*Proceedings of the Royal Society of Edinburgh*, page 1 of 21
DOI:10.1017/prm.2023.112

# The space of commuting elements in a Lie group and maps between classifying spaces

**Daisuke Kishimoto**
Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, Japan (kishimoto@math.kyushu-u.ac.jp)

**Masahiro Takeda**
Institute for Liberal Arts and Sciences, Kyoto University, Kyoto 606-8316, Japan (takeda.masahiro.87u@kyoto-u.ac.jp)

**Mitsunobu Tsutaya**
Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, Japan (tsutaya@math.kyushu-u.ac.jp)

(Received 27 March 2023; accepted 26 September 2023)

Let $\pi$ be a discrete group, and let $G$ be a compact-connected Lie group. Then, there is a map $\Theta\colon \mathrm{Hom}(\pi, G)_0 \to \mathrm{map}_*(B\pi, BG)_0$ between the null components of the spaces of homomorphisms and based maps, which sends a homomorphism to the induced map between classifying spaces. Atiyah and Bott studied this map for $\pi$ a surface group, and showed that it is surjective in rational cohomology. In this paper, we prove that the map $\Theta$ is surjective in rational cohomology for $\pi = \mathbb{Z}^m$ and the classical group $G$ except for $SO(2n)$, and that it is not surjective for $\pi = \mathbb{Z}^m$ with $m \geqslant 3$ and $G = SO(2n)$ with $n \geqslant 4$. As an application, we consider the surjectivity of the map $\Theta$ in rational cohomology for $\pi$ a finitely generated nilpotent group. We also consider the dimension of the cokernel of the map $\Theta$ in rational homotopy groups for $\pi = \mathbb{Z}^m$ and the classical groups $G$ except for $SO(2n)$.

*Keywords:* space of commuting elements; Lie group; classifying space; flat connection; rational cohomology

2020 *Mathematics Subject Classification:* 55R37; 57T10

## 1. Introduction

Given two topological groups $G$ and $H$, there is a natural map

$$\widehat{\Theta}\colon \mathrm{Hom}(G, H) \to \mathrm{map}_*(BG, BH)$$

sending a homomorphism to its induced map between classifying spaces, where $\mathrm{Hom}(G, H)$ and $\mathrm{map}_*(BG, BH)$ denote the spaces of homomorphisms and based maps, respectively. If $G$ and $H$ are discrete, then the map $\widehat{\Theta}$ in $\pi_0$ is a well-known bijection:

$$\mathrm{Hom}(G, H) \cong [BG, BH]_*.$$

However, the map $\widehat{\Theta}$ in $\pi_0$ is not bijective in general. Indeed, if $G = H = U(n)$, then Sullivan [28] constructed a map between classifying spaces, called the unstable Adams operation, which is not in the image of the map $\widehat{\Theta}$ in $\pi_0$, even rationally. Since then, the map $\widehat{\Theta}$ in $\pi_0$ has been intensely studied for both $G$ and $H$ being Lie groups completed at a prime, which led to a new development of algebraic topology and has been producing a variety of applications. See surveys [11, 17, 20] for details. Clearly, the map $\widehat{\Theta}$ is of particular importance not only in $\pi_0$. However, not much is known about higher homotopical structures of the map $\widehat{\Theta}$ such as homotopy groups and (co)homology of dimension $\geqslant 1$.

We describe two interpretations of the map $\widehat{\Theta}$. The first one is from algebraic topology. Stasheff [27] introduced an $A_\infty$-map between topological monoids, which is defined by replacing the equality in the definition of a homomorphism by coherent higher homotopies with respect to the associativity of the multiplications. He also showed that to each $A_\infty$-map, we can assign a map between classifying spaces, and so there is a map

$$\mathcal{A}_\infty(G,H) \to \mathrm{map}_*(BG,BH)$$

where $\mathcal{A}_\infty(G,H)$ denotes the space of $A_\infty$-maps between topological groups $G, H$. It is proved in [14, 29] that this map is a weak homotopy equivalence, and since the map $\widehat{\Theta}$ factors through this map, we can interpret that the map $\widehat{\Theta}$ depicts the difference of homomorphisms, solid objects, and $A_\infty$-maps, soft objects, between topological groups.

