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NATURAL KIND SEMANTICS FOR A CLASSICAL ESSENTIALIST THEORY OF KINDS

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Abstract. The aim of this paper is to provide a complete Natural Kind Semantics for an Essentialist Theory of Kinds. The theory is formulated in two-sorted first order monadic modal logic with identity. The natural kind semantics is based on Rudolf Willes Theory of Concept Lattices. The semantics is then used to explain several consequences of the theory, including results about the *specificity* (species–genus) relations between kinds, the definitions of kinds in terms of genera and *specific differences* and the existence of *negative* kinds. First, I show under which conditions the *Hierarchy* principle, which has been subjected to counterexamples in the literature, holds. I also show that a different principle about the species–genus relations between kinds, namely *Kant's Law*, follows from the essentialist theory. Second, I introduce two new operations for kinds and show that they can be used to provide traditional definitions of kinds in terms of genera and specific differences. Finally, I show that these operations of specific difference induce, for each kind, a uniquely specified contrary kind and a uniquely specified subcontrary kind, which can be used as semantic values for non-classical predicate negations of kind terms.

§1. Introduction. The aim of this paper is to provide a complete Natural Kind Semantics for an Essentialist Theory of Kinds. The theory is formulated in two-sorted first order monadic modal classical logic with identity. The natural kind semantics is based on Rudolf Willes Theory of Concept Lattices. The semantics is used to explain several consequences of the theory, including results about the *specificity* (species–genus) relations between kinds, the definitions of kinds in terms of genera and *specific differences* and the existence of *negative* kinds.

The structure of the paper is as follows. In Section 2, I discuss basic properties of the specificity (species–genus) relations between kinds. I review Thomason's original algebraic (lattice-theoretic) models for kinds and I argue that the approach has several problems. In particular, the models are undecided with respect to the membership and identity conditions of kinds and so it is not clear which contemporary theory of kinds the approach favors. In Section 3, I introduce the language and deductive calculus, based on classical logic, for an Essentialist Theory of Natural Kinds. The specific axioms for this theory reflect the essentialist membership conditions of kinds. In Section 4, I introduce another lattice-theoretic class of models based on Wille's Theory

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of Concept Lattices and use them to provide a semantics for the essentialist theory. The remaining sections study the consequences of the essentialist theory by using the new semantics. Whereas the first two aim at solving the problems that Thomason's original models had, the last two provide novel applications concerning the definitions of kinds in terms of genera and specific differences. In Section 5, I show under which conditions the *Hierarchy* condition holds, which explains how the counterexamples from the literature arise. I also show that a different principle concerning the species–genus relations between kinds, namely *Kant's Law*, holds. This principle states that the extension of a kind and its essence are inversely related. In Section 6, I show that the model provides an explanation of the classical account of definitions of kinds in terms of genera and specific differences. In Section 7, I show that these specific differences induce operations of term negation, distinct from propositional negation, that behave non-classically. Finally, Section 9 (Appendix) contains a completeness proof for the Natural Kind Semantics.

§2. Species-genus relations.

2.1. Specificity relations. Kinds, be they vernacular or scientific, seem to be ordered by how *specific* they are. Some kinds are more specific than others and dually, these kinds are more general than the first. Consider the following examples:

(1) The kind *Hammer* is more specific (less general) than the kind *Tool*.
(2) The kind *Lithium-6* is more specific (less general) than the kind *Lithium*.
(3) The kind *Protein* is more general (less specific) than the kind *Albumin*.

The study of these specificity relations goes back at least to Aristotle's discussion of species and genera. To honour tradition, let us call this the *species–genus relation* between kinds. As Hacking [1 7] notes, this is a formal binary relation between kinds. The relation holds whenever one of the kinds is more specific than the other. We can distinguish between the *(improper) species of relation* $\leq$, which is plausibly a *partial order*, and the *proper species of relation* $<$, which is plausibly asymmetric and transitive. We will use the term 'species–genus' to refer to the improper one (but both notions are interdefinable). Throughout the paper, we will write $K \leq K'$ to mean that kind K is a species of kind K'. The terms 'species' and 'genus' will be used as referring to the relata of the species–genus relation. Note that the use of the term 'species' in biology is quite different [17]. The term 'species' in biology is generally used to refer to a monadic (higher-order) property, not to a relation. One says that *Canis familiaris* is a biological species, not that *Canis familiaris* is a biological species of another entity. Thus we have a distinction between the *logical* notion of species and the *biological* one. We are concerned with the former, which applies to non-biological entities too.

One may argue that we do not need to study in depth the specificity relations between kinds, since good old predication already explains them. Monadic first-order (modal) logic is generally treated as a good approximation to Aristotelian syllogistics (if we ignore the issues related to existential import), so we can take kind terms to be monadic predicates. Then a sentence such as '*Hammer* is a species of *Tool*' will get translated as $\forall x(Hammer(x) \rightarrow Tool(x))$. That the species–genus relation is a preorder follows from the interaction between the material conditional and the universal quantifier. From the semantic point of view, kinds are represented extensionally as sets. The species–genus relation then is represented by the partial

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order of set-theoretical inclusion. Thus ‘Human is a species of Animal’ is true iff $[Human] \subseteq [Animal]$. The limitations of bare first-order languages for referring to kinds are well-known. For instance, in the first-order setting we cannot quantify over kinds or make claims about whether two kinds are identical. There is no obvious way of paraphrasing sentences like ‘Two new kinds of subatomic particles have been discovered’ directly as universally quantifying over individuals. The expression ‘is a species of’ does not denote a relation either, because it would have to hold between first-order monadic predicates, requiring second-order predication.

A natural logical framework to talk about kinds is higher-order logic, like the one built up over functional types that is used for doing Montague semantics. Kind terms would be represented as higher-order monadic predicates of different types, i.e., as being of type $\langle \sigma, t \rangle$, σ being here a metavariable for an arbitrary type.[1] Then one way to model the species–genus relation would be as predication, i.e., $Mammal(Human)$ is true iff $[Mammal]([Human]) = 1$. The term ‘is a species of’ can be made to denote relations of different types now. But this ‘species-as-predications’ strategy does not work. The species–genus relation and higher-order predication must be distinct because they have different formal properties. For example, whereas predication holds between individuals and properties, the species–genus relation does not. Although $Human(Socrates)$ is true, it is not the case that Socrates is a species of *Human*. Socrates himself is not a kind and therefore cannot be a relatum of the species–genus relation. This issue can be solved by making the proper species–genus relation take only higher-order entities as relata.

There are more important problems though. For example, the species–genus relation is transitive, but predication is not. If *Human* is a species of *Mammal* and *Mammal* is a species of $Animal$, then it follows that $Human$ is a species of $Animal$. But even if $Mammal(Human)$ and $Animal(Mammal)$ are true, it does not follow that $Animal(Human)$ is true. In fact, this last formula is ill-formed in most of the higher-order logics used for semantics, because the terms do not have the appropriate types. For example, $Human : \langle e, t \rangle$, $Mammal : \langle \langle e, t \rangle, t \rangle$ and $Animal : \langle \langle \langle e, t \rangle, t \rangle, t \rangle$, so the term $Animal$ is expecting a term of type $\langle \langle e, t \rangle, t \rangle$, not of type $\langle e, t \rangle$ and so it does not apply to $Human$. The best way to solve the problem is to lift the first-order strategy to the higher-order setting. We define the species–genus relations as $\lambda X_{\langle \sigma, t \rangle} \lambda Y_{\langle \sigma, t \rangle}.\forall x_\sigma (X(x) \rightarrow Y(x))$, which is of type $\langle \langle \sigma, t \rangle, \langle \langle \sigma, t \rangle, t \rangle \rangle$. Semantically, we use again the set-theoretic inclusions between the denotations of the predicates. Mutatis mutandis, the same goes for modal logic, we just replace sets of objects by functions from worlds to sets of objects.

Be it first-order or higher-order, I think that this approach is roughly speaking correct. Nevertheless, my main concern is not with the language or with the deductive calculus, but with the *semantics*. Standard semantics represents the denotations of kind terms as sets of objects, or as functions from possible worlds to sets of objects. But kinds are associated also with some properties shared by their instances. The main role of kinds is to *classify* objects according to some *criteria*. Often these criteria are based on properties (relations) shared by these objects. The contemporary theories of kinds have different views on this (see [4]). According to *Natural Kind Essentialism* (see [3,

[1] There is a deep discussion on whether kind terms behave like singular terms of type e or as monadic predicates of type $\langle \sigma, t \rangle$. We cannot consider this issue in this paper. Since our aim is to deal with the *classificatory* role of kinds, I have assumed the latter view.

10, 25, 34]), an object belongs to a kind iff it exemplifies essentially all the properties that form the general essence of the kind. According to causal theories, like Boyd's *Homeostatic Property Clusters* view [5] or Khalidi's theory [23], if an object belongs to a kind then it must have some properties that are in certain causal relations (or clustered by some causal mechanisms). Other theories, such as Dupré's *Promiscuous Realism* [9], impose weaker membership conditions, but still appeal to properties. Analogously, whether one kind is more specific than another does not depend only on its instances, it also depends on the properties shared by these. For example, we would expect that if K is a species of K', then whichever properties are shared by K'-s are also shared by K-s. However, the standard semantics does not reflect these facts.

In this paper I will follow the first-order strategy but I will *change the semantics*. This change will force us to pay attention to the properties of the species–genus relation itself. In contrast to the standard semantics, this new *Natural Kind Semantics* will appeal both to objects *and* to their common attributes. This semantics mirrors the commitments of Natural Kind Essentialism. In fact, some of these commitments are not endorsed by causal (and other) theories.

2.2. Thomason's lattice model of kinds. One may wonder whether there is any interesting condition that the species–genus structure of kinds satisfies. Thomason's insight was that a system of kinds $(L, \leq)$, where L is the set of kinds and $\leq$ is the partial order *is a species of*, forms an algebraic structure known as a *lattice* (see [35]).[2]

Recall that a lattice L is a partial order closed under the binary operations of *meet* $\sqcap$ and *join* $\sqcup$, which satisfy conditions analogous to conjunction and disjunction:

$$z \leq x \sqcap y \iff z \leq x \text{ and } z \leq y,$$
$$x \sqcup y \leq z \iff x \leq z \text{ and } y \leq z.$$

The meet $K \sqcap K'$ of two kinds is their greatest lower bound, i.e., their most general common species. The join $K \sqcup K'$ of two kinds is their least upper bound, i.e., their most specific common genus. A lattice is *bounded* iff it contains a highest element 1, called the *top*, and a lowest element 0, called the *bottom*. We will call 1 the *Summum Genus* and 0 the *Null Kind*. So every kind is a species of the Summum Genus and a genus of the Null Kind. A lattice is *complete* iff every (possibly infinite) subset of elements A has a join $\bigsqcup A$ and a meet $\bigsqcap A$.[3]

Thomason proposes that kinds form a bounded lattice that satisfies the *Hierarchy* condition:

Definition 2.1. *Let* $(L, \leq)$ *be a bounded lattice. Then L is* hierarchical iff *for all* $x, y \in L$, $x \leq y$ *or* $y \leq x$ *or* $x \sqcap y = 0$.

In other words, if two distinct kinds overlap then one of them must be a proper species of the other. A *tree* (with bottom) is a lattice where for each element $x \neq 0$ the set $\uparrow x = \{ y \in L \mid x \leq y \}$ is a chain (i.e., every pair of elements are comparable). Thus hierarchical lattices are trees with a bottom element attached. A clear example of

[2] The basic notions that we will make use of can be found in any standard introduction to lattice theory, such as [8]. Posets will be depicted as in Figure 5.

[3] In a lattice every finite subset has meet and join, so whether a lattice is complete only matters if the domain is *infinite*. E.g., if there were infinitely many atoms and one atom for every atomic number n, there would be infinitely many chemical elements. Although all the results in the paper apply to infinite lattices, the reader can safely ignore completeness.

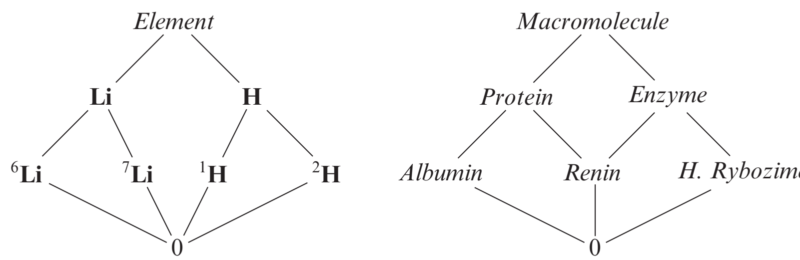

Fig. 1. Tree and general lattice.

a tree is provided by the chemical elements and the isotopes (see the left of Figure 1). Every chemical element (*Lithium*, *Hydrogen*, ...) branches into its isotopes, and no two distinct non-trivial chemical kinds overlap unless they are an element and one of its isotopes. Nevertheless, several philosophers of science (see [19, 22, 31, 36]) have argued that we cannot assume that *Hierarchy* always holds (see the right of Figure 1). We will go back to this issue in Section 5.

To sum up, Thomason's proposal gives an answer to the question regarding the order structure of kinds, namely, that kinds form a lattice that satisfies the *Hierarchy Principle*. Although this is a good starting point, it has several limitations as an explanation of the structure of kinds.

First of all, it gives no clues about the *membership conditions of kinds*. The main role of kinds is to *classify* other entities, such as objects. To classify these one makes use of *sorting criteria*. Under a realist account of kinds, these sorting criteria should correspond to the membership conditions of kinds, i.e., to the conditions that need to be satisfied for an object to belong to the corresponding kind. In fact, the main competing theories of natural kinds are usually distinguished by the membership conditions they assign to kinds. For instance, according to *Natural Kind Essentialism* [3, 10, 25, 34], an object belongs to a kind iff it essentially exemplifies all the attributes that form the general essence of the kind. According to causal theories, like Boyd's *Homeostatic Property Clusters* view [5] and more recent ones [23], the membership conditions of kinds are fuzzier. However, if an object belongs to a kind then it must exemplify some properties that tend to co-occur due to certain causal relations or mechanisms. The point is that without any mention of membership conditions we cannot know which theory of kinds a given class of structures favors. Thomason's approach posits kinds, but mentions neither objects nor membership conditions. Although this neutrality may be interesting, it comes with the price of not being able to explain one of the most basic features of kinds, namely the conditions under which kinds classify their members.

Second, we are not given any reason for why kinds should be structured as a lattice, being closed under meets and joins. One can suggest that given any domain of objects being classified, something like a most general kind (a *Summum Genus*) must exist. Nevertheless, it is not clear why for any two kinds a most specific common genus and a most general common species should exist too. Thomason simply posits that these entities exist without giving any further arguments.

Third, as some counterexamples by Tobin [36] Ruphy [31], Khalidi [22] and Hendry [19] show (see Section 5), the *Hierarchy* condition fails in many domains. Thomason's model can easily deal with them by dropping this assumption. Nevertheless, it should be preferable to have some explanation for why the principle fails. Tobin has already given such an explanation in terms of overlappings between kinds. But Thomason's model cannot explain why these crossings happen, because it is silent about the conditions under which some object belongs to several kinds.

Finally, the approach gives no clues about how the species–genus relation between kinds relates to their instances or to the properties shared by them. One expects that, if one kind is a species of another, then the extension of the former should be included in the extension of the latter. Another condition that should plausibly be satisfied is that if one kind is a species of another, then whatever attributes are common to the members of the genus should be common to the members of the species too. But Thomason's model is, again, silent in these regards. The new algebraic models will deal with these problems. But first, let us introduce the essentialist theory.

