\beta_V : C(V) \longrightarrow C(G^*_{\mathrm{aut}}(\Gamma)) \otimes C(V), \qquad g_i \longmapsto \sum_{k=1}^{n} u_{ki} \otimes g_k,$$

$$\beta_E : C(E) \longrightarrow C(G^*_{\mathrm{aut}}(\Gamma)) \otimes C(E), \qquad f_j \longmapsto \sum_{l=1}^{m} u_{s(e_l)s(e_j)}u_{r(e_l)r(e_j)} \otimes f_l,$$

define actions of $G^*_{\mathrm{aut}}(\Gamma)$ on $C(V)$ and $C(E)$, respectively. Those actions inspired us to deduce what an action of a compact matrix quantum group on $C^*(\Gamma)$ should look like. However, note that edges in the commutative C^*-algebra $C(E)$ of continuous functions on E are represented as projections unlike in the case of $C^*(\Gamma)$. Therefore, the quantum symmetry group of $C^*(\Gamma)$ is $G^+_{\mathrm{aut}}(\Gamma)$ rather than $G^*_{\mathrm{aut}}(\Gamma)$. On the other hand, if we consider the quotient of $C^*(\Gamma)$ by the relations $s_e = s_e^*$, its quantum symmetry group is $G^*_{\mathrm{aut}}(\Gamma)$. Indeed, selfadjointness of s_e yields

$$\sum_{l=1}^{m} u_{s(e_j)s(e_l)}u_{r(e_j)r(e_l)} \otimes s_{e_l} = \alpha(s_{e_j}) = \alpha(s_{e_j})^* = \sum_{l=1}^{m} u_{r(e_j)r(e_l)}u_{s(e_j)s(e_l)} \otimes s_{e_l},$$

from which we obtain relations (3.5) by multiplication with $(1 \otimes s^*_{e_i})$ from the left.

References

[1] T. Banica, *Quantum automorphism groups of homogeneous graphs.* J. Funct. Anal. **224**(2005), 243–280. http://dx.doi.org/10.1016/j.jfa.2004.11.002

[2] T. Banica and J. Bichon, *Quantum automorphism groups of vertex-transitive graphs of order* ≤ 11. J. Algebraic Combin. **26**(2007), no. 1, 83–105. http://dx.doi.org/10.1007/s10801-006-0049-9
[3] T. Banica, J. Bichon, and G. Chenevier, *Graphs having no quantum symmetry.* Ann. Inst. Fourier (Grenoble) **57**(2007), 955–971. http://dx.doi.org/10.5802/aif.2282
[4] J. Bhowmick, D. Goswami, and A. Skalski, *Quantum isometry groups of 0-dimensional manifolds.* Trans. Amer. Math. Soc. **363**(2011), no. 2, 901–921. http://dx.doi.org/10.1090/S0002-9947-2010-05141-4
[5] J. Bichon, *Quantum automorphism groups of finite graphs.* Proc. Amer. Math. Soc. **131**(2003), 665–673. http://dx.doi.org/10.1090/S0002-9939-02-06798-9
[6] ———, *Free wreath product by the quantum permutation group.* Algebr. Represent. Theory **7**(2004), 343–362. http://dx.doi.org/10.1023/B:ALGE.0000042148.97035.ca
[7] A. Chassaniol, *Quantum automorphism group of the lexicographic product of finite regular graphs.* J. Algebra **456**(2016), 23–45. http://dx.doi.org/10.1016/j.jalgebra.2016.01.036
[8] J. Cuntz and W. Krieger, *A class of* C^**-algebras and topological Markov chains.* Invent. Math. **56**(1980), no. 3, 251–268. http://dx.doi.org/10.1007/BF01390048
[9] S. Eilers, G. Restorff, E. Ruiz, and A. Sørensen, *The complete classification of unital graph* C^**-algebras: Geometric and strong.* arxiv:1611.07120
[10] M. B. Fulton, *The quantum automorphism group and undirected trees.* Ph.D. Thesis, Virginia Polytechnic Institute and State University, 2006.
[11] D. Goswami and J. Bhowmick, *Quantum isometry groups*. Infosys Science Foundation Series in Mathematical Science, Springer, New Delhi, 2016. http://dx.doi.org/10.1007/978-81-322-3667-2
[12] S. Joardar and A. Mandal, *Quantum symmetry of graph* C^**-algebras associated with connected graphs*. 2017. arxiv:1711.04253
[13] S. Neshveyev and L. Tuset, *Compact quantum groups and their representation categories*. Cours Spécialisés, 20, Société Mathématique de France, Paris, 2013.
[14] P. Podleś, *Symmetries of quantum spaces. Subgroups and quotient spaces of quantum* $\mathrm{SU}(2)$ *and* $\mathrm{SO}(3)$ *groups.* Comm. Math. Phys. **170**(1995), 1–20. http://dx.doi.org/10.1007/BF02099436
[15] I. Raeburn, *Graph algebras*. CBMS Regional Conference Series in Mathematics, 103, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2005. http://dx.doi.org/10.1090/cbms/103
[16] R. Speicher and M. Weber, *Quantum groups with partial commutation relations.* arxiv:1603.09192
[17] T. Timmermann, *An invitation to quantum groups and duality. From Hopf algebras to multiplicative unitaries and beyond.* EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2008. http://dx.doi.org/10.4171/043
[18] S. Wang, *Quantum symmetry groups of finite spaces.* Comm. Math. Phys. **195**(1998), 195–211. http://dx.doi.org/10.1007/s002200050385
[19] S. L. Woronowicz, *Compact matrix pseudogroups.* Comm. Math. Phys. **111**(1987), 613–665. http://dx.doi.org/10.1007/BF01219077
[20] ———, *A remark on compact matrix quantum groups.* Lett. Math. Phys. **21**(1991), no. 1, 35–39. http://dx.doi.org/10.1007/BF00414633

Saarland University, Fachbereich Mathematik, 66041 Saarbrücken, Germany
e-mail: simon.schmidt@math.uni-sb.de weber@math.uni-sb.de