The second interpretation is from bundle theory. Let $\pi$ be a finitely generated discrete group, and let $G$ be a compact-connected Lie group. Let $\mathrm{Hom}(\pi,G)_0$ and $\mathrm{map}_*(B\pi,BG)_0$ denote the path components of $\mathrm{Hom}(\pi,G)$ and $\mathrm{map}_*(B\pi,BG)$ containing trivial maps, respectively. In this paper, we study the natural map

$$\Theta\colon \mathrm{Hom}(\pi,G)_0 \to \mathrm{map}_*(B\pi,BG)_0$$

which is the restriction of the map $\widehat{\Theta}$. If $B\pi$ has the homotopy type of a manifold $M$, then $\mathrm{Hom}(\pi,G)_0$ and $\mathrm{map}_*(B\pi,BG)_0$ are identified with the based moduli spaces of flat connections and all connections on the trivial $G$-bundle over $M$, denoted by $\mathrm{Flat}(M,G)_0$ and $\mathcal{C}(M,G)_0$, respectively. Under this identification, the map $\Theta$ can be interpreted as the inclusion:

$$\mathrm{Flat}(M,G)_0 \to \mathcal{C}(M,G)_0.$$

Atiyah and Bott [4] studied the map $\Theta$ for a surface group $\pi$ through the above flat bundle interpretation in the context of gauge theory because flat connections are solutions to the Yang–Mills equation over a Riemann surface. In particular, they used Morse theory to prove that the map $\Theta$ is surjective in rational cohomology whenever $\pi$ is a surface group. However, their proof is so specialized to surface groups that it does not apply to other groups $\pi$. Then, we ask:

Question 1.1. Is the map $\Theta$ surjective in rational cohomology whenever $B\pi$ is of the homotopy type of a manifold?

In this paper, we study the above question in the special case $\pi = \mathbb{Z}^m$. The space $\mathrm{Hom}(\mathbb{Z}^m, G)$ is called the space of commuting elements in $G$ because there is a natural homeomorphism

$$\mathrm{Hom}(\mathbb{Z}^m, G) \cong \{(g_1, \ldots, g_m) \in G^m \mid g_i g_j = g_j g_i \text{ for } 1 \leqslant i, j \leqslant m\}$$

where we will not distinguish these two spaces. Recently, several results on the space of commuting elements in a Lie group have been obtained from a view of algebraic topology [**1**–**3**, **5**–**7**, **12**, **15**, **16**, **21**, **22**, **25**, **26**]. In particular, the first and the second authors gave a minimal generating set of the rational cohomology of $\mathrm{Hom}(\mathbb{Z}^m, G)_0$ when $G$ is the classical group except for $SO(2n)$. Using this generating set, we will prove:

THEOREM 1.2. *If $G$ is the classical group except for $SO(2n)$, then the map*

$$\Theta\colon \mathrm{Hom}(\mathbb{Z}^m, G)_0 \to \mathrm{map}_*(B\mathbb{Z}^m, BG)_0$$

*is surjective in rational cohomology.*

As an application of theorem 1.2, we will prove the following theorem. We refer to [**19**] for the localization of nilpotent groups.

THEOREM 1.3. *Let $\pi$ be a finitely generated nilpotent group, and let $G$ be the classical group except for $U(1)$ and $SO(2n)$. Then, the map*

$$\Theta\colon \mathrm{Hom}(\pi, G)_0 \to \mathrm{map}_*(B\pi, BG)_0$$

*is surjective in rational cohomology if and only if the rationalization $\pi_{(0)}$ is abelian.*

As a corollary, we will obtain:

COROLLARY 1.4. *Let $M$ be a nilmanifold, and let $G$ be the classical group except for $SO(2n)$. Then, the inclusion*

$$\mathrm{Flat}(M, G)_0 \to \mathcal{C}(M, G)_0$$

*is surjective in rational cohomology if and only if $M$ is a torus.*

As a corollary to theorem 1.2, we will show that the map $\Theta$ is surjective in rational cohomology for $G = SO(2n)$ with $n = 2, 3$ (corollary 3.10). On the contrary, as mentioned above, the result of Atiyah and Bott [**4**] implies that the map $\Theta$ is surjective in rational cohomology for $m = 2$ and $G = SO(2n)$ with any $n \geqslant 2$. Then, we may expect that the map $\Theta$ is also surjective in rational cohomology for $m \geqslant 3$ and $n \geqslant 4$. However, the surjectivity breaks as:

THEOREM 1.5. *For $m \geqslant 3$ and $n \geqslant 4$, the map*

$$\Theta\colon \mathrm{Hom}(\mathbb{Z}^m, SO(2n))_0 \to \mathrm{map}_*(B\mathbb{Z}^m, BSO(2n))_0$$

*is not surjective in rational cohomology.*

We will also consider the map $\Theta$ in rational homotopy groups. It is proved in [21] that $\mathrm{Hom}(\mathbb{Z}^m, G)_0$ is rationally hyperbolic, and so the total dimension of its rational homotopy groups is infinite. On the contrary, we will see in § 4 that the rational homotopy group of $\mathrm{map}_*(B\mathbb{Z}^m, BG)_0$ is finite dimensional. Then, the map $\Theta$ for $\pi = \mathbb{Z}^m$ cannot be injective. On the contrary, we can consider the surjectivity of the map $\Theta$ in rational homotopy groups by looking at its cokernel. Baird and Ramras [7] gave a lower bound for the dimension of the cokernel of the map $\Theta$ for $G = \mathrm{GL}_n(\mathbb{C})$ in rational homotopy groups. In particular, for $\pi = \mathbb{Z}^m$, they proved that the dimension of the cokernel of the map

$$\Theta_*\colon \pi_i(\mathrm{Hom}(\mathbb{Z}^m, \mathrm{GL}_n(\mathbb{C}))_0) \otimes \mathbb{Q} \to \pi_i(\mathrm{map}_*(B\mathbb{Z}^m, B\mathrm{GL}_n(\mathbb{C}))_0) \otimes \mathbb{Q}$$

is bounded below by $\sum_{i<k\leqslant n} \binom{m}{2i-k}$ whenever $n \geqslant (m+i)/2$. By using theorem 1.2, we can improve this result as follows.

THEOREM 1.6. *Let $c_i(m, G)$ be the dimension of the cokernel of the map*

$$\Theta_*\colon \pi_i(\mathrm{Hom}(\mathbb{Z}^m, G)_0) \otimes \mathbb{Q} \to \pi_i(\mathrm{map}_*(B\mathbb{Z}^m, BG)_0) \otimes \mathbb{Q}.$$

(1) *For $G = U(n), SU(n)$, we have*

$$c_i(m, G) = \sum_{i<k\leqslant n} \binom{m}{2i-k}.$$

(2) *For $G = Sp(n), SO(2n+1)$, we have*

$$c_i(m, G) \geqslant \sum_{i/3<k\leqslant n} \binom{m}{4i-k}$$

*where the equality holds for $i \leqslant 2n+3$.*

Remarks on theorem 1.6 are in order. By [16], $\pi_1(\mathrm{Hom}(\mathbb{Z}^m, G)_0)$ is abelian, and it is easy to see that $\pi_1(\mathrm{map}_*(B\mathbb{Z}^m, BG))$ is abelian too. Then, $\pi_1 \otimes \mathbb{Q}$ in theorem 1.6 makes sense. By [8], the $G = U(n)$ case is equivalent to the $G = \mathrm{GL}_n(\mathbb{C})$ case, and so we can see that the lower bound of Baird and Ramras [7] for $\pi = \mathbb{Z}^m$ mentioned above is attained by theorem 1.6.