§3. An essentialist theory of kinds.

3.1. The language. The theory we will study is to be called the *Logic of Natural Kind Essentialism* or shortly **NKE**. The language is an extension of that of first-order *two-sorted* monadic classical modal predicate logic with identity. The two sorts are O, for individual *objects*, and P, for *attributes*. The sets of variables and constants of one sort are disjoint from those of the other sort. The monadic predicates represent natural kind terms and therefore will denote kinds.

DEFINITION 3.1. *The* language of **NKE** *contains the following items*:[4]

(1) *An infinite set of variables for objects* $\mathrm{Var}_O = \{x_O, y_O, z_O, \dots\}$.
(2) *An infinite set of variables for attributes* $\mathrm{Var}_P = \{X_P, Y_P, Z_P, \dots\}$.
(3) *A non-empty set of constants for objects* $\mathrm{Con}_O = \{a_O, b_O, c_O, \dots\}$.
(4) *A non-empty set of constants for attributes* $\mathrm{Con}_P = \{F_P, G_P, H_P, \dots\}$.
(5) *A set of monadic natural kind predicates* $\mathrm{Pred} = \{K, K', K'', \dots\}$.
(6) *Symbols of identity* $=$, *and* $\mathrm{Ess}(_{O,P})$, $\mathrm{Exe}(_{O,P})$ *of exemplification*.
(7) *Connectives* $\neg$, $\rightarrow$, *necessity operator* $\Box$, *universal quantifier* $\forall$.
(8) *Auxiliary symbols* (,).

As usual, an *object term* is either a variable for objects or a constant for objects, an *attribute term* is either a variable for attributes or a constant for attributes, and a *term* is either an object term or an attribute term. Usually, we will use lower-case $t_O, t'_O, \dots$ for object terms and upper-case $T_P, T'_P, \dots$ for attribute terms to spot the relevant sorts. For a term of arbitrary sort i we use lower-case $t_i, t'_i, \dots$. We will drop sort indexes whenever ambiguities do not arise. The only interesting additions to the language are the symbols $\mathrm{Ess}(_{O,P})$ and $\mathrm{Exe}(_{O,P})$ of essential exemplification and of exemplification, respectively. Note that in many-sorted logic usually each predicate has a sort, but our kind predicates do not have sorts.

DEFINITION 3.2. *The set of* atomic formulas *is defined as follows*:

(1) *If* t *is an object term and* T *is an attribute term, then* $\mathrm{Ess}(t, T)$ *and* $\mathrm{Exe}(t, T)$ *are atomic formulas.*

[4] By 'infinite' we restrict our attention, as usual, to countable infinity. We will use the same symbol '=' both for the object-language identities and for syntactical metalinguistic identities between expressions. Regarding parentheses, usual conventions apply.

(2) *If* t_i *is a term of sort* i *and* K *is a predicate, then* $K(t_i)$ *is an atomic formula.*
(3) *If* t_i, t_i' *are terms of the same sort* i, *then* $t_i = t_i'$ *is an atomic formula.*

Most theories of kinds distinguish between the natural kinds, the entities that belong to these kinds (or that are classified by them), and the properties and relations (or criteria) that these entities need to share in order for them to belong to the kinds. The atomic formulas reflect the relations between these entities.

I will call entities denoted by terms of sort O *objects*, entities denoted by terms of sort P *attributes* and entities denoted by natural kind predicates *kinds*. We say that an object *exemplifies essentially* an attribute. For instance, $\text{Ess}(\textit{Socrates}, \textit{Rational})$ is read 'Socrates is essentially rational'. The terms 'object', 'attribute', 'kind' and 'exemplification' come from J. Lowe's Four Category Ontology [25]. To be clear, I will only use Lowe's terminology, not his full theory. The essentialist theory **NKE** to be discussed is exhaustively described in Section 3.2. An object may *belong to the extension of* a kind. For example, $\textit{Cat}(\textit{Simone})$ is read 'Simone is a cat'. An attribute may *belong to the general essence* of a kind. For instance, $\textit{Albumin}(\textit{Water-Soluble})$ is read 'Albumin is water-soluble'.[5] We also have *identity statements*, such as $\textit{Cicero} = \textit{Cicero}$ to be read 'Cicero is Cicero'. These only express identities between entities of the same category, e.g., objects and attributes cannot be compared. Finally, there are two plausible ways to define the species–genus relation. On the one hand, we could define the species–genus relation as *extension containment*, so that K is a species of K' iff every object that is a K is also a K'. On the other hand, we could also define the relation à la Leibniz as *essence containment*, so that K is a species of K' iff every attribute in the essence of K' is in the essence of K:

$$K \leq_{\text{ext}} K' := \forall x(K(x) \to K'(x)),$$
$$K \leq_{\text{int}} K' := \forall X(K'(X) \to K(X)).$$

We will later on see that these two definitions are equivalent due to *Kant's Law*. Summing up, we can read 'is' as identity (entity–entity), species–genus (kind–kind), essential exemplification (object–attribute), membership to the kind (object–kind) or membership to the essence of a kind (attribute–kind).

The set of formulas is defined by recursion in the standard way:

Definition 3.3. *The set of* formulas *is defined by recursion as follows*:

(1) *Every atomic formula is a formula.*
(2) *If* φ *and* ψ *are formulas, then* $\neg\varphi$, $\Box\varphi$ *and* $\varphi \to \psi$ *are formulas.*
(3) *If* φ *is a formula and* x *is an object variable, then* $\forall x\varphi$ *is a formula.*
(4) *If* φ *is a formula and* X *is an attribute variable, then* $\forall X\varphi$ *is a formula.*

To simplify notation, we will use an index $i \in \{O, P\}$ and individual metavariables α_i to write $\forall\alpha_i\varphi$ instead of the pair of formulas $\forall x\varphi$ and $\forall X\varphi$. The rest of the logical connectives $\vee$, $\wedge$, $\leftrightarrow$, $\top$, $\bot$, $\exists$, $\Diamond$ are defined as usual. The notions of *free variable*, of a term *being free* for a variable in a formula, of *substitution* $\varphi[t_i/\alpha_i]$ of all (free) occurrences of a variable α_i of sort i by a term t_i of same sort i in a formula φ, and so on, are defined standardly. Note that substitutions are allowed only between terms of the *same* sort.

[5] Lowe calls the object–attribute relation *exemplification*, the object–kind relation *instantiation* and the attribute–kind relation *characterization*.

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Before moving on, let me make some comments regarding the expressive resources of this language. First, it is *first-order* and thus does not allow quantification over kinds. To quantify over kinds we would need a higher-order language.

Second, based on Fine's criticisms of the modal view of essence [12], many essentialists reject the definition of essential properties in terms of de re modality. I account for this by assuming a primitive symbol of *essential exemplification*. We can define *accidental exemplification* as $\text{Acc}(x, X) := \text{Exe}(x, X) \wedge \neg\text{Ess}(x, X)$.[6]

Third, there are other logics that could be used for representing reasoning about kinds. For instance, we could use the Aristotelian *Syllogistic Logic* developed by Martin [26] (based on work by Corcoran [7] and Smiley [32]). This choice has some plausibility. On the one hand, it is very close to Aristotle's original logic, which was developed with an essentialist conception of species and genera in mind. On the other hand, Martin provided a complete semantics for the logic using *meet-semilattices*, which is very close to Thomason's original proposal. The basic vocabulary of Syllogistic Logic includes some terms a, b, ...that denote kinds and some symbols A, E, I, O that link terms to form formulas, and that express the relations between propositions according to the Quantificational Square of Opposition.

However, this logic would not be expressive enough for our purposes. We want to reason naturally not only about kinds, but about the objects and attributes that figure in their essentialist membership conditions and we want to formulate explicitly these membership conditions (to be able to compare the essentialist theory to others). Moreover, it is unclear how we would represent identity and quantificational statements in such a framework. In order to talk about objects and attributes in Syllogistic Logic, we would have to represent them as kinds as well (e.g., as infimae species and maximal genera). This is strange since we distinguish between an object and the kinds it belongs to. We do so even if the kind has the object as its only instance. We also want to discuss principles that link the species–genus relations between kinds to the exemplification relations between objects and attributes (e.g., Hierarchy and Kant's Law; see Section 5). There are other interesting features regarding kinds, such as their closure under certain operations, that it is not clear how they could be accommodated in this framework. Furthermore, Syllogistic Logic is not Frege–Russell classical logic. Some theorems of the latter (e.g., distributivity) are not theorems of the former. But essentialist theories seem to assume as background a classical (modal) predicate logic.

3.2. A classical essentialist theory of kinds. I will now formulate an essentialist theory of kinds in classical logic. There are many essentialist theories, such as those by Ellis [10], Lowe [25], Bird [3] and Tahko [34]. Although there are important differences among them, I assume that they share at least some assumptions. The first one is about the *membership conditions of natural kinds*:

Essentialist Membership Conditions: An object x belongs to a kind K iff x essentially exemplifies all the attributes $X_1, \dots, X_n$ in the essence of K.

The natural attributes $X_1, \dots, X_n$ form the *general essence* of the kind. This is the core assumption of essentialism, namely that every kind has a general essence that fixes its membership conditions. Causal theories and promiscuous realism would not take

[6] Kit Fine's *Logic of Essence* [13], where essence is represented by an operator, is more expressive. It would be interesting to combine the semantics of this paper with that logic.

this claim as an axiom. Now, this claim is distinct from the familiar thesis put forward by Kripke in [24]:

Kripkean Individual Essentialism: If an object x belongs to a kind K, then x belongs to K necessarily.

We accept this as the second assumption. Our third assumption is dual to the first:

General essence: An attribute X belongs to the essence of a kind K iff X is essentially exemplified by all the objects $x_1, \dots, x_n$ in the extension of K.

It says that an attribute belongs to the essence of a kind iff it is exemplified essentially by every member of the kind. Finally, we have a claim about the *identity conditions of natural kinds*:

Essentialist Identity Conditions: Two essentialist natural kinds K and K' are identical iff they have the same general essence.

We will not be able to formulate this thesis in the object language, for it lacks higher-order quantification and does not allow identity statements between natural kind terms. Nevertheless, this claim will be shown to be preserved by the semantics. I think that these assumptions would be accepted by any natural kind essentialist.

A Hilbert-style deductive calculus for **NKE** is given by adding some axioms to a standard Barcan system for two-sorted first-order monadic classical modal predicate logic with identity (we use that from Hughes and Cresswell [20]):

Definition 3.4. *The* Theory of Natural Kind Essentialism ***NKE*** *has the following axioms and inference rules, where* $i \in \{O, P\}$:

(1) *Every propositional tautology from modal logic* $S5$. [*Tautology*]
(2) $\vdash \forall \alpha_i \varphi \to \varphi[t_i/\alpha_i]$, *where* t_i *is free for* α_i *in* φ. [$CP2$]
(3) $\vdash \alpha_i = \alpha_i$. [Ref]
(4) $\vdash \alpha_i = \beta_i \to (\varphi \to \varphi')$, *where* φ *is atomic.*[7] [*Leibniz Law*]
(5) $\vdash \forall \alpha_i \Box \varphi \to \Box \forall \alpha_i \varphi$. [*Barcan*]
(6) $\vdash \forall x(K(x) \leftrightarrow \forall X(K(X) \to \mathrm{Ess}(x, X)))$. [*Membership Conditions*]
(7) $\vdash \forall X(K(X) \leftrightarrow \forall x(K(x) \to \mathrm{Ess}(x, X)))$. [*General Essence*]
(8) $\vdash \forall x \forall X(\mathrm{Ess}(x, X) \to \Box \mathrm{Exe}(x, X))$. [*Essence-De Re Necessity*]
(9) $\vdash \forall x \forall X(\mathrm{Ess}(x, X) \to \Box \mathrm{Ess}(x, X))$. [*Necessary Essentialism*]
(10) $\vdash \forall x(K(x) \to \Box K(x))$. [*Kripkean Individual Essentialism*]
(11) $\vdash \forall X(K(X) \to \Box K(X))$. [*Kripkean Attribute Essentialism*]
(12) *If* $\vdash \varphi$, *then* $\vdash \Box \varphi$. [*Nec*]
(13) *If* $\vdash \varphi$ *and* $\vdash \varphi \to \psi$, *then* $\vdash \psi$. [*MP*]
(14) *If* $\vdash \varphi \to \psi$ *and* α_i *does not occur free in* φ, *then* $\vdash \varphi \to \forall \alpha_i \psi$. [*CUG*]

The only new axioms are *Membership Conditions*, *General Essence*, *Essence-De Re Necessity*, *Necessary Essentialism* and *Kripkean Individual/Attribute Essentialism*. The first two and the last one are the formal analogues of the essentialist assumptions. The third one says that essential exemplification entails de re necessary exemplification. We did not include the converse, to reflect Kit Fine's thesis [12] that essence is stronger

7 As usual, here φ' is obtained from φ by replacing some occurrences of α by β.

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than necessity. The last one says that essential exemplification is necessary. Its converse holds by modal axiom (T).

PROPOSITION 3.1. *The following are theorems of **NKE**:*

(1) $\vdash \forall x \forall y(x = y \to \Box x = y)$. [*Necessity of Identity*]
(2) $\vdash \forall x \forall y(x \neq y \to \Box x \neq y)$. [*Necessity of Distinctness*]
(3) $\vdash \forall x \forall X(\neg \mathrm{Ess}(x, X) \to \Box \neg \mathrm{Ess}(x, X))$. [Neg. *Necessary Essentialism*]
(4) $\vdash \forall x(\neg K(x) \to \Box \neg K(x))$. [Neg. *Kripkean Individual Essentialism*]
(5) $\vdash \forall X(\neg K(X) \to \Box \neg K(X))$. [Neg. *Kripkean Attribute Essentialism*]

Proof. (1) and (2) are proven as usual, using the rule if $\vdash \Diamond\varphi \to \psi$ then $\vdash \varphi \to \Box\psi$ holds in S5. (3)–(5) are analogous, applying the rule to the corresponding axioms. □

3.3. Objections to the essentialist theory. Before we move on to the semantics, let me address an objection to the theory. The axioms are strong enough to exclude the main competitors of kind essentialism. For instance, causal accounts of natural kinds à la Boyd [5], and those that impose weaker membership conditions for kinds, would not accept the axioms. First, *Membership Conditions* requires all the instances of the kind to have all the properties associated with the kind, which is not the case according to these theories. Second, later on I will represent the extensions and the essences of kinds by using sets. A fortiori, kinds will have clear-cut identity conditions (namely those of *Essentialist Identity Conditions*), which cannot explain the fact that, according to causal theories, kinds have vague boundaries.[8]

The aim of this paper is not to argue for a specific theory of kinds, but to study what one such theory entails. That being said, I think that the results that follow put some pressure on causal theories of kinds by demanding more from them. First, the semantics will explain why kinds have the specificity structure that they seem to have. It explains why the *Hierarchy* condition discussed in the literature holds only for some domains and explains the link between the specificity relations, the objects and the attributes through *Kant's Law* (see Section 5). It is unclear to me how the claims about the vague boundaries of kinds or the causal relations that need to hold between the properties shared by the members of the kind can be reconciled with these basic observations about how some kinds are more specific than others. For instance, *Kant's Law* does not seem to follow if we assume weaker membership conditions, as required by causal accounts (see Section 5). Moreover, despite their alleged closeness to scientific practice, existing causal accounts are generally silent on how kinds are related to their members. I find this disturbing, considering that the role of kinds is to *classify* their members. This is not a mere curiosity, it affects the formal models chosen. For instance, partitions, lattices and trees are among the most used mathematical models of classification (see [30]). In contrast, formal models of causal relations, like structural equation models (e.g., Khalidi's SEM account [23]), are not used much for classification purposes. The former mathematical structures seem to better capture the patterns in the classifications of chemical elements, minerals and species into clades than the ones from causal models. Besides, this approach is closer to usual semantics for natural kind terms in natural language, which at first sight seems to be at odds with causal accounts. These problems show that causal accounts have a lot to explain about

[8] One may also argue that the phylogenetic taxonomic research programme [11] is incompatible with (origin) essentialism. This debate falls outside the scope of this paper.