## 2. The map $\Phi$

Hereafter, let $G$ be a compact-connected Lie group with maximal torus $T$, and let $W$ denote the Weyl group of $G$. We define a map

$$\Phi\colon G/T \times_W T^m \to \mathrm{Hom}(\mathbb{Z}^m, G)_0$$

by $\Phi(gT, (g_1, \ldots, g_m)) = (gg_1g^{-1}, \ldots, gg_mg^{-1})$ for $g \in G$ and $g_1, \ldots, g_m \in T$, where $T^m$ denotes the direct product of $m$ copies of $T$, instead of a torus of dimension $m$. In this section, we will define maps involving the map $\Phi$ and show their properties.

https://doi.org/10.1017/prm.2023.112 Published online by Cambridge University Press

First, we recall the following result of Baird [5]. It is well known that there is a natural isomorphism:

$$H^*(G/T \times_W T^m; \mathbb{Q}) \cong H^*(G/T \times T^m; \mathbb{Q})^W$$

and so we will not distinguish them. Baird [5] proved:

THEOREM 2.1. *The map*

$$\Phi^*\colon H^*(\mathrm{Hom}(\mathbb{Z}^m, G)_0; \mathbb{Q}) \to H^*(G/T \times T^m; \mathbb{Q})^W$$

*is an isomorphism.*

By using theorem 2.1, the first and the second authors [21] gave a minimal generating set of the rational cohomology of $\mathrm{Hom}(\mathbb{Z}^m, G)_0$ when $G$ is the classical group except for $SO(2n)$, which we recall in the next section.

In order to define maps involving the map $\Phi$, we need the functoriality of classifying spaces. Then, we employ the Milnor construction [24] as a model for the classifying space. Let

$$EG = \lim_{n\to\infty} \underbrace{G * \cdots * G}_{n}$$

where $X * Y$ denotes the join of spaces $X$ and $Y$. Following Milnor [24], we denote a point of $EG$ by

$$t_1 g_1 \oplus t_2 g_2 \oplus \cdots$$

such that $t_i \geqslant 0$, $\sum_{n\geqslant 1} t_n = 1$ with only finitely many $t_i$ being non-zero, and $s_1 g_1 \oplus s_2 g_2 \oplus \cdots = t_1 h_1 \oplus t_2 h_2 \oplus \cdots$ if $s_k = t_k = 0$ ($g_k \neq h_k$, possibly) and for $i \neq k$, $s_i = t_i$ and $g_i = h_i$, where $e$ denotes the identity element of $G$. Then, $G$ acts freely on $EG$ by

$$(t_1 g_1 \oplus t_2 g_2 \oplus \cdots) \cdot g = t_1 g_1 g \oplus t_2 g_2 g \oplus \cdots .$$

We define the classifying space of $G$ by

$$BG = EG/G.$$

Note that the inclusion $ET \to EG$ induces a map $\iota\colon BT \to EG/T$ which is a homotopy equivalence because both $ET$ and $EG$ are contractible. We record a simple fact which follows immediately from the definition of the Milnor construction.

LEMMA 2.2. *The natural map $BT \to BG$ factors as the composite:*

$$BT \xrightarrow{\iota} EG/T \to EG/G = BG.$$

https://doi.org/10.1017/prm.2023.112 Published online by Cambridge University Press

Now, we define a map

$$\phi\colon G/T\times BT\to BG,\quad (gT,[t_1g_1\oplus t_2g_2\oplus\cdots])\mapsto [t_1gg_1\oplus t_2gg_2\oplus\cdots]$$

for $g\in G$ and $[t_1g_1\oplus t_2g_2\oplus\cdots]\in BT$. Since $T$ is abelian, we have

$$[t_1ghg_1\oplus t_2ghg_2\oplus\cdots]=[t_1gg_1h\oplus t_2gg_2h\oplus\cdots]=[t_1gg_1\oplus t_2gg_2\oplus\cdots]$$

for $h\in T$, implying that the map $\phi$ is well-defined. We also define

$$\bar{\alpha}\colon G/T\to EG/T,\quad gT\mapsto [1g\oplus 0e\oplus 0e\oplus\cdots].$$

Let $\alpha$ denote the composite $G/T\xrightarrow{\bar{\alpha}}EG/T\xrightarrow{\iota^{-1}}BT$. Then, there is a homotopy fibration $G/T\xrightarrow{\alpha}BT\to BG$.