Table 1. *Example I Tobin's model: context*

	Water-soluble	Secreted in kidneys	RNA molecule	Catalyst Catalyst	Chain of aminoacids
a	X				X
b		X		X	X
c			X	X	

the structure of kinds. That being said, even causal theorists can find this model useful as a coarse approximation to what they take to be the true picture.

§4. Natural kind semantics. Instead of using general lattices, I will use lattices from Wille's *Theory of Formal Concept Lattices* [15][9] to provide a complete semantics for the **NKE** theory (completeness is shown in Section 9 [Appendix]). In what follows, the names of some notions from the Theory of Concept Lattices will be replaced by others more suitable to our current purposes. One can find some papers in the literature making related proposals. For instance, Mormann [29] proposes Wille's theory as a good candidate for explicating Armstrong's realism.[10] The models used here will also differ from those used in Freund's *Logic of Sortals* [14] (based on previous work by Cocchiarella [6]) in representing kinds as bi-dimensional entities that involve both a set of objects *and* a set of attributes shared by these. Nevertheless, a detailed comparison of the present approach to these others is beyond the scope of this paper.

4.1. Natural kind semantics. The semantics for **NKE** is not the standard one. I will give a different semantics by making use of Rudolf Wille's *theory of concept lattices*. Instead of mapping the monadic predicates to arbitrary subsets of a domain of individuals, they will be mapped to elements in a lattice, to be interpreted as essentialist *natural kinds*.

Definition 4.1. *A* formal context *is a structure* (S, Q, I), *where* S *and* Q *are sets and* $I \subseteq S \times Q$ *is a binary relation*.

We call S the set of *individual objects*, Q the set of *attributes* and I the relation of *essential exemplification*. Each context will be represented by a table, as in Table 1. The rows correspond to objects and the columns to attributes. For example, S is a set of atoms and Q is a set of atomic number values. Or S is a set of particular tools and Q is the set of functions they fulfill. Note that the properties can be relational monadic, e.g., *descends from x*, *performs function g*, and so on.

The first crucial definition is that of the notions of extension and intension:

Definition 4.2. *Let* (S, Q, I) *be a context*, $A \subseteq S$, *and* $B \subseteq Q$. *Then we define the functions* $e\colon \wp(Q) \to \wp(S)$ *and* $i\colon \wp(S) \to \wp(Q)$ *as follows*:

(1) $e(B) := \{x \in S \mid xIP \text{ for all } P \in B\}$ *is the* extension function.
(2) $i(A) := \{P \in Q \mid xIP \text{ for all } x \in A\}$ *is the* intension function.

[9] Concept lattices, semilattices and trees are popular models of *classifications* (see [30]).
[10] In fact, the basic theses in this paper were strongly inspired by the claims made by Mormann in [29]. There are still other related proposals (see Swoyer's Leibnizian calculus in [33]).

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Clearly, A is an extension for some B iff $A = ei(A)$ and B is the intension for some A iff $B = ie(B)$.[11] Second, we will represent kinds as certain pairs of sets:

DEFINITION 4.3. *Let (S, Q, I) be a context, $A \subseteq S$, and $B \subseteq Q$. Then (A, B) is a* natural kind *iff $A = e(B)$ and $i(A) = B$.*

Kinds are represented by pairs of sets: a set of objects A, the *extension*, and a set of attributes B, the *intension*.[12] The extension contains all the objects that exemplify all the attributes in the intension. The intension represents the *general essence* of the kind, and it contains all the attributes that are exemplified by all the objects in the extension. We will sometimes abbreviate the extension A of kind K as ext_K and its intension B as int_K. For instance, if the domain contains atoms, we have kinds $Lithium = (\{x, y\}, \{Z = 3\})$, $Gold = (\{q\}, \{Z = 79\})$ and the isotope $Oxygen\text{-}16 = (\{z\}, \{Z = 8, N = 8\})$. Basic properties of these operators are:

PROPOSITION 4.1. *Let (S, Q, I) be a context, $A, A' \subseteq S$, and $B, B' \subseteq Q$. Then:*

(1) $A \subseteq e(B)$ iff $B \subseteq i(A)$. [*Galois Connection*]
(2) *If $A \subseteq A'$ then $i(A') \subseteq i(A)$.* [*Antitonicity i*]
(3) *If $B \subseteq B'$ then $e(B') \subseteq e(B)$.* [*Antitonicity e*]
(4) *$A \subseteq ei(A)$ and $B \subseteq ie(B)$.* [*Extensiveness ei, ie*]
(5) *If $A \subseteq A'$ then $ei(A) \subseteq ei(A')$.* [*Monotonicity ei*]
(6) *If $B \subseteq B'$ then $ie(B) \subseteq ie(B')$.* [*Monotonicity ie*]
(7) *$eiei(A) = ei(A)$ and $ieie(B) = ie(B)$.* [*Idempotence ei, ie*]
(8) *$iei(A) = i(A)$ and $eie(B) = e(B)$.*
(9) *$i(\bigcup_{i \in I} A_i) = \bigcap_{i \in I} i(A_i)$ and $e(\bigcup_{i \in I} B_i) = \bigcap_{i \in I} e(B_i)$.*

Recall that we can define the species–genus relation as *extension containment*, i.e., (A, B) is a species of (A', B') iff $A \subseteq A'$. From these properties half of the main theorem of the theory of concept lattices follows:[13]

THEOREM 4.1 (Fundamental Theorem of Concept Lattices). *Let L^* be the set of kinds of the context (S, Q, I). If (A, B) and (A', B') are kinds, define $(A, B) \leq (A', B') := A \subseteq A'$. Then $(L^*, \leq)$ is a complete lattice, where:*

(1) $(A, B) \leq (A', B')$ iff $A \subseteq A'$ iff $B' \subseteq B$.
(2) $\bigsqcap_{i \in I} (A_i, B_i) = (\bigcap A_i, ie(\bigcup B_i))$.
(3) $\bigsqcup_{i \in I} (A_i, B_i) = (ei(\bigcup A_i), \bigcap B_i)$.
(4) $1 = (S, i(S))$ *and* $0 = (e(Q), Q)$.
(5) xIP iff $x \in e(P)$ iff $P \in i(x)$.

The collection of all kinds forms a lattice. Lattices L^* are usually called *concept lattices*, but we will call them *lattices of kinds*. Note that the whole lattice and order structures follow from the way in which we represented kinds as pairs of sets.

The most general kind, the Summum Genus, has the form $1 = (S, i(S))$. It contains every object in its extension, and it has empty intension iff there is no attribute that

[11] Extensions and intensions are also called *Galois-closed sets* in the literature.
[12] Wille calls such a pair (A, B) a *formal concept*. His choice of the terms 'extension' and 'intension' is likely due to how these terms were used by the Port-Royal account of concepts.
[13] In fact, the relation between concept lattices and complete lattices is far more interesting than that, since *every complete lattice* is isomorphic to a concept lattice [15].

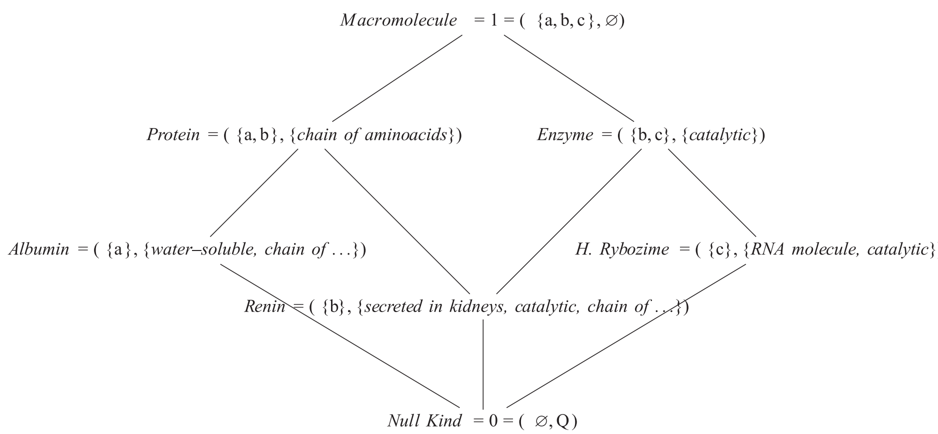

Fig. 2. Example I Tobin's model: lattice of kinds.

is exemplified by every object. The most specific kind, the Null Kind, has the form $0 = (e(Q), Q)$. It contains every attribute in its intension, and it has empty extension iff there is no object that exemplifies every attribute. Moreover, each object x induces a kind $K_x = (ei(x), i(x))$ and each attribute P induces a kind too $K_P = (e(P), ie(P))$. These are the smallest kinds that contain x (or P) in their extension (or intension), respectively. To honour tradition, we will call the kinds K_x the *infimae species* and the kinds K_P the *maximal genera*.

We can consider the following example by Tobin [36]. A context is given in Table 1 and the lattice of kinds in Figure 2. In contrast with Thomason's model, we now have information about the members of the kinds and about the properties they share. For example, the kind *Albumin* contains just one member, molecule *a*. Its intension contains two properties, namely *being water soluble* and *being a chain of aminoacids*. Kinds *Protein* and *Enzyme* overlap because they have a common member, molecule *b*, which belongs to their most general common species *Renin*.

We will now use formal contexts to define models for the theory. We use a Kripkean fixed-domain semantics. We also require an exemplification relation for each possible world to distinguish between essential and non-essential exemplification. Nevertheless, the natural kinds will be the ones induced by the relation of essential exemplification. Thus we will treat both constant terms and natural kind predicates as *rigid* expressions whose semantic value is invariant.[14]

Definition 4.4. *A* skeleton *is a structure* $M = (W, R, S, Q, I_{\text{Ess}}, \{I^w_{\text{Exe}}\}_{w \in W})$ *s.t.*:

(1) *W is a non-empty set of* possible worlds.
(2) *$R \subseteq W \times W$ is an equivalence relation of* accessibility.
(3) *S and Q are non-empty disjoint sets of* objects *and* attributes.
(4) *$I_{\text{Ess}} \subseteq S \times Q$ and $I^w_{\text{Exe}} \subseteq S \times Q$ for each $w \in W$, are the* essential exemplification *and* w-exemplification, *which satisfy $I_{\text{Ess}} \subseteq I^w_{\text{Exe}}$ for each $w \in W$.*

The domains are disjoint and non-empty, since our logic is two-sorted non-free classical logic. The lattice of kinds is the lattice L^* of the essentialist context (S, Q, I_{Ess}).

[14] The claim that natural kind terms are rigid designators was put forward by Kripke [24].

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By rigidity, the extension of a kind contains all its possible instances. Note that we also have a concept lattice L_w in each possible world. A pair in such a lattice is not an essentialist kind, for it contains in its extension the objects that *w*-exemplify (not essentially) all the attributes in the intension. So each essentialist kind $K = (A, B)$ induces a *modal intension* function f_K that sends each world w to the pair $(e_w(A), i_w e_w(B))$. These 'local approximations' to kinds could be used to model how the extension of a kind 'changes' from one world to another, but we will not consider them in detail.

DEFINITION 4.5. *A* model *based on skeleton M is a structure* $\boldsymbol{M} = (M, \boldsymbol{J})$ *where* $\boldsymbol{J}$ *is an* interpretation function *that satisfies*:

(1) *If* a *is an object constant, then* $\boldsymbol{J}(a) \in S$.
(2) *If F is an attribute constant, then* $\boldsymbol{J}(F) \in Q$.
(3) *If K is a natural kind predicate, then* $\boldsymbol{J}(K) \in L^*$.

As usual, a *variable assignment* g is a function from variables to entities in the context that preserves differences in sort, i.e., $g(x) \in S$ and $g(X) \in Q$. The *u*α*-variant* $g[u/\alpha_i]$ of g is defined as usual. Given a model **M** and an assignment g, we define the *semantic value* $[.]_g$ as the function that extends both the assignment and interpretation, i.e., $[\alpha_i]_g := g(\alpha_i)$, $[a]_g := \mathbf{J}(a)$, $[F]_g := \mathbf{J}(F)$ and $[K]_g := \mathbf{J}(K)$. Recall that $\text{ext}_{[K]}$, $\text{int}_{[K]}$ are shorthands for the extension and intension of kind $[K]$. We now define truth in the usual Tarskian manner.

DEFINITION 4.6. *Let* $\boldsymbol{M}$ *be a model. We define the relation* φ is true in world w in **M** under assignment g, $\boldsymbol{M}, g, w \models \varphi$, *by recursion as follows*:

(1) $\boldsymbol{M}, g, w \models t_i = t_i'$ iff $[t_i]_g = [t_i']_g$.
(2) $\boldsymbol{M}, g, w \models \text{Ess}(t, T)$ iff $[t]_g I_{\text{Ess}} [T]_g$.
(3) $\boldsymbol{M}, g, w \models \text{Exe}(t, T)$ iff $[t]_g I^w_{\text{Exe}} [T]_g$.
(4) $\boldsymbol{M}, g, w \models K(t)$ iff $[t]_g \in \text{ext}_{[K]}$, *where* t *is an object term*.
(5) $\boldsymbol{M}, g, w \models K(T)$ iff $[T]_g \in \text{int}_{[K]}$, *where T is an attribute term*.
(6) $\boldsymbol{M}, g, w \models \neg\varphi$ iff $\boldsymbol{M}, g, w \models \varphi$ *does not hold*.
(7) $\boldsymbol{M}, g, w \models \varphi \to \psi$ iff $\boldsymbol{M}, g, w \models \varphi$ *does not hold or* $\boldsymbol{M}, g, w \models \psi$ *holds*.
(8) $\boldsymbol{M}, g, w \models \Box\varphi$ iff $\boldsymbol{M}, g, w' \models \varphi$ *for every* w' *s.t.* wRw'.
(9) $\boldsymbol{M}, g, w \models \forall\alpha_i\varphi$ iff $\boldsymbol{M}, g[u/\alpha], w \models \varphi$ *for all* $u \in S$ (*or* $u \in Q$), *where* α_i *is an object* (*attribute*) *variable*.

The only new conditions are the atomic ones. $\text{Ess}(t, T)$ is true in *w* iff the object denoted by *t* essentially exemplifies the attribute denoted by *T*. $\text{Exe}(t, T)$ is true in *w* iff the object denoted by *t* exemplifies in *w* the attribute denoted by *T*. $K(t)$ is true in *w* iff the object denoted by *t* belongs to the kind denoted by *K*, i.e., if it is a member of its extension. $K(T)$ is true iff the attribute denoted by *T* belongs to the general essence of the kind denoted by *K*, i.e., if it is a member of its intension. We obtain the usual truth conditions for the rest of the formulas. Figure 3 shows some formulas true in Tobin's model (Figure 2).

The definitions regarding validity, logical consequence, and so on are standard. For instance, φ is *valid in* $\boldsymbol{M}$, $\mathbf{M} \models \varphi$ iff $\mathbf{M}, g, w \models \varphi$ for every world *w* and assignment *g*. It is *valid* $\models \varphi$ iff valid in every model. The Free Variables Lemma and the Substitution Lemma hold as usual. This finishes the semantics.

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$$\mathbf{M}_{Tobin}, g, w \models \mathrm{Ess}(c, Catalytic)$$
$$\mathbf{M}_{Tobin}, g, w \models Enzyme(Catalytic)$$
$$\mathbf{M}_{Tobin}, g, w \models \forall x Macromolecule\,(x)$$
$$\mathbf{M}_{Tobin}, g, w \models Albumin \ \leq_{ext} \ Protein$$
$$\mathbf{M}_{Tobin}, g, w \models \exists X\,(Protein(X) \wedge Albumin\,(X))$$
$$\mathbf{M}_{Tobin}, g, w \models \forall x (Renin(x) \leftrightarrow (Protein \ \sqcap_O \ Enzyme\,(x)))$$
$$\mathbf{M}_{Tobin}, g, w \models \forall x (Protein(x) \leftrightarrow (Albumin \ \sqcup_O \ Renin\,(x)))$$

Fig. 3. Formulas true in Tobin's model.

4.2. Basic consequences of the essentialist theory. We can study several consequences of the theory through the natural kind semantics. First, we should check that the new axioms are valid:

PROPOSITION 4.2. *The following hold*:

(1) $\models \forall x(K(x) \leftrightarrow \forall X(K(X) \rightarrow \mathrm{Ess}(x, X)))$. [*Membership Conditions*]
(2) $\models \forall X(K(X) \leftrightarrow \forall x(K(x) \rightarrow \mathrm{Ess}(x, X)))$. [*General Essence*]
(3) $\models \forall x \forall X(\mathrm{Ess}(x, X) \rightarrow \Box \mathrm{Exe}(x, X))$. [*Essence-De Re Necessity*]
(4) $\models \forall x \forall X(\mathrm{Ess}(x, X) \rightarrow \Box \mathrm{Ess}(x, X))$. [*Necessary Essentialism*]
(5) $\models \forall x(K(x) \rightarrow \Box K(x))$. [*Kripkean Individual Essentialism*]
(6) $\models \forall X(K(X) \rightarrow \Box K(X))$. [*Kripkean Attribute Essentialism*]
(7) *If K and K' are monadic, then* $[K]_g = [K']_g$ iff $\mathrm{int}_{[K]_g} = \mathrm{int}_{[K']_g}$.

Proof. (1) Let $\mathbf{M}, g$ and w be given and $u \in S$ an object. Assume that $\mathbf{M}, g[u/x], w \models K(x)$ and so $u \in \mathrm{ext}_{[K]}$. Then $\mathbf{M}, g[u/x, P/X], w \models K(X) \rightarrow \mathrm{Ess}(x, X)$ for every $P \in Q$ iff if $P \in \mathrm{int}_{[K]}$ then $uI_{\mathrm{Ess}}P$. But if the antecedent holds, we get $uI_{\mathrm{Ess}}P$. Conversely, assume $\mathbf{M}, g[u/x] \models \forall X(K(X) \rightarrow \mathrm{Ess}(x, X))$. So for all $P \in Q$, if $\mathbf{M}, g[u/x, P/X] \models K(X)$ then $\mathbf{M}, g[u/x, P/X] \models \mathrm{Ess}(x, X))$. Thus if $P \in \mathrm{int}_{[K]}$ then $uI_{\mathrm{Ess}}P$ and $u \in \mathrm{ext}_{[K]}$. So $\mathbf{M}, g[u/x] \models K(x)$. (2) is similar. (3)–(6) Note that $I_{\mathrm{Ess}} \subseteq I^{w}_{\mathrm{Exe}}$ for every world w and that the truth-conditions of $\mathrm{Ess}(x, X)$, $K(\alpha_i)$ do not depend on the possible world. (7) holds in every concept lattice. □

The last one is a metalinguistic claim which encodes the Essentialist Identity Conditions for Kinds, i.e., that two kinds are identical iff they have the same essence. Using Soundness it can be checked that the following are not theorems:

PROPOSITION 4.3. *The following hold*:

(1) $\not\vdash \forall x \exists X \mathrm{Ess}(x, X)$. [*No-Bare Particulars*]
(2) $\not\vdash \forall X \exists x \mathrm{Ess}(x, X)$. [*Exemplification*]
(3) $\not\vdash \forall x \forall X \forall Y((\mathrm{Ess}(x, X) \leftrightarrow \mathrm{Ess}(x, Y)) \rightarrow X = Y)$. [*Coextensionality*]
(4) $\not\vdash \forall x \forall y \forall X((\mathrm{Ess}(x, X) \leftrightarrow \mathrm{Ess}(y, X)) \rightarrow x = y)$. [*PII*]

The first one is the *Principle of No-Bare Particulars*, which requires that every object essentially exemplifies some attribute. The second one is the *Principle of Exemplification*, which requires that every attribute be essentially exemplified by some object. The third one is the *Principle of Coextensionality*, which gives extensional identity conditions to attributes. Two attributes are identical iff they are essentially

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exemplified by the same objects. Lastly, the *Principle of the Identity of Indiscernibles* restricted to essential attributes requires any two objects that have the same essential attributes to be identical. Counterexamples are easy to construct. Adding any of these as axioms to **NKE** would provide more specific essentialist theories. Although the first two are plausible principles, we have not assumed them in order to keep the essentialist theory as general as possible.

The simplest way to express in the language the operations between kinds is to define the following symbols:

DEFINITION 4.7. *We define the following abbreviations in the language*:

(1) Null $\text{Kind}_O(x) := \forall X \text{Ess}(x, X)$.
(2) Summum $\text{Genus}_O(x) := x = x$.
(3) $K \sqcap_O K'(x) := K(x) \wedge K'(x)$.
(4) $K \sqcup_O K'(x) := \forall X((K(X) \wedge K'(X)) \rightarrow \text{Ess}(x, X))$.
(5) Null $\text{Kind}_P(X) := X = X$.
(6) Summum $\text{Genus}_P(X) := \forall x \text{Ess}(x, X)$.
(7) $K \sqcap_P K'(X) := \forall x((K(x) \wedge K'(x)) \rightarrow \text{Ess}(x, X))$.
(8) $K \sqcup_P K'(X) := K(X) \wedge K'(X)$.

Note that these are abbreviations of formulas, they are not predicates. Notice also that they have to make sense of the difference in truth conditions for each sort. For example, every object belongs to the extension of the Summum Genus, but an attribute belongs to its intension iff it is essentially exemplified by every object.

4.3. Objections to natural kind semantics. In this section I will address several objections to the semantics. First, one may object to the existence of the Null Kind. The existence of the Null Kind is just as objectionable as the existence of the empty set, the null part or the number zero. All these entities play a theoretical role in simplifying the corresponding algebraic structures. In any case, the Null Kind has a semantic role to play. If we are to give semantic values to necessarily empty kind terms, such as 'Square Circle', we need the Null Kind. In principle, calling it 'Null' does not make its extension empty. Recall that the Null Kind is represented by the pair $0 = (e(Q), Q)$. It is uniquely characterized as the kind that has in its extension all the objects that essentially exemplify all the attributes in the domain. Plausibly, there will be incompatible attributes (i.e., no object can have both essentially), so its extension will be empty. But I do not see why it follows from this that we should object to its existence.

Second, some philosophers of science have argued that not every domain has a most general kind, i.e., a Summum Genus. For instance, in the counterexample we depicted in Figure 2, Tobin says that *Macromolecule* is not really a kind [36]. The main reason given by Tobin to reject *Macromolecule* as a kind, in contrast to *Protein* or *Enzyme*, is that 'Grouping them together under this category masks important differences and similarities between the members'. One may be tempted to exclude it because its intension contains only properties that are common to all the objects in the domain and therefore does not help us establish differences.

The argument seems to be that we should only affirm the existence of those kinds that allow us to track theoretically interesting differences between the objects. Since there are no such differences between the instances of the Summum Genus, we should not claim that it exists. But if there is no Summum Genus, it is hard to say what the classification is a classification *of*. When we classify objects we already know (or conjecture) that they are similar in some respects and we are trying to separate them by finding salient respects in which they differ. But I do not see why focusing on the differences should be a good reason to ignore the common properties. In any case, there is experimental work from cognitive psychology that explains why we tend to ignore most general kinds. As explained by Tversky [37], when we classify a domain of objects we tend to ignore those properties that are common to all the objects, because they do not help us classifying them. If all pieces of furniture in a room are made of wood, *being made of wood* is not relevant to classify them as tables, chairs, and so on, so we tend to ignore that property. But as Tversky explains, there is also the *Extension Effect*. If we later on include metallic pieces of furniture in the domain, we tend to consider *being made of wood* as relevant for classification. The lesson is that once we have fixed the domain of objects, we are psychologically inclined to ignore the properties that are common to all of them because we are looking for the differences. But this selection on some particular attributes is just temporary, for we are equally inclined to take these attributes as salient once new objects are considered.

Third, traditionally Infimae Species are the extensionally smallest kinds strictly above the individuals. But what have been called 'Infimae Species' in the semantics often have just *one* individual in the extension. Every object x corresponds to a kind of the form $(ei(x), i(x))$ (the same goes for attributes and maximal genera, with form $(e(R), ie(R)))$. But it sounds strange to say that, for an object like Socrates, there is a kind that contains in its extension all the entities that share with Socrates a certain set of essential attributes. One may think at first that such pairs of sets should be deleted from the collection of natural kinds.

However, such a move would be too radical. If there is only one instance of each kind, then deleting these Infimae Species from the model amounts to deleting entities that are representing kinds. Consider Tobin's example in Figure 2. The kind *Albumin* $= (\{a\}, \{$*water-soluble, chain of, ...*$\})$ has just one instance a. It is one of the Infimae Species. Deleting it from the family of kinds is deleting the pair of sets that represents the kind *Albumin*. One may think that collections with just one instance are intuitively not natural kinds. The thought may be that kinds with just one instance cannot support inductive reasoning. But what supports inductive inferences are the properties shared by the instances of the kind, not the instances themselves, and these may have plenty of properties.

Furthermore, I think that the traditional view is mistaken in emphasizing the role of those most specific kinds that are strictly more general than what I called 'Infimae Species'. Consider the case where the lattice of kinds includes infinitely descending chains of even more specific kinds $\cdots < K'' < K' < K$.[15] Then what I called 'Infimae Species' will still exist, but there will be no Infimae Species in the traditional sense. For each such kind, there will always exist a strictly more specific kind below it. In contrast, it does not matter how big the lattice is, for what I have called 'Infimae Species' and 'Maximal Genera' will always exist. Moreover, these are 'building blocks' of the lattice, for each kind is the join of its Infimae Species (the ones that are more specific than it) and the meet of its Maximal Genera. However you call them, the kinds that are crucial

[15] Such a scenario was considered by Kant in his *Logic Lectures*.

for the structure of the whole system of kinds are what I called 'Infimae Species' and 'Maximal Genera'.

Finally, one may argue that the theory and semantics are not faithful representations of natural kind essentialism. By the anti-symmetry of the species–genus relations, coextensional kinds are identical, i.e., if $A = A'$ then $(A, B) = (A', B')$. But two distinct kinds (e.g., *Mammal* and *Rat*) may be accidentally coextensional (e.g., if some extinction event happens).

However, recall that the semantics has a fixed domain and that the kind terms are rigid. Thus the extension of a kind consists of all its *possible* instances. Kinds *Mammal* and *Rat* have different possible instances, and thus they are not identical. The identity conditions of kinds are a consequence of essentialist assumptions. If an object belongs to a kind, then it essentially exemplifies all the attributes in the essence of the kind, i.e., $A \subseteq e(B)$. If an object essentially exemplifies all the attributes in the kind then it belongs to the kind, i.e., $e(B) \subseteq A$. These hold by the axiom of *Membership Conditions*. If an attribute is in the essence of the kind then it is essentially exemplified by all the members of the kind, i.e., $B \subseteq i(A)$. If an attribute is essentially exemplified by all the members of the kind, then it belongs to the essence of the kind, i.e., $i(A) \subseteq B$. These hold by the axiom of *General Essence*. But then $A = e(B)$ and $B = i(A)$ and therefore if K and K' are coextensional, they have the same essence and a fortiori they must be identical.

§5. Hierarchy and Kant's Law.

In this section I will argue that, although *Hierarchy* fails, the semantics shows the conditions under which it holds. Moreover, I will present another principle about the species–genus relations that follows from **NKE** namely *Kant's Law*.

5.1. Hierarchy.

In his discussion about the species–genus relations between kinds, Thomason suggested that these are *hierarchical*, that is to say, that kinds are arranged forming a tree-like pattern. The condition was also discussed by Ellis [10] and Hacking [17]. Recall that it can be stated for a lattice as:

Hierarchy: For all $K, K' \in L$, either $K \leq K'$ or $K' \leq K$ or $K \sqcap K' = 0$.

Two incomparable kinds do not overlap, so they belong to different branches of the tree. *Hierarchy* implies that kinds are nested, which results in the shape of a tree. This condition goes back to the *Porphyryan Tree of Categories* (see Figure 4), where the root is the *Summum Genus* and the leaves are the *Infimae Species*.[16] The Porphyryan Tree satisfies the condition. For instance, *Living* and *Substance* overlap and in fact *Living* $<$ *Substance*.

However, this condition has recently come under attack by philosophers of science. Hacking [17] argued that, although *Rubidium-47* is a species of *Boson* and of *Rubidium*, *Rubidium* is not a species of *Boson* and *Boson* is not a species of *Rubidium*, so the condition does not hold. Recently, other philosophers of science, such as Tobin [36], Ruphy [31], Khalidi [22] and Hendry [19], have presented specific counterexamples from biochemistry, nuclear physics and astrophysics. For example, *Albumin* and *Renin*

[16] This is not historically accurate. Tree diagrams for representing logical relations appear for the first time several centuries later (see [18]).

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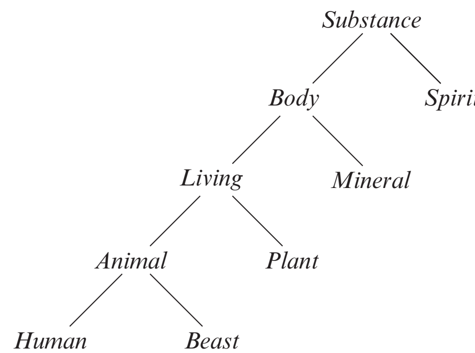

Fig. 4. Porphyryan tree.

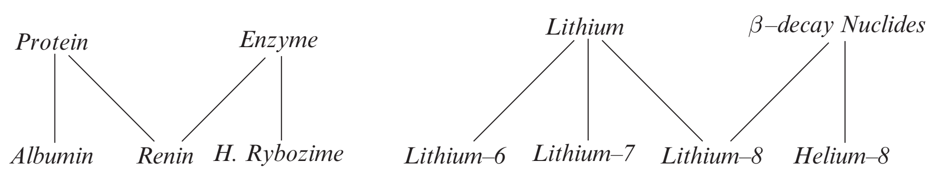

Fig. 5. Counterexamples to hierarchy.

are proteins, *Renin* and *Hairpin Ribozyme* are enzymes, but *Albumin* is not an enzyme and *Hairpin Ribozyme* is not a protein. Thus the kinds *Protein* and *Enzyme* overlap. For another example, *Lithium-6*, *Lithium-7* and *Lithium-8* are species of *Lithium*, but *Lithium-8* and *Helium-8* are species of *β-Decay Nuclides*. So the kinds *Lithium* and *β-Decay Nuclides* overlap. The resulting orders are not trees because the branches overlap (see Figure 5).