Lemma 2.3. *There is a map $\widehat{\phi}\colon G/T\times BT\to BT$ satisfying the homotopy commutative diagram:*

$$\begin{tikzcd}
G/T\vee BT \arrow[r] \arrow[d, "\alpha\vee 1"] & G/T\times BT \arrow[r, equal] \arrow[d, "\widehat{\phi}"] & G/T\times BT \arrow[d, "\phi"] \\
BT \arrow[r, equal] & BT \arrow[r] & BG.
\end{tikzcd}$$

*Proof.* Define a map:

$$\bar{\phi}\colon G/T\times BT\to EG/T,\quad (gT,[t_1g_1\oplus t_2g_2\oplus\cdots])\mapsto [t_1gg_1\oplus t_2gg_2\oplus\cdots].$$

Quite similarly to the map $\phi$, we can see that the map $\bar{\phi}$ is well-defined. Let $\widehat{\phi}$ denote the composite:

$$G/T\times BT\xrightarrow{\bar{\phi}}EG/T\xrightarrow{\iota^{-1}}BT.$$

Then, by lemma 2.2, the right square of the diagram in the statement is homotopy commutative. We also have

$$\bar{\phi}(gT,[1e\oplus 0e\oplus 0e\oplus\cdots])=[1g\oplus 0e\oplus 0e\oplus\cdots]=\bar{\alpha}(gT)$$

and

$$\bar{\phi}(eT,[t_1g_1\oplus t_2g_2\oplus\cdots])=[t_1g_1\oplus t_2g_2\oplus\cdots]=\iota([t_1g_1\oplus t_2g_2\oplus\cdots])$$

for $[t_1g_1\oplus t_2g_2\oplus\cdots]\in BT$, where $[1e\oplus 0e\oplus 0e\oplus\cdots]$ is the basepoint of $BT$. Then, the left square is homotopy commutative too, finishing the proof. $\square$

We may think of the map $\widehat{\phi}$ as a higher version of the map defined by conjugation in [**10**]. We define a map

$$\widehat{\Phi}\colon G/T\times_W \mathrm{map}_*(B\mathbb{Z}^m,BT)_0\to \mathrm{map}_*(B\mathbb{Z}^m,BG)_0$$

by $\widehat{\Phi}(gT,f)(x)=\phi(gT,f(x))$ for $g\in G,\ f\in\mathrm{map}_*(B\mathbb{Z}^m,BT)_0$ and $x\in B\mathbb{Z}^m$. Since there is a natural homomorphism $\mathrm{Hom}(\mathbb{Z}^m,T)_0\cong T^m$, we will not distinguish them.

https://doi.org/10.1017/prm.2023.112 Published online by Cambridge University Press

LEMMA 2.4. *There is a commutative diagram:*

$$\begin{tikzcd} G/T \times_W \mathrm{Hom}(\mathbb{Z}^m, T)_0 \arrow[r, "\Phi"] \arrow[d, "1\times\Theta"'] & \mathrm{Hom}(\mathbb{Z}^m, G)_0 \arrow[d, "\Theta"] \\ G/T \times_W \mathrm{map}_*(B\mathbb{Z}^m, BT)_0 \arrow[r, "\widehat{\Phi}"] & \mathrm{map}_*(B\mathbb{Z}^m, BG)_0. \end{tikzcd}$$