According to Tobin [36], many of these counterexamples are the result of either *intrataxonomic crossings*, which are overlappings between kinds of the same taxonomic system, or *intertaxonomic crossings*, which are overlappings of different taxonomic systems over the same domain. In the latter case, since taxonomic pluralism in science is the rule rather than the exception, this results in systematic violations of *Hierarchy*. The upshot of these criticisms is that this assumption is too strong. The order structure of kinds is more complex than that of a tree.

To be fair to Thomason, his lattice-theoretic model does not really need the principle. Nevertheless, our current proposal tells us which conditions objects and attributes need to satisfy in order for *Hierarchy* to hold. The constraint we need is that the extensions of attributes should form a *set-theoretic hierarchy*:[17]

Definition 5.1. *Let* (S, Q, I) *be a context. Then it is a* hierarchical context iff $\forall P, R \in Q$ $e(P) \subseteq e(R)$ *or* $e(R) \subseteq e(P)$ *or* $e(P) \cap e(R) = e(Q)$.

The following can be proven:

Proposition 5.1. *The context* (S, Q, I) *is hierarchical* iff L^* *is hierarchical.*

[17] Such families of sets are also used as models of classification (see [30]).

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Since the extensions of attributes often overlap without being nested, this condition usually fails. This is the case precisely in the counterexamples given by Tobin. For instance, the extensions of the attributes *being a catalyst* and *being a chain of aminoacids* overlap, but they are not included into each other.

The previous proposition provides the means to express *Hierarchy* in the logic. Define $X \sqsubseteq Y := \forall x(\text{Ess}(x, X) \to \text{Ess}(x, Y))$ for inclusion of attributes and $\mathit{Null}(X, Y) := \forall x((\text{Ess}(x, X) \wedge \text{Ess}(x, Y)) \leftrightarrow \forall Z\text{Ess}(x, Z))$, which says that the intersection of the extensions of two attributes is the extension of Q (which is usually empty). Then the principle says:

$$\nvDash \forall X \forall Y(X \sqsubseteq Y \vee Y \sqsubseteq X \vee \mathit{Null}(X, Y)). \text{ [Hierarchy]}$$

The formula is not valid because there are models where kinds overlap non-trivially (e.g., Tobin's example in Table 1). But in some models *Hierarchy does* hold. For instance, in the example of the classification of atoms into chemical elements and isotopes (Figure 1), the attributes form a hierarchy.

Even if it does not always hold, *Hierarchy* has something to teach us, for it shows an interesting difference between the algebraic structure of classical propositions and that of kinds. In some lattices of kinds the *distributive law*, which is the equation $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$, fails. As Thomason himself argued, trees provide counterexamples to the distributivity law. For example, consider the tree of chemical kinds in Figure 1. We have ${}^6\mathbf{Li} \sqcup (\mathbf{Li} \sqcap {}^2\mathbf{H}) = {}^6\mathbf{Li} \sqcup 0 = {}^6\mathbf{Li}$ but $({}^6\mathbf{Li} \sqcup \mathbf{Li}) \sqcap ({}^6\mathbf{Li} \sqcup {}^2\mathbf{H}) = \mathbf{Li} \sqcap \mathit{Element} = \mathbf{Li}$. Since the distributivity condition generally fails for trees, we should not assume that it holds in our models for kinds. A fortiori, since every Boolean algebra is distributive, this entails that in contrast with classical propositions, kinds do not form Boolean algebras.

Summing up, trees are very special lattices. *Hierarchy* requires the overlapping extensions of attributes to be nested. We can avoid the counterexamples related to *Hierarchy* by acknowledging that only some specific domains are hierarchically structured. Since the condition holds for some domains this also shows that we should not expect the lattice of kinds to be Boolean, and not even distributive.

5.2. Kant's Law. I will propose now a principle about the species genus relations that we can replace *Hierarchy* with. In contrast with the latter, this principle *is* a theorem of essentialism. It was discussed by Kant (and Leibniz; see [33]) in his *Jäsche Lectures on Logic* [21]:

> Every concept, as a *partial concept*, is contained *in* the presentation of things; as a *ground of cognition*, i.e., *as a characteristic*, it has these things contained *under it*. In the former regard, every concept has an *intension* [content]; in the latter, it has an *extension*. Intension and extension of a concept have an inverse relation to each other. The more a concept contains under it, the less it contains in it.

The previous quote describes a classical principle that is also known as *Kant's Law* or *Kant's Duality* (see [33]). We can obtain the relevant principle by replacing systematically the word 'concept' by the word 'kind' in the quote. The principle says that the more general the kind is, the less the number of attributes common to all its members will be. Dually, the more specific the kind is, the greater the number of attributes shared by its instances will be:

Kant's Law: The extension of a kind is inversely related to its general essence.

Roughly speaking, the principle says that the more general the kind is (the more instances it has), the poorer or 'shallower' its general essence will be. Dually, the more specific the kind is (the fewer instances it has), the richer its essence will be. This explains why the membership conditions of more specific kinds are more demanding than those of more general kinds, for an object needs to have more essential properties to be an instance of the former. The following is a theorem:

$$\vdash \forall x(K(x) \to K'(x)) \leftrightarrow \forall X(K'(X) \to K(X)). \text{ [Kant's Law]}$$

Using the definitions of the species–genus relation in Section 3 we get:

$$\vdash (K \leq_{\text{ext}} K') \leftrightarrow (K \leq_{\text{int}} K'). \text{ [Kant's Law]}$$

Thus the two definitions of the species–genus relation we gave are equivalent. Semantically, it follows directly from the definition of the species–genus relation.

Contrary to *Hierarchy*, *Kant's Law* is compatible with the existence of non-trivially overlapping kinds. For instance, the kinds *Protein* and *Enzyme* overlap because they have a common species *Renin*. Since some proteins are not enzymes and some enzymes are not proteins, this is incompatible with *Hierarchy*. But the most that Kants Law predicts is that there will be fewer instances of *Renin* than of *Protein* or of *Enzyme* and that the instances of *Renin* will share all the essential attributes common to both *Protein* and *Enzyme* (and possibly more).

Kant's Law describes the systematic relationship that holds between the extension and the essence of a kind. It also shows why representing kinds only as sets of objects or only as sets of properties is explanatorily limited. Such approaches cannot explain why it is plausible to assume that the extension of a kind grows inversely to its essence. The principle also shows, as Mill explained [28], why specific differences are to be expected, since if one kind is a proper species of another, then the former is expected to have 'more' properties in its essence.

Moreover, recall that causal theories reject the left-to-right half of the axiom of *Membership Conditions*, i.e., they reject that all the objects in the kind have all the properties associated to it. To prove the right-to-left direction of *Kant's Law*, one needs to use this part of the axiom of *Membership Conditions*. For suppose that the right half of *Kant's Law* holds. Assume that x is a K. To show that x is a K', by the right-to-left direction of *Membership Conditions*, we need to prove that x has all the attributes in the essence of K'. The essence of K' is included into the essence of K, so if x has all the attributes in the essence of K, we are done. But this is precisely what the left-to-right half of *Membership Conditions* says. So if *Kant's Law* holds, causal theories will not be able to show why it is so, which makes causal theories less explanatory than essentialist ones.

To sum up the discussion so far, recall the problems with Thomason's model:

(1) It gives no clues about the membership conditions of kinds.
(2) It does not explain why kinds should be closed under meets and joins.
(3) Even if it does not assume *Hierarchy*, it does not explain why it fails.
(4) It gives no clues about how the species–genus relation between kinds relates to their instances or to the properties shared by them.

About the first one, in Thomason's model there is no mention of objects or of shared attributes. Thomason's model does not explain in what sense kinds classify their instances or in virtue of what they do so. However, by using our semantics for an essentialist theory, we get instead membership conditions for natural kinds. As for the second problem, that kinds are closed under meets and joins follows from the main theorem of Concept Lattices and the definition of a natural kind in Section 4.1. Although we cannot prove *inside* **NKE** that the meets and joins of kinds exist (we lack higher-order quantification), we still have the metalinguistic result that in any given model kinds must form a lattice. Since **NKE** is complete with respect to this semantics, we have indirect support for the claim that kinds do form a lattice. The third objection is not really a problem, for as we saw *Hierarchy* is not an essentialist theorem and we can say in which models is satisfied. For the last objection, *Kant's Law* provides an interesting link between species–genus relations and the membership conditions of kinds that is an essentialist theorem.

§6. Definitions in terms of genera and specific differences.

6.1. Two operations of specific difference. The next application of the semantics concerns the *Aristotelian Theory of Definitions*. According to the traditional theory, one can abstract from a species by deleting part of its essence and correspondingly enlarging the extension. Dually, one can determine or specify a genus by adding attributes to its essence and correspondingly restricting its extension. To define a species one gives a genus and a differentia or specific difference. To take the standard example, the species *Human* is defined from the genus *Animal* and the specific difference *rational*, by adding the latter attribute to the essence of *Animal*. The aim of this section is to introduce new operations to show that kinds can be given Aristotelian definitions in terms of genera and specific differences.

Generally speaking, one can interpret the meets and joins of kinds as operations of *logical specification* and *logical generalization*, respectively. To specify, suppose that we take a kind $K = (A, B)$ and that we want to divide it into a species by enriching the intension B. We take a set of attributes B' which forms an intension, we add it to B and we obtain the closest natural kind, which has intension $ie(B \cup B')$. This is just the meet $K \sqcap K'$, where $K' = (e(B'), B')$. Dually, to generalize, we take a kind K, we select some attributes in B which form an intension by overlapping B with another intension B', and then we take the corresponding kind. Again, this is just the join $K \sqcup K'$. The meet and join of kinds can be understood as logical determination and abstraction operations that form new kinds from others.

However, to consider specific differences we need new operations that are well-defined in the lattice. If K is a proper species of K', then we have $B' \subset B$ and therefore the remainder $B - B'$ includes those attributes that make the K-s specifically K-s among the K'-s. But one may object that we are not subtracting two kinds to get a new kind. Given kinds K and K', we want a subtraction operation that always gives us the kind which contains the difference $B - B'$ in its intension. Since $B - B'$ is not necessarily the intension of a kind, what we need to do is to close this set to get an intension. The difference in extensions works analogously. Suppose that K is a species of K'. We can define the specific difference as the kind $K' \rightsquigarrow K := (e(B - B'), ie(B - B'))$ or as the kind $K' \setminus K := (ei(A' - A), i(A' - A))$. In other words, we can either take the attributes of the species that are not in the genus and then obtain the corresponding

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kind, or we can take the objects in the genus that are not in the species and then obtain the corresponding kind. Whereas the former is the kind induced by the intensional closure over a difference, the latter is the kind induced by the extensional closure over a difference. We can think about the former as the intensional way to subtract a species from a genus and about the latter as the extensional way to subtract the species from the genus.

I will first give an abstract characterization of the new operations and then I will show that they can be found in every concept lattice:

DEFINITION 6.1. *Let L be a complete lattice. Then* $\rightsquigarrow : L \times L \to L$ *is a* specific conditional iff *it satisfies* (1)–(5). *Dually,* $\setminus : L \times L \to L$ *is a* specific difference iff *it satisfies* (6)–(10):

(1) $x \sqcap y = x \sqcap (x \rightsquigarrow y)$. [*Modus Ponens*]
(2) $x \leq y \rightsquigarrow z$ iff $y \leq x \rightsquigarrow z$. [*Duality of the Conditional*]
(3) $(x \rightsquigarrow y) \sqcap (y \rightsquigarrow z) \leq x \rightsquigarrow z$. [*Transitivity*]
(4) $(x_1 \sqcup x_2 \sqcup \cdots) \rightsquigarrow y = (x_1 \rightsquigarrow y) \sqcap (x_2 \rightsquigarrow y) \sqcap \cdots$. [*De Morgan I*]
(5) $x \rightsquigarrow (x \rightsquigarrow y) \leq x \rightsquigarrow y$. [*Contraction*]
(6) $x \sqcup y = x \sqcup (y \setminus x)$. [*Dual Modus Ponens*]
(7) $z \setminus y \leq x$ iff $z \setminus x \leq y$. [*Duality of the Difference*]
(8) $(z \setminus x) \leq (z \setminus y) \sqcup (y \setminus x)$. [*Triangular Inequality*]
(9) $y \setminus (x_1 \sqcap x_2 \sqcap \cdots) = (y \setminus x_1) \sqcup (y \setminus x_2) \sqcup \cdots$. [*De Morgan II*]
(10) $y \setminus x \leq (y \setminus x) \setminus x$. [*Dual Contraction*]

Plausible alternative names for ‘specific conditional’ and ‘specific difference’ are *intensional difference* and *extensional difference*, respectively. We will only focus on the properties of the conditional, since those of the difference follow by duality:

PROPOSITION 6.1. *Let $(L, \rightsquigarrow)$ be a complete lattice with specific conditional. Then*:

(1) $x = 1 \rightsquigarrow x$. [*Constant*]
(2) $x \leq y$ iff $x \rightsquigarrow y = 1$. [*Tautology*]
(3) *If* $x \leq y$ *then* $z \rightsquigarrow x \leq z \rightsquigarrow y$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$. [*Monotonicity$_I$*]
(4) *If* $x \leq y, z \leq w$ *then* $w \rightsquigarrow x \leq z \rightsquigarrow y$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow w$. [*Monotonicity$_{II}$*]
(5) $x \rightsquigarrow x = 1$. [*Identity*]
(6) $x \leq y \rightsquigarrow x$ *and* $y = y \sqcap (x \rightsquigarrow y)$. [*Weakening*]
(7) $x \leq y$ iff $x = y \sqcap (y \rightsquigarrow x)$ iff $x \leq x \rightsquigarrow y$. [*Order*]
(8) $x \rightsquigarrow y \leq x \rightsquigarrow (z \rightsquigarrow y)$.
(9) *If* $x \leq y \rightsquigarrow z$ *then* $x \sqcap y \leq z$. [*Half-Galois*]
(10) $x \rightsquigarrow (x \rightsquigarrow y) = x \rightsquigarrow y$. [*Contraction Eq.*]

Proof. (1) $x = x \sqcap 1 = 1 \sqcap (1 \rightsquigarrow x) = 1 \rightsquigarrow x$ by Modus Ponens. (2) $x \leq y = 1 \rightsquigarrow y$ iff $1 \leq x \rightsquigarrow y$ by Constant and Duality of Conditional. (3) Let $x \leq y$. By Transitivity and Tautology $z \rightsquigarrow x = (z \rightsquigarrow x) \sqcap 1 = (z \rightsquigarrow x) \sqcap (x \rightsquigarrow y) \leq (z \rightsquigarrow y)$. By De Morgan $(y \rightsquigarrow z) \sqcap (x \rightsquigarrow z) = (x \sqcup y) \rightsquigarrow z = y \rightsquigarrow z$. (4) Let $x \leq y$, $z \leq w$. By Monotonicity$_I$, $w \rightsquigarrow x \leq w \rightsquigarrow y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$, so by De Morgan $w \rightsquigarrow x = (w \sqcup z) \rightsquigarrow x = (w \rightsquigarrow x) \sqcap (z \rightsquigarrow x) \leq (w \rightsquigarrow y) \sqcap (z \rightsquigarrow y) \leq z \rightsquigarrow y$. We apply this again to obtain the other. (5) By Tautology. (6) By Identity and Duality of Conditional. (7) By Modus Ponens and Weakening. (8) By Weakening and Monotonicity. (9) If $x \leq y \rightsquigarrow z$ then $y \sqcap x \leq y \sqcap (y \rightsquigarrow z) = y \sqcap z \leq z$. (10) By Weakening and Contraction. □

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The conditional fails to satisfy the following properties:[18]

(1) $x \rightsquigarrow (y \sqcap z) = (x \rightsquigarrow y) \sqcap (x \rightsquigarrow z)$. [Dist. Conditional-Meet]
(2) If $x \sqcap y \leq z$ then $x \leq y \rightsquigarrow z$. [Half-Galois]
(3) $x \rightsquigarrow (y \rightsquigarrow z) \leq (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)$. [Dist. Conditional-Conditional]
(4) $x \rightsquigarrow (y \rightsquigarrow z) \leq y \rightsquigarrow (x \rightsquigarrow z)$. [Permutation]

A counterexample for the first two is the context $S = \{x, y, z\}$, $Q = \{p, r, t\}$ where $i(x) = \{p, t\}$, $i(y) = \{r\}$ and $i(z) = \{p\}$. The remaining intensions are $\varnothing$ and Q. Then $0 = y \sqcap z \leq x$ but $y \not\leq z \rightsquigarrow x = x$, so Half-Galois fails. Furthermore, $y \rightsquigarrow (y \sqcap z) = y \rightsquigarrow 0 = x$ but $(y \rightsquigarrow y) \sqcap (y \rightsquigarrow z) = 1 \sqcap 1 = 1$, so the conditional does not Distribute over Meets either.