*Proof.* By definition, we have

$$\Theta(f)([t_1g_1 \oplus t_2g_2 \oplus \cdots]) = [t_1f(g_1) \oplus t_2f(g_2) \oplus \cdots]$$

for $f \in \mathrm{Hom}(\mathbb{Z}^m, T)_0$ and $[t_1g_1 \oplus t_2g_2 \oplus \cdots] \in B\mathbb{Z}^m$. Then, we get

$$\begin{aligned} &\widehat{\Phi} \circ (1 \times \Theta)(gT, f)([t_1g_1 \oplus t_2g_2 \oplus \cdots]) \\ &\quad = [t_1gf(g_1)g^{-1} \oplus t_2gf(g_2)g^{-1} \oplus \cdots] \\ &\quad = \Theta \circ \Phi(gT, f)([t_1g_1 \oplus t_2g_2 \oplus \cdots]) \end{aligned}$$

for $g \in G$, $f \in \mathrm{Hom}(\mathbb{Z}^m, T)_0$ and $[t_1g_1 \oplus t_2g_2 \oplus \cdots] \in B\mathbb{Z}^m$. Thus, the proof is finished. □

Next, we consider the evaluation map:

$$\omega\colon \mathrm{map}_*(X, Y)_0 \times X \to Y, \quad (f, x) \mapsto f(x).$$

Note that the map $\phi\colon G/T \times BT \to BG$ factors through $G/T \times_W BT$. We denote the map $G/T \times_W BT \to BG$ by the same symbol $\phi$.

LEMMA 2.5. *There is a commutative diagram:*

$$\begin{tikzcd} G/T \times_W \mathrm{map}_*(B\mathbb{Z}^m, BT)_0 \times B\mathbb{Z}^m \arrow[r, "\widehat{\Phi}\times 1"] \arrow[d, "1\times\omega"'] & \mathrm{map}_*(B\mathbb{Z}^m, BG)_0 \times B\mathbb{Z}^m \arrow[d, "\omega"] \\ G/T \times_W BT \arrow[r, "\phi"] & BG. \end{tikzcd}$$

*Proof.* For $g \in G$, $f \in \mathrm{map}_*(B\mathbb{Z}^m, BT)_0$ and $x \in B\mathbb{Z}^m$, we have

$$\omega \circ (\widehat{\Phi} \times 1)(gT, f, x) = \phi(gT, f(x)) = \phi(gT, \omega(f, x)) = \phi \circ (1 \times \omega)(gT, f, x).$$

Thus, the statement is proved. □

## 3. Rational cohomology

In this section, we will prove theorems 1.2 and 1.5, and we will apply theorem 1.2 to prove theorem 1.3. To prove theorem 1.2, we will employ the generating set of the rational cohomology of $\mathrm{Hom}(\mathbb{Z}^m, G)_0$ given in [**21**], and to prove theorem 1.5, we will consider a specific element of $H^*(\mathrm{Hom}(\mathbb{Z}^m, SO(2n))_0) \cong H^*(SO(2n)/T \times T^m)^W$.

https://doi.org/10.1017/prm.2023.112 Published online by Cambridge University Press

### 3.1. Cohomology generators

Hereafter, the coefficients of (co)homology will be always in $\mathbb{Q}$, and we will suppose that $G$ is of rank $n$, unless otherwise specified. First, we set notation on cohomology. Since $G$ is of rank $n$, the cohomology of $BT$ is given by

$$H^*(BT) = \mathbb{Q}[x_1, \dots, x_n], \quad |x_i| = 2.$$

We also have that the cohomology of $T^m$ is given by

$$H^*(T^m) = \Lambda(y_1^1, \dots, y_n^1, \dots, y_1^m, \dots, y_n^m), \quad |y_i^j| = 1$$

such that $y_i^k = \pi_k^*(\sigma(\overset{k}{x_i}))$, where $\pi_k \colon B\mathbb{Z}^m \to B\mathbb{Z}$ is the $k$-th projection and $\sigma$ denotes the cohomology suspension. Let $[m] = \{1, 2, \dots, m\}$. For $I = \{i_1 < \cdots < i_k\} \subset [m]$, we set:

$$y_i^I = y_i^{i_1} \dots y_i^{i_k}.$$

It is well known that the map $\alpha \colon G/T \to BT$ induces an isomorphism:

$$H^*(G/T) \cong H^*(BT)/(\widetilde{H}^*(BT)^W).$$

We denote $\alpha^*(x_i)$ by the same symbol $x_i$, and so $H^*(G/T)$ is generated by $x_1, \dots, x_n$.

Now, we recall the minimal generating set of the rational cohomology of $\mathrm{Hom}(\mathbb{Z}^m, G)_0$ given in [**21**]. For $d \geqslant 1$ and $I \subset [m]$, we define

$$z(d, I) = x_1^{d-1} y_1^I + \cdots + x_n^{d-1} y_n^I \in H^*(G/T \times T^m)$$

and let

$$\mathcal{S}(m, U(n)) = \{z(d, I) \mid d \geqslant 1, \emptyset \neq I \subset [m], d + |I| - 1 \leqslant n\}$$

where we have $|z(d, I)| = 2d + |I| - 2$. We also let:

$$\mathcal{S}(m, SU(n)) = \{z(d, I) \in \mathcal{S}(m, U(n)) \mid d \geqslant 2 \text{ or } |I| \geqslant 2\}$$

where $x_1 + \cdots + x_n = 0$ and $y_1^i + \cdots + y_n^i = 0$ for $i = 1, \dots, m$. Since $W$ is the symmetric group on $[n]$ for $G = U(n), SU(n)$ such that for $\sigma \in W$, $\sigma(x_i) = x_{\sigma(i)}$ and $\sigma(y_i^j) = y_{\sigma(i)}^j$, we have

$$\mathcal{S}(m, G) \subset H^*(G/T \times T^m)^W.$$

For an integer $k$, let $\epsilon(k) = 0$ for $k$ even and $\epsilon(k) = 1$ for $k$ odd. We define

$$w(d, I) = x_1^{2d+\epsilon(|I|)-2} y_1^I + \cdots + x_n^{2d+\epsilon(|I|)-2} y_n^I \in H^*(G/T \times T^m)$$

and let

$$\mathcal{S}(m, Sp(n)) = \{w(d, I) \mid d \geqslant 1, \emptyset \neq I \subset [m], 2d + |I| + \epsilon(|I|) - 2 \leqslant 2n\}$$

where we have $|w(d,I)| = 4d + |I| + 2\epsilon(|I|) - 4$. We also let

$$\mathcal{S}(m, SO(2n+1)) = \mathcal{S}(m, Sp(n)).$$

Since $W$ is the signed symmetric group on $[n]$ for $G = Sp(n), SO(2n+1)$ such that for $\sigma \in W$, $(\pm\sigma)(x_i) = \pm x_{\sigma(i)}$ and $(\pm\sigma)(y_i^j) = \pm y_{\sigma(i)}^j$, we have

$$\mathcal{S}(m, G) \subset H^*(G/T \times T^m)^W.$$

The following theorem is proved in [**21**].

Theorem 3.1. *If $G$ is the classical group except for $SO(2n)$, $(\Phi^*)^{-1}(\mathcal{S}(m,G))$ is a minimal generating set of the rational cohomology of* $\mathrm{Hom}(\mathbb{Z}^m, G)_0$.

### 3.2. Proof of theorem 1.2

First, we consider the map $\widehat{\phi}\colon G/T \times BT \to BT$ of lemma 2.3 in cohomology.

Lemma 3.2. *For each $x_i \in H^*(BT)$, we have*

$$\widehat{\phi}^*(x_i) = x_i \times 1 + 1 \times x_i.$$

*Proof.* The statement immediately follows from the left square of the homotopy commutative diagram in lemma 2.3. $\square$

Next, we consider the map $\Theta\colon \mathrm{Hom}(\mathbb{Z}^m, T)_0 \to \mathrm{map}_*(B\mathbb{Z}^m, BT)_0$.