A counterexample for the other two is the context $S = \{x, y, z, a, b, c, d, e, f, g, h, l\}$, $Q = \{p, q, r, s, t, n\}$ and $i(x) = \{p, q, t\}$, $i(y) = \{s, t, n\}$, $i(z) = \{p, q, r, n\}$, $i(a) = \{p, q, r\}$, $i(b) = \{p, q\}$, $i(c) = \{p\}$, $i(d) = \{q\}$, $i(e) = \{r\}$, $i(f) = \{p, r\}$, $i(g) = \{t\}$, $i(h) = \{n\}$, $i(l) = \{s, n\}$. The remaining intensions are $\{p, q\}$, $\{p, r\}$, $\{s, n\}$, $\{p, r, n\}$, $\{s, t, n\}$, $\{p, q, t\}$, $\{p, q, r, n\}$, Q and $\varnothing$. The intension of $y \rightsquigarrow z$ is $\{p, q, r\}$, that of $x \rightsquigarrow z$ is $\{p, q, r, n\}$, that of $x \rightsquigarrow y$ is $\{s, n\}$, that of $x \rightsquigarrow (y \rightsquigarrow z)$ is $\{r\}$ and that of $(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)$ is $\{p, q, r\}$. Thus the intension of $(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)$ is not included in the intension of $x \rightsquigarrow (y \rightsquigarrow z)$, so Distribution fails. Since the intension of $y \rightsquigarrow (x \rightsquigarrow z)$ is $\{p, q, r\}$, Permutation fails too.

The last step is to check that the operations in the concept lattices are indeed specific conditionals and differences. Although this fact could be proven directly, we are going to obtain it as a corollary of a new slightly more general result that applies to complete lattices. It turns out that the two operations can be defined in *every complete lattice relative to some chosen join-dense and meet-dense subsets*:

Definition 6.2. *Let L be a complete lattice, $x \in L$ and $X \subseteq L$. Then* x *is* join-irreducible iff *if $x = y \sqcup z$ then $x = y$ or $x = z$. X is* join-dense *in L* iff *for each $x \in L$ there is an $A \subseteq X$ such that $x = \bigsqcup A$.*

We obtain by duality the notions of *meet-irreducible element* and *meet-dense* subset. A join (meet)-dense subset makes every element in the lattice a join (meet)-combination of elements in the set.[19] We now show that each meet-dense (join-dense) subset induces its own specific conditional (difference):

Proposition 6.2. *Let L be a complete lattice, $X \subseteq L$ join-dense, and $Y \subseteq L$ meet-dense. Let $j(x) :=\downarrow x \cap X$ and $m(x) :=\uparrow x \cap Y$. Then $x \rightsquigarrow y := \bigsqcap(m(y) - m(x))$ is a specific conditional and $x \setminus y := \bigsqcup(j(x) - j(y))$ is a specific difference.*

Proof. For Modus Ponens, let $x \rightsquigarrow y := \bigsqcap(m(y) - m(x))$. First, $x \sqcap y = \bigsqcap m(x) \sqcap \bigsqcap m(y) = \bigsqcap[m(x) \cup m(y)] = \bigsqcap[m(x) \cup (m(y) - m(x))] = \bigsqcap m(x) \sqcap \bigsqcap(m(y) - m(x)) = x \sqcap (x \rightsquigarrow y)$. For the Duality of the Conditional we prove by reductio. Assume that $x \leq y \rightsquigarrow z$. Now suppose for reductio that there is a $q \in m(z) - m(x) \not\subseteq m(y)$. Therefore $q \in m(z) - m(y)$ and since by our original assumption

[18] The lattice is Heyting (complete) iff satisfies the other half of Galois iff the conditional distributes over meets iff finite meets distribute over arbitrary joins. To avoid clutter, in the counterexamples we use the same symbol, e.g., x, for both the object and the corresponding kind.

[19] Basic properties of dense subsets can be found in [8].

$x \leq \prod(m(z) - m(y)) = y \rightsquigarrow z \leq q$ we have $q \in m(x)$, which contradicts the assumption. Thus $m(z) - m(x) \subseteq m(y)$; therefore $y = \prod m(y) \leq \prod(m(z) - m(x)) = x \rightsquigarrow z$. For Transitivity, $x \rightsquigarrow y \sqcap y \rightsquigarrow z = \prod[m(y) - m(x)] \sqcap \prod[m(z) - m(y)] = \prod[m(y) - m(x) \cup m(z) - m(y)] \leq \prod(m(z) - m(x)) = x \rightsquigarrow z$, since $m(z) - m(x) \subseteq m(z) - m(y) \cup m(y) - m(x)$. For De Morgan, $(x_1 \sqcup x_2 \sqcup \cdots) \rightsquigarrow y = \prod(m(y) - m(x_1 \sqcup x_2 \sqcup \cdots)) = \prod(m(y) - m(x_1) \cap m(x_2) \cap \cdots) = \prod(m(y) - m(x_1) \cup m(y) - m(x_2) \cup \cdots) = \prod(m(y) - m(x_1)) \sqcap \prod(m(y) - m(x_2)) \sqcap \cdots = x_1 \rightsquigarrow y \sqcap x_2 \rightsquigarrow y \sqcap \cdots$. For Contraction, $m(y) - m(x) \subseteq m(x \rightsquigarrow y)$. Since $(A - B) - B = A - B$ for sets, we have $m(y) - m(x) = (m(y) - m(x)) - m(x) \subseteq m(x \rightsquigarrow y) - m(x)$; therefore $x \rightsquigarrow (x \rightsquigarrow y) = \prod(m(x \rightsquigarrow y) - m(x)) \leq \prod(m(y) - m(x)) = x \rightsquigarrow y$. The proof for the specific difference is dual. □

A complete lattice has many join-dense (meet-dense) subsets, including the lattice itself and any superset of a join-dense subset. However, sometimes a specific choice is the most natural one. In the case of concept-lattices, the set of join-irreducible elements (the infimae species) forms a join-dense subset and the set of meet-irreducible elements (the maximal genera) forms a meet-dense subset [15].[20] One can check that, in fact, these induce the specific conditional and difference operations in the concept lattice. Our join-irreducibles have the form $(ei(x), i(x))$, whereas the meet-irreducibles have the form $(e(R), ie(R))$:

Corollary 6.1. *Let L be a concept lattice, $K \rightsquigarrow K' := (e(B' - B), ie(B' - B))$ and $K \setminus K' := (ei(A - A'), i(A - A'))$. Then $K \rightsquigarrow K' = \prod(m(K) - m(K))$ and $K \setminus K' = \bigsqcup(j(K) - j(K'))$.*

Proof. We prove it for the difference. If A is an extension, then $x \in A$ iff $ei(x) \subseteq A$. We have $\bigsqcup(j(K) - j(K')) = \bigsqcup\{ (ei(x), i(x)) \in L \mid ei(x) \subseteq A$ and $ei(x) \nsubseteq A' \}$. Therefore, the extension of this kind is $ei(\bigcup\{ ei(x) \mid x \in A - A' \})$. But $A - A' \subseteq \bigcup\{ ei(x) \mid x \in A - A' \}$; therefore $ei(A - A') \subseteq ei(\bigcup\{ ei(x) \mid x \in A - A' \})$. Since each $x \in A - A'$ is such that $ei(x) \subseteq ei(A - A')$, we get $\bigcup\{ ei(x) \mid x \in A - A' \} \subseteq ei(A - A')$; therefore $ei(\bigcup\{ ei(x) \mid x \in A - A' \}) \subseteq eiei(A - A') = ei(A - A')$. So $\bigsqcup(j(K) - j(K')) = K \setminus K'$. □

It should not be a surprise that the operations can be defined using the maximal genera and infimae species. Let K be a species of K' and consider the specific conditional $K \rightsquigarrow K' := (e(B' - B), ie(B' - B)) = \prod(m(K) - m(K))$. The set $m(K')$ contains all the maximal genera that K' is a species of. But these correspond to the attributes in the intension of K'. So the specific conditional $K \rightsquigarrow K'$ is obtained by taking the most general kind that is a species of all these maximal genera, i.e., mutatis mutandis, by taking all the attributes in K' that are not in K and then adding the attributes needed to get the corresponding kind.

6.2. Definitions in terms of genera and specific differences. Suppose that K is a proper species of K'. By *Kant's Law* we know that the essence of the species is strictly richer than that of the genus. But if so, there is a difference in their essences. There are some attributes that make K-s specifically K-s among the K'-s. This is precisely the *specific difference* of the species with respect to its genus. John Stuart Mill summarizes this fact in *A System of Logic*:

[20] This fact is the core of the proof of the converse of the Fundamental Theorem of Concept Lattices, which says that each complete lattice is isomorphic to a concept lattice (see [15]).

> From the fact that the genus includes the species, in other words denotes more than the species, or is predicable of a greater number of individuals, it follows that the species must connote more than the genus. It must connote all the attributes which the genus connotes, or there would be nothing to prevent it from denoting individuals not included in the genus. And it must connote something besides, otherwise it would include the whole genus. (...) This surplus of connotation - this which the species connotes over and above the connotation of the genus - is the Differentia, or specific difference; or, to state the same proposition in other words, the Differentia is that which must be added to the connotation of the genus, to complete the connotation of the species. [28]

What Mill says is that the specific difference is what must be added to the essence of the genus in order to obtain the species. But how can we make precise this idea of 'adding'? Let us interpret $K' \rightsquigarrow K$ as 'the intensional difference of K with respect to K''. From the previous section we have:

$$K \leq K' \text{ iff } K = K' \sqcap (K' \rightsquigarrow K).$$

So a species K is the overlapping of its genus K' with the intensional difference of K with respect to K'. We have $(A, B) = (A', B') \sqcap (e(B - B'), ie(B - B')) = (A' \cap e(B - B'), ie(B' \cup ie(B - B'))) = (A' \cap e(B - B'), i(e(B') \cap e(B - B'))) = (A' \cap e(B - B'), ie(B' \cup (B - B')))$. The species is obtained by taking the attributes of the essence of the species that are 'specific' to them (i.e., not shared by other instances of the genus outside the species) and adding them to those in the essence of the genus (and closing this set to get an essence). For instance, the species *Human* is obtained by adding to the essence of *Animal* those properties that distinguish humans from other animals, e.g., *being rational* (and whatever other properties are needed to get the essence of *Human*). As described by Mill, we add to the essence of the genus those properties that are specific to the members of the species.

But why should we go from the genus to the species? Given the species and another sort of difference, we could equally get the genus. Let us interpret $K' \setminus K$ as 'the extensional difference of K' with respect to K'. We also have

$$K \leq K' \text{ iff } K' = K \sqcup (K' \setminus K).$$

A genus K' is the sum of its species K with the extensional difference of K' with respect to K. We have $(A', B') = (A, B) \sqcup (ei(A' - A), i(A' - A)) = (ei(A \cup ei(A' - A)), B \cap i(A' - A)) = (e(i(A) \cap i(A' - A)), B \cap i(A' - A)) = (ei(A \cup (A' - A)), B \cap i(A' - A))$. The genus is obtained by adding to the extension of the species those objects that belong to the genus but not to the species, and then taking the closure of this set (i.e., adding the missing objects that share the essence too). For instance, the genus *Animal* is obtained by adding to the extension of the species *Human* all the non-human animals (and closing the set). For another example, consider again Tobin's in Figure 2. We have that $Protein \setminus Albumin = Renin$, so the extensional difference of genus *Protein* with respect to its species *Albumin* is the species *Renin*.

Briefly put, there are two ways to subtract one kind from another. We can *define a species intensionally* by adding properties to the essence of its genus, or we can *define the genus extensionally* by adding objects to the extension of the species. These two operations are distinct and dual to each other. When we get a species by overlapping the genus with the intensional difference, we cannot recover the genus by joining the intensional difference to the species, we have to use the extensional difference. Combining both operations, we get back to where we started.

Other classical theses hold. For instance, there is a sense in which one can get every species as a 'division' of the Summum Genus. Since $K \leq 1$ we have $K = 1 \sqcap (1 \rightsquigarrow K) = 1 \rightsquigarrow K$. If we were given the Summum Genus 1 and the intensional specific differences of each kind with respect to it, we could obtain each kind by overlapping the Summum Genus with the corresponding specific difference. If the lattice satisfies *Hierarchy*, this has consequences for the differences too. If K is a proper species of K', the intensional difference $K' \rightsquigarrow K$ of K with respect to K' is simply K. In a tree we lose information regarding what makes a species *intensionally* different from other species of the same genus. In contrast, if the genus K' branches into non-null species K and K^*, these two will be disjoint. Thus a kind K and its extensional difference with respect to the genus K' form a *partition* of the genus K'. In a tree, the genera get partitioned into their proper species, and they can be recovered by adding to the latter the extensional differences.

§7. Negative kinds.

7.1. Contrariety, subcontrariety and contradictoriness. In this last section I want to address two objections against the algebraic structure of kinds. Before that, let us end describing the lattice of kinds. Kinds are closed under *negations* too. I show that these negations are induced by the operations of specific difference.

Aristotle made two distinctions concerning negation that are relevant for our purposes. On the one hand, he distinguished between *propositional* and *predicate negations*. The sentence 'x is not mortal' has two different readings, depending on the scope of negation. The propositional reading takes the scope of negation to be the proposition *that x is mortal*, so that the sentence expresses the proposition *that it is not the case that x is mortal*. The predicate reading takes the scope of negation to be restricted to the predicate 'mortal', or to the property of *being mortal*. If so, the sentence expresses the proposition *that x is not-mortal*. In classical logic, both readings are usually represented as $\neg Mortal(x)$.

If the negation of the predicate behaves classically, there is no need to distinguish between both readings, for they are logically equivalent. However, for some classes of predicates we need a more nuanced distinction between predicate negations. This issue is related to the other relevant Aristotelian distinction between *contrary*, *subcontrary* and *contradictory terms*. This distinction, described by the Quantificational Square of Opposition,[21] is usually applied to sentences and propositions. Nevertheless, it applies to other terms and their denotations as well, such as predicates and properties. Two predicates are contraries iff there is no object that satisfies them both, but some objects can satisfy neither. Two predicates are subcontraries iff every object satisfies at least one of them, but some objects can satisfy both. Two predicates are contradictories iff there is no object that satisfies them both and every object satisfies at least one of them. There are plenty of examples of contrary predicates in natural languages. For

21 There are other interpretations of the Square (and Hexagon) of Opposition [2], e.g., the *Hexagon of Inner and Outer Negations* [27] describes propositional and predicate negations.

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instance, *antonymous adjectives* denote contrary properties: small-big, bright-dark, friends-enemies, and so on. Nothing can be both small and big, but some things are neither small nor big. Natural examples of subcontrary predicates are harder to come by. Subcontrariety appears whenever we have two predicates that overlap in extension and jointly cover the whole domain of entities. Some examples would be monadic predicates obtained from a relational predicate and its converse, like employer-employee, host-guest, and so on. If the domain is all the people working in a company, 'employer' and 'employee' would be subcontraries, given that everyone must be either an employer or an employee, but some people are both employers and employees.

In some cases, as in the case of antonymous predicates, for every predicate P there is a natural choice for a unique (non-contradictory) contrary predicate P^*. If so, we can treat the mapping that sends each predicate to its contrary itself as a new negation operation. An important difference between this negation and the mapping that sends each predicate to its contradictory predicate is precisely that the former is *non-classical*. For instance, as we have seen, the Law of Excluded Middle fails for a predicate and its contrary. Moreover, predicate negation entails propositional negation, but the converse is false. For example, 'x is tall' entails 'it is not the case that x is short', but the converse does not hold.[22] Of course, this does not entail that the background logic is non-classical. Propositional negation can still be classical. The difference is that we have an additional negation for a certain class of predicates that is non-classical.

Our previous first-order representation of 'x is not mortal' as $\neg Mortal(x)$ does not distinguish between the two readings if the negated predicate is intended to denote a property that is only *contrary* to the one denoted by 'mortal' and not contradictory to it, like the property *Immortal*. If we use the standard semantics, we can pick an appropriate subset for our new predicate 'Immortal', since all the subsets of the domain can be denotations of predicates. However, in the natural kind semantics, for all we have said so far, given a kind K there may not be a pair (A, B) that satisfies our definition of kind and behaves as the negation of K. The purpose of the following section is to show that such candidates *do* exist.

7.2. Propositional and predicate negations. I will show now that the operations of specific difference introduced in the previous sections induce their own negations. In other words, they entail the existence of two sorts of *negative kinds*. As Wille [38] argues, there are two natural negation-like operations in a concept lattice:

$$K^c = (e(B^c), ie(B^c)), \qquad K^s = (ei(A^c), i(A^c)).$$

Whereas the one on the left takes the objects sharing those attributes that are not in K, the one on the right takes the attributes shared by all the objects that are not in K. Wille abstracts these operations to study the resulting lattices:

Definition 7.1. *Let* $(L, \sqcap, \sqcup, {}^c, {}^s, 0, 1)$ *be a bounded lattice and* ${}^c, {}^s : L \to L$ *two monadic operators. Then* L *is a* dicomplemented lattice iff:

(1) $x^{ss} \leq x$. [*Intensiveness*]
(2) *If* $x \leq y$ *then* $y^s \leq x^s$. [*Antitonicity*]

22 This entailment is present in Kant's distinction between *negative* and *infinite* quality judgements in his Jäsche Lectures on Logic [21]. Whereas negative judgements correspond to propositional negations, infinite judgements correspond to predicate negations (see [27]).

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(3) $x = (x \sqcap y) \sqcup (x \sqcap y^s)$.
(4) $x \leq x^{cc}$. [*Extensiveness*]
(5) *If* $x \leq y$ *then* $y^c \leq x^c$. [*Antitonicity*]
(6) $x = (x \sqcup y) \sqcap (x \sqcup y^c)$.

The operation ^{s} is the *weak negation* or *weak supplement* and ^{c} is the *weak opposition* or *weak complement*. The most interesting example of a dicomplemented lattice for us is given by the negation and opposition in a concept lattice.

PROPOSITION 7.1. *Let L be a dicomplemented lattice. Then*:

(1) $x^{ccc} = x^c \leq x^s = x^{sss}$.
(2) $(x^s)^c \leq x^{ssss} = x^{ss} \leq x \leq x^{cc} = x^{cccc} \leq (x^c)^s$.
(3) $x \sqcap x^c = 0$ *and* $x \sqcup x^s = 1$.
(4) $(x \sqcup y)^c = x^c \sqcap y^c$ *and* $(x \sqcap y)^s = x^s \sqcup y^s$.
(5) $0^c = 1 = 0^s$ *and* $1^c = 0 = 1^s$.

Now we show that these negations can be obtained from the new operations of specific difference I introduced in Section 6:

PROPOSITION 7.2. *Let* $(L, \rightsquigarrow)$ *be a complete lattice with specific conditional. Then* $x^c = x \rightsquigarrow 0$ *satisfies*:

(1) $y \leq x^c$ iff $x \leq y^c$. [*Duality of Negation*]
(2) *If* $x \leq y$ *then* $y^c \leq x^c$.
(3) $x \leq x^{cc}$.
(4) $x \rightsquigarrow x^c = x^c \leq x \rightsquigarrow y$. [*Explosion*]
(5) $x^c \sqcup y \leq x \rightsquigarrow y$. [*Disjunctive Syllogism*]
(6) $x = (x \sqcup y) \sqcap (x \sqcup y^c)$.
(7) $(x_1 \sqcup x_2 \sqcup \cdots)^c = x_1^c \sqcap x_2^c \sqcap \cdots$. [*De Morgan Opposition*]
(8) *If* $x \rightsquigarrow y = 1$ *and* $y^c = 1$ *then* $x^c = 1$. [*Modus Tollens*]

Proof. (1) By the Duality of the Conditional. (2)–(4) Follow from Monotonicity, Contraction and the Duality of Negation. (5) By Explosion and Weakening. (6) We just need to prove half of it. By Disjunctive Syllogism, De Morgan and Identity $(x \sqcup y^c) \leq y \rightsquigarrow x = (y \rightsquigarrow x) \sqcap 1 = (y \rightsquigarrow x) \sqcap (x \rightsquigarrow x) = (x \sqcup y) \rightsquigarrow x$; therefore by Modus Ponens $(x \sqcup y^c) \sqcap (x \sqcup y) \leq x$. (7) By De Morgan. (8) If $x \rightsquigarrow y = y^c = 1$, by (3) $y \leq y^{cc} = 0$ so $1 = x \rightsquigarrow y = x \rightsquigarrow 0 = x^c$. □

Therefore, if $\rightsquigarrow$ is a specific conditional, then $x^c = x \rightsquigarrow 0$ is a weak opposition. Dually, if $\setminus$ is a specific difference, $x^s = 1 \setminus x$ is a weak negation. In this way we obtain Wille's negations as special cases of the differences. It is easy to check that, in the case of concept lattices, the weak opposition is $K^c = (e(B^c), ie(B^c))$ and the weak negation is $K^s = (ei(A^c), i(A^c))$, as expected.

To distinguish between propositional and predicate negations, we introduce the new operations of kinds in the logic:

DEFINITION 7.2. *We define the following abbreviations of symbols*:

(1) $K \rightsquigarrow_O K'(x) := \forall X((K'(X) \wedge \neg K(X)) \rightarrow \mathrm{Ess}(x, X))$.
(2) $K \setminus_O K'(x) := \forall X(\forall y((K(y) \wedge \neg K'(y)) \rightarrow \mathrm{Ess}(y, X)) \rightarrow \mathrm{Ess}(x, X))$.
(3) $K \rightsquigarrow_P K'(X) := \forall x((\forall Y(K'(Y) \wedge \neg K(Y)) \rightarrow \mathrm{Ess}(x, Y)) \rightarrow \mathrm{Ess}(x, X))$.
(4) $K \setminus_P K'(X) := \forall x((K(x) \wedge \neg K'(x)) \rightarrow \mathrm{Ess}(x, X))$.

(5) $K^c_O(x) := K \rightsquigarrow_O \text{Null Kind}_O(x)$, *is the O-weak opposition of K.*
(6) $K^s_O(x) := \text{Summum Genus}_O \setminus K(x)$, *is the O-weak negation of K.*
(7) $K^c_P(X) := K \rightsquigarrow_P \text{Null Kind}_P(X)$, *is the P-weak opposition of K.*
(8) $K^s_P(X) := \text{Summum Genus}_P \setminus K(X)$, *is the P-weak negation of K.*

For the weak opposition, one considers all the attributes that are not in the essence of a kind and takes the objects exemplifying all those attributes. The meet of the two kinds is the Null Kind, but their join need not be the Summum Genus, since weak opposition does not usually satisfy *Predicate Excluded Middle*. If $K \sqcup K^c = (ei(A \cup e(B^c)), B \cap ie(B^c) \neq 1$, there is an object x that is neither a K nor a K^c. If an object does not exemplify an attribute in the essence B of K, it does not follow that it exemplifies all the attributes that are not in B. For instance, define $Immortal(x) := Mortal^c(x)$. If a stone lacks one of the properties needed to be mortal, it does not follow that it has all the properties to be immortal, for immortal things are alive and the stone is not. The operation of weak opposition provides for each kind a unique contrary kind.

For the weak negation, one considers all the objects that are not in the extension of a kind and then takes all the attributes that all these objects exemplify. The join of a kind and its opposite is the Summum Genus, but their meet need not be the Null Kind, since weak negation does not usually satisfy *Predicate Non-Contradiction*. If $K \sqcap K^s = (A \cap ei(A^c), ie(B \cap i(A^c))) \neq 0$, there is an object that is both a K and K^s. If an object has all the properties needed to be a K, it can still have all the properties needed to be a K^s. For instance, define $Employer(x) := Employee^s(x)$. If someone has all the properties to be an employer, she can still have all the properties needed to be an employee. The weak negation provides for each kind a unique subcontrary kind.

7.3. The non-classical structure of kinds. Here I address two last objections to the algebraic structure of kinds. First, authors such as Armstrong [1] have argued that natural properties are not closed under disjunctions or negations. For instance, although red things are similar enough to each other, not red things (e.g., the Eiffel Tower and Julius Caesar) are not similar to each other. By analogy, natural kinds are not closed under disjunctions or negations. But in this semantics they are.

According to this semantics, kinds are closed under conjunction, disjunction and negation-like operations. But kinds are not represented simply as internally unstructured sets. They are represented as pairs of sets that need to satisfy the constraint that the extension and intension should be adequately related. The 'disjunction' of two kinds is the most specific kind that is a genus of both. In fact, if we look at some of the examples given before we will see that the resulting 'disjunctions' of kinds are quite natural. For example, in the tree of chemical elements *Lithium* is the 'disjunction' of its isotopes. Now consider:

$$\nvDash \forall x(K \sqcup_O K'(x) \leftrightarrow K(x) \vee K'(x)),$$
$$\nvDash \forall X(K \sqcap_P K'(X) \leftrightarrow K(X) \vee K'(X)).$$

The formulas are not valid. In the first one, the left–right direction does not hold. Some objects have all the attributes that are in both essences but lack at least one attribute from each of the essences. So the extension of the join of two kinds is not simply the union of their extensions. Dually, the extension of the meet of two kinds is the intersection of their extensions. But the essence of the meet usually contains *more* essential properties than the union of the essences. The same goes for negations. The negation and the opposition of a kind usually do not have as extension the *complement* of the extension of the original kind. If there is no essential property common to all things that are not atoms of lithium, then *Lithium*^{s} is simply the Summum Genus. The objections against the closure of kinds under those operations assume a standard semantics where the extensions of the disjunctions and negations of kinds are the union and complement of the extensions of the original kinds, which is not the case according to this semantics.

Second, one may argue that the algebraic structure is too weak. The lattice of kinds is not Boolean and not even distributive. But plausibly, just as propositions described by classical logic, kinds should form a distributive or Boolean structure. Moreover, one may think that if kinds have non-classical structure, the logic should be non-classical too, which would make our semantics at odds with classical logic.

As argued in Section 5, most trees are not even distributive lattices. But there are accepted hierarchical (tree-like) classifications both in scientific and ordinary discourse; thus we should not assume that kinds form a distributive lattice. Since every Boolean lattice is distributive, we cannot assume the lattice of kinds to be Boolean either. Furthermore, kinds having non-classical structure does not prevent *propositions* from having Boolean structure. This is where the Aristotelian distinctions introduced in the previous section play their part. The negations ^{c}, ^{s} over kinds correspond to predicate negations. Their behaviour is non-classical. The opposite of a kind is contrary but usually not contradictory to it, e.g., $\forall x(K(x) \vee K^c(x))$ is not a theorem. The negation of a kind is subcontrary but usually not contradictory to it, e.g., $\forall x \neg(K(x) \wedge K^s(x))$ is not a theorem. However, the non-classical behaviour of predicate negations does not affect the classical features of propositional negation $\neg$. For example, it still holds that either x belongs to the extension of K or it does not, since $\vdash \forall x(K(x) \vee \neg K(x))$ is an instance of *Excluded Middle*. And it still holds that x cannot both belong and not belong to the extension of K, since $\vdash \forall x \neg(K(x) \wedge \neg K(x))$ is an instance of *Non-Contradiction*.

§8. Conclusion. The aim of this paper was to provide and study a complete Natural Kind Semantics, based on Rudolf Willes Theory of Concept Lattices, for an Essentialist Theory of Kinds formulated in two-sorted first order monadic modal logic. I considered first a related class of algebraic models proposed by Thomason and I presented some objections to them. Then I introduced a new semantics which preserves the essentialist membership conditions of kinds. The semantics was applied to several issues. First, I showed under which conditions the *Hierarchy* condition fails. I also suggested a different principle about the species–genus relations between kinds, namely *Kant's Law*, which follows from the essentialist theory. Second, I introduced two new operations and showed how they can be used to provide traditional definitions of kinds in terms of genera and specific differences. Finally, I showed that these operations of specific difference induce, for each kind, a uniquely specified contrary kind and a uniquely specified subcontrary kind, which show in what sense the structure of kinds is non-classical.

§9. Appendix. Soundness of **NKE** is proven by induction on the length of the derivation, as usual. The only new cases concern the new axioms, which we have already shown to be sound. We omit the proof.

Given that the semantics is not standard, we cannot get completeness as a corollary of standard completeness. Nevertheless, we can give a completeness proof by making minor modifications to the one by Henkin and Kripke. Recall that it has two parts.

First, we need to show that every consistent theory can be extended to a maximally consistent Henkin theory. We recall the basic definitions. A set of formulas Δ is a *theory* iff if $\Delta \vdash \varphi$ then $\varphi \in \Delta$. It is *consistent* iff it does not hold that $\Delta \vdash \bot$. It is *maximally consistent* iff Δ is consistent and if Δ' is consistent and $\Delta \subseteq \Delta'$ then $\Delta = \Delta'$. It is *Henkin* (omega-complete) iff for every sentence $\exists x\varphi$ there is a closed object term $t \in Term_O$ such that $\Delta \vdash \exists x\varphi(x) \to \varphi(t)$ and for every sentence $\exists X\varphi(X)$ there is a closed attribute term $T \in Term_P$ such that $\Delta \vdash \exists X\varphi(X) \to \varphi(T)$. Note that Henkin theories require witnesses of both sorts. A *saturated* theory is a maximally consistent Henkin theory.

We just recall the thorny step specific to modal logic (see [16]). In the extension of a consistent theory to a saturated one, one needs to add, one by one, witnesses for each quantified sentence and guarantee in each step that the resulting set is consistent. In the non-modal case, witnesses are usually taken to be fresh terms, which requires expansions of the language. Since the logic is modal, given a saturated set w which does not contain $\Box\varphi$, we need to guarantee that there is a saturated set w' which extends the set $w^* = \{\varphi \mid \Box\varphi \in w\} \cup \{\neg\varphi\}$. The problem is that in order to construct w' from w^* we need to add new terms, but by this point all the terms of the language are already in w^*. Since we have the Barcan axiom and the logic is S5, we can use Thomason's strategy to check that saturated sets already satisfy this condition, safely reusing terms that already occurred in the construction before. Since this part of the proof is standard, we omit it too.

Second, for such a Henkin theory we need to construct its canonical model. In our case, the canonical model for the theory will contain a context that induces a concept lattice. The context will be obtained from the sets of terms in the language. We then check that the kind predicates denote elements in its lattice.

Definition 9.1. *Let Γ be a consistent theory and Δ its saturated extension. We define the skeleton $(W, R, S, Q, I_{\text{Ess}}, \{I^w_{\text{Exe}}\}_{w\in W})$ as follows*:

1. *W is the set of all saturated theories* w *which satisfy*:
   - (a) $t_i = t'_i \in \Delta$ iff $t_i = t'_i \in w$.
   - (b) $\text{Ess}(t, T) \in \Delta$ iff $\text{Ess}(t, T) \in w$.
   - (c) $K(t_i) \in \Delta$ iff $K(t_i) \in w$.
2. wRw' iff $\{\varphi \mid \Box\varphi \in w\} \subseteq w'$.
3. *$S = Term_O$ is the set of closed object terms.*
4. *$Q = Term_P$ is the set of closed attribute terms.*
5. $tI_{\text{Ess}}T$ iff $\Delta \vdash \text{Ess}(t, T)$.
6. $tI^w_{\text{Exe}}T$ iff $w \vdash \text{Exe}(t, T)$.

Proposition 9.1. *The structure $(W, R, S, Q, I_{\text{Ess}}, \{I^w_{\text{Exe}}\}_{w\in W})$ is a skeleton.*

Proof. W is non-empty, for it includes Δ. That R is an equivalence follows as usual from S5 axioms. By definition, the sets of terms of different sort must be disjoint and non-empty. If $tI_{\text{Ess}}T$ then $\Delta \vdash \text{Ess}(t, T)$, so $w \vdash \text{Ess}(t, T)$. But by instantiating over t and T the axiom $\vdash \forall x\forall X(\text{Ess}(x, X) \to \Box\text{Exe}(x, X))$ we get $w \vdash \Box\text{Exe}(t, T)$. By modal axiom (T) we have $w \vdash \text{Exe}(t, T)$ and so $tI^w_{\text{Exe}}T$. □

Since (S, Q, I_{Ess}) is a context, it induces a *lattice of kinds L*.

Definition 9.2. *Let Γ and Δ be as before and $(W, R, S, Q, I_{\text{Ess}}, \{I^w_{\text{Exe}}\}_{w\in W})$ the skeleton as described. Then we define the model M as follows*:

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(1) $J(a) := a$ *for every object constant* $a \in \mathrm{Con}_O$.
(2) $J(F) := F$ *for every attribute constant* $F \in \mathrm{Con}_P$.
(3) $J(K) := (\{t \in Term_O \mid \Delta \vdash K(t)\}, \{T \in Term_P \mid \Delta \vdash K(T)\})$.

Thus as usual, each closed term denotes itself. A natural kind predicate for objects denotes a pair of sets (A, B) such that A contains exactly those object terms t for which it holds, according to theory Δ, that $K(t)$.

Lemma 9.1. *The function J is an interpretation function. In particular, if K is a natural kind predicate, then $J(K) \in L$.*

Proof. We split the proof into two. (i) First we prove that A is an extension and B an intension. We prove that $A = \{t \in Term_O \mid \Delta \vdash K(t)\}$ is an extension, i.e., $ei(A) \subseteq A$. Let $t' \in ei(A)$. By instantiating over t' the axiom of Membership Conditions we have $\Delta \vdash K(t') \leftrightarrow \forall X(K(X) \to \mathrm{Ess}(t', X))$. Assume for reductio the negation of the right-hand side, that $\Delta \vdash \exists X(K(X) \wedge \neg\mathrm{Ess}(t', X))$. Since Δ is Henkin, there is a witness T for the sentence, i.e., $\Delta \vdash K(T) \wedge \neg\mathrm{Ess}(t', T)$.

We now prove that if $\Delta \vdash K(T)$ then $T \in i(A)$. Assume $\Delta \vdash K(T)$ and let $t \in A$. By instantiating over T the axiom of General Essence we get $\Delta \vdash (\forall x(K(x) \to \mathrm{Ess}(x, T)) \leftrightarrow K(T)$ and so by Modus Ponens and instantiating over t we get $\Delta \vdash K(t) \to \mathrm{Ess}(t, T)$. Since $t \in A$ by definition we have $\Delta \vdash K(t)$ and so by Modus Ponens we get $\Delta \vdash \mathrm{Ess}(t, T)$. This proves that if $\Delta \vdash K(T)$ then $T \in i(A)$. But since we already have the antecedent we get that $T \in i(A)$. Now since $t' \in ei(A)$ we have $\Delta \vdash \mathrm{Ess}(t', T)$, which contradicts $\Delta \vdash \neg\mathrm{Ess}(t', T)$.

Thus we have proved that $\Delta \vdash \forall X(K(X) \to \mathrm{Ess}(t', X))$. From this it follows that $\Delta \vdash K(t')$ and therefore that $t' \in A$. So $ei(A) \subseteq A$, which proves that A is an extension. The proof that $B = \{T \in Term_P \mid \Delta \vdash K(T)\}$ is an intension is analogous and makes use of the axiom of General Essence.

(ii) We now prove that $A = e(B)$, from this it follows that $B = i(A)$. If $t \in A$ and $T \in B$ then $\Delta \vdash K(t)$ and $\Delta \vdash K(T)$. By applying the axiom of Membership Conditions $\Delta \vdash \forall x(K(x) \leftrightarrow \forall X(K(X) \to \mathrm{Ess}(x, X)))$ we get $\Delta \vdash \mathrm{Ess}(t, T)$, so $A \subseteq e(B)$. Conversely, let $t \in e(B)$. So if $\Delta \vdash K(T)$ then tIT, i.e., $\Delta \vdash \mathrm{Ess}(t, T)$. We prove that $t \in A$, i.e., that $\Delta \vdash K(t)$. By the axiom of Membership Conditions, we only need to prove that $\Delta \vdash \forall X(K(X) \to \mathrm{Ess}(t, X))$. So assume that $\Delta \vdash \exists X(K(X) \wedge \neg\mathrm{Ess}(t, X))$. Since Δ is Henkin, we have a witness T' such that $\Delta \vdash K(T') \wedge \neg\mathrm{Ess}(t, T')$. But since $t \in e(B)$ we get $\Delta \vdash \mathrm{Ess}(t, T')$, which contradicts the conjunction just obtained. Therefore $\Delta \vdash \forall X(K(X) \to \mathrm{Ess}(t, X))$ and by Membership Conditions $\Delta \vdash K(t)$, i.e., $t \in A$. So $A = e(B)$. □

Note that the proof appeals to the theory Δ being Henkin. Before identifying as congruent co-denoting terms, we need some auxiliary results about quotients:

Definition 9.3. *Let (S, Q, I) be a context and $\approx \subseteq S \cup Q \times S \cup Q$ an equivalence relating objects with objects and attributes with attributes. It is a* congruence *over the context* iff *for every $a, b \in S$ and $F, G \in Q$, if $a \approx b$, $F \approx G$ and aIF then bIG.*

We use angle brackets for equivalence classes, i.e., $\langle a \rangle := \{b \in S \mid a \approx b\}$. If (S, Q, I) is a context and $\approx$ a congruence over it, then $(\mathbf{S}, \mathbf{Q}, \mathbf{I})$ is its *quotient context*, where $\mathbf{S} = S\,|_{\approx}$, $\mathbf{Q} = Q\,|_{\approx}$ and $\langle a \rangle \mathbf{I} \langle F \rangle$ iff for every $b \in \langle a \rangle$ and every $G \in \langle F \rangle$ we have bIG. We denote the lattice of kinds of the quotient context as $\mathbf{L}$.

Proposition 9.2. *Let* $\mathbf{(S, Q, I)}$ *be the quotient context of* (S, Q, I). *Then*:

(1) *If* $a \approx b$, *then* $i(a) = i(b)$. [*Intensional Indiscernibility*]
(2) *If* $F \approx G$, *then* $e(F) = e(G)$. [*Extensional Indiscernibility*]
(3) *If* $K = (A, B) \in L$, $a \approx b$ *and* $a \in A$, *then* $b \in A$. [*Kind Indiscernibility*]
(4) *If* $K = (A, B) \in L$, $F \approx G$ *and* $F \in B$, *then* $G \in B$. [*Kind Indiscernibility*]
(5) *If* $K = (A, B) \in L$, *then* $\boldsymbol{K} = (\{\langle a \rangle \in \boldsymbol{S} \mid a \in A\}, \{\langle F \rangle \in \boldsymbol{Q} \mid F \in B\}) \in \boldsymbol{L}$.

Proof. (1) and (2) follow from the reflexivity of the congruence. (3) and (4) follow from (1) and (2), respectively. For (5), let $K = (A, B) = (\text{ext}_K, \text{int}_K) \in L$. We prove that $(A', B') = (\{\langle a \rangle \mid a \in \text{ext}_K\}, \{\langle F \rangle \mid F \in \text{int}_K\})$ is a kind in $\mathbf{L}$. We first prove that $\mathbf{i}(A') = B'$: $\langle G \rangle \in \mathbf{i}(A')$ iff $\langle a \rangle \mathbf{I} \langle G \rangle$ for every $a \in A$ iff aIG for every $a \in A$ iff $G \in B = \text{int}_K$ iff $\langle G \rangle \in B'$. We now prove that A' is an extension. Let $\langle a \rangle \in \mathbf{ei}(A')$. If $F \in B$ then $\langle F \rangle \in B' = \mathbf{i}(A')$. Therefore $\langle a \rangle \mathbf{I} \langle F \rangle$, so aIF. This shows that $a \in e(B) = A$ and therefore that $\langle a \rangle \in A'$. □

Thus if K is a kind in the original lattice, the pair of sets $\mathbf{K}$ is a kind in the lattice of the quotient context. Its extension is the set of all the equivalence classes of the objects in the extension of K. Let t_i and t_i' be closed terms, we define:

$$t_i \approx t_i' \Leftrightarrow \Delta \vdash t_i = t_i',$$

so $t_i \approx t_i'$ iff $w \vdash t_i = t_i'$, for all $w \in W$. We check that it is a congruence:

Lemma 9.2. $\approx$ *is a congruence over the contexts* (S, Q, I_{Ess}) *and* (S, Q, I^w_{Exe}):

(1) *If* $t \approx t'$, $T \approx T'$ *and* $\Delta \vdash \text{Ess}(t, T)$, *then* $\Delta \vdash \text{Ess}(t', T')$.
(2) *If* $t \approx t'$, $T \approx T'$ *and* $w \vdash \text{Exe}(t, T)$, *then* $w \vdash \text{Exe}(t', T')$.
(3) *If* $t \approx t'$ *and* $\Delta \vdash K(t)$, *then* $\Delta \vdash K(t')$.
(4) *If* $T \approx T'$ *and* $\Delta \vdash K(T)$, *then* $\Delta \vdash K(T')$.

Proof. That $\approx$ is an equivalence follows from the axioms of identity. If $t \approx t'$, $T \approx T'$ and tIT we have $\Delta \vdash t = t'$, $\Delta \vdash T = T'$ and $\Delta \vdash \text{Ess}(t, T)$. By Leibniz Law we have $\Delta \vdash t = t' \to (\text{Ess}(t, T) \to \text{Ess}(t', T))$ and therefore $\Delta \vdash \text{Ess}(t', T)$. Applying Leibniz Law to $T = T'$ with atomic formula $\text{Ess}(t', T)$ we get $\Delta \vdash \text{Ess}(t', T')$. (2) is proven analogously. (3) and (4) follow from the previous proposition. □

The structure $(W, R, \mathbf{S}, \mathbf{Q}, \mathbf{I}_{\text{Ess}}, \{\mathbf{I}^w_{\text{Exe}}\}_{w \in W})$ is the *canonical skeleton of* Δ.

Definition 9.4. *Let* $(W, R, \boldsymbol{S}, \boldsymbol{Q}, \boldsymbol{I}_{\text{Ess}}, \{\boldsymbol{I}^w_{\text{Exe}}\}_{w \in W})$ *be the canonical skeleton. The* canonical model *of* Δ *is obtained by adding the interpretation function* J^*:

(1) $J^*(a) := \langle a \rangle$, *for each object constant* $a \in \text{Con}_O$.
(2) $J^*(F) := \langle F \rangle$, *for each attribute constant* $F \in \text{Con}_P$.
(3) $J^*(K) := (\{\langle t \rangle \mid t \in \text{ext}_{[K]}\}, \{\langle T \rangle \mid T \in \text{int}_{[K]}\})$.

Thus every closed term t_i denotes its equivalence class $[t_i] = \langle t_i \rangle$. From the previous lemmas about quotients it follows that the canonical model is a *model*. What remains is to prove the *Truth Lemma* to check that it is also *canonical*. From this the completeness of **NKE** follows:

Proposition 9.3 (Truth Lemma). *Let* Γ *and* Δ *be as before. Let* $\boldsymbol{M}$ *be the canonical model of* Δ, $w \in W$ *and* φ *a sentence. Then* $\boldsymbol{M}, g, w \models \varphi$ iff $w \vdash \varphi$.

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Proof. The proof is as usual over the complexity of φ. The new cases correspond to the new atomic formulas. But these are immediate. First, $\mathbf{M}, g, w \models \text{Ess}(t, t')$ iff $\langle t \rangle = [t]^* \mathbf{I}[t']^* = \langle t' \rangle$ iff $t I_{\text{Ess}} t'$ iff $\Delta \vdash \text{Ess}(t, t')$ iff $w \vdash \text{Ess}(t, t')$. Second, $\mathbf{M}, g, w \models \text{Exe}(t, t')$ iff $\langle t \rangle = [t]^* \mathbf{I}^w_{\text{Exe}}[t']^* = \langle t' \rangle$ iff $t I^w_{\text{Exe}} t'$ iff $w \vdash \text{Exe}(t, t')$. Third, $\mathbf{M}, g, w \models K(t)$ iff $[t]^* = \langle t \rangle \in \text{ext}_{[K]^*}$ iff $t \in \text{ext}_{[K]}$ iff $\Delta \vdash K(t)$ iff $w \vdash K(t)$ (similarly for the sort P). The cases for connectives use the inductive hypotheses and maximal consistency. The case for the universal quantifier uses also the Henkin property, the Substitution Lemma, the Lemma of Alphabetic Variants and the classical properties of the quantifier. In the step of modal formulas the proof is as usual. But recall that we have that if $\Box\psi$ is not in w, then there is a saturated set w' such that $\{\gamma \mid \Box\gamma \in w\} \cup \{\neg\psi\} \subseteq w'$, and we need to check that w' is indeed in W. For that we use the theorems in Section 3.2: Case $t = t' \in \Delta$ iff $t = t' \in w'$ is proven as usual using (Necessity of Identity) and (Necessity of Distinctness). Case $\text{Ess}(t, T) \in \Delta$ iff $\text{Ess}(t, T) \in w'$ is proven analogously using (Necessary Essentialism) and (Neg. Necessary Essentialism). Lastly, cases of the form $K(t_i) \in \Delta$ iff $K(t_i) \in w'$ are proven using (Kripkean Individual/Attribute Essentialism) and (Neg. Kripkean Individual/Attribute Essentialism). $\Box$

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