Lemma 3.3. *The map $\Theta\colon \mathrm{Hom}(\mathbb{Z}, T)_0 \to \mathrm{map}_*(B\mathbb{Z}, BT)_0$ is a homotopy equivalence.*

*Proof.* For a topological group $K$ with a non-degenerate unit, there is a homomorphism $(K * K)/K \cong \widetilde{\Sigma} K$ such that the composite

$$\widetilde{\Sigma} K \simeq (K * K)/K \to BK$$

is identified with the adjoint of the natural homotopy equivalence $K \simeq \Omega BK$, where $\widetilde{\Sigma}$ denotes the unreduced suspension. By definition, $\widetilde{\Sigma}\mathbb{Z}$ is homotopy equivalent to a wedge of infinitely many copies of $S^1$, and the map $\widetilde{\Sigma}\mathbb{Z} \to B\mathbb{Z}$ is identified with the fold map onto $S^1$. Thus, the composite $\widetilde{\Sigma}\{0,1\} \to \widetilde{\Sigma}\mathbb{Z} \to B\mathbb{Z}$ is a homotopy equivalence. Note that for any homomorphism $f\colon \mathbb{Z} \to T$, there is a commutative diagram:

$$\begin{tikzcd}
\widetilde{\Sigma}\{0,1\} \arrow[r] \arrow[d] & \widetilde{\Sigma}\mathbb{Z} \arrow[r, "\widetilde{\Sigma} f"] \arrow[d] & \widetilde{\Sigma} T \arrow[d] \\
B\mathbb{Z} \arrow[r, equal] & B\mathbb{Z} \arrow[r, "Bf"] & BT.
\end{tikzcd}$$

https://doi.org/10.1017/prm.2023.112 Published online by Cambridge University Press

Thus, since $\mathrm{Hom}(\mathbb{Z}, T)_0 = \mathrm{map}_*(\{0,1\}, T)_0$, we get a commutative diagram:

$$\begin{tikzcd}
\mathrm{map}_*(\{0,1\},T)_0 \arrow[r, "\widetilde{\Sigma}"] \arrow[d, Rightarrow] & \mathrm{map}_*(\widetilde{\Sigma}\{0,1\},\widetilde{\Sigma}T)_* \arrow[r] & \mathrm{map}_*(\widetilde{\Sigma}\{0,1\},BT)_0 \\
\mathrm{Hom}(\mathbb{Z},T)_0 \arrow[rr, "\Theta"] & & \mathrm{map}_*(B\mathbb{Z},BT)_0. \arrow[u, "\simeq"']
\end{tikzcd}$$

Clearly, the composite of the top maps is identified with the homotopy equivalence $\mathrm{map}_*(\{0,1\}, \Omega BT)_0 \cong \mathrm{map}_*(\Sigma\{0,1\}, BT)_0$. Then, the bottom map is a homotopy equivalence too, completing the proof. □

Lemma 3.4. *The map* $\Theta\colon \mathrm{Hom}(\mathbb{Z}^m, T)_0 \to \mathrm{map}_*(B\mathbb{Z}^m, BT)_0$ *is a homotopy equivalence.*

*Proof.* Let $F_m$ be the free group of rank $m$. Clearly, we have

$$\mathrm{Hom}(F_m, T)_0 \cong (\mathrm{Hom}(\mathbb{Z}, T)_0)^m.$$

Since $BF_m$ is homotopy equivalent to a wedge of $m$ copies of $S^1$, we also have

$$\mathrm{map}_*(BF_m, BT)_0 \simeq (\mathrm{map}_*(B\mathbb{Z}, BT)_0)^m.$$

It is easy to see that through these equivalences, the map $\Theta\colon \mathrm{Hom}(F_m, T)_0 \to \mathrm{map}_*(BF_m, BT)_0$ is identified with the product of $m$ copies of the map $\Theta\colon \mathrm{Hom}(\mathbb{Z}, T)_0 \to \mathrm{map}_*(B\mathbb{Z}, BT)_0$. Thus, by lemma 3.3, the map $\Theta\colon \mathrm{Hom}(F_m, T)_0 \to \mathrm{map}_*(BF_m, BT)_0$ is a homotopy equivalence. Now, we consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{Z}^m,T)_0 \arrow[r, "\Theta"] \arrow[d] & \mathrm{map}_*(B\mathbb{Z}^m,BT)_0 \arrow[d] \\
\mathrm{Hom}(F_m,T)_0 \arrow[r, "\Theta"] & \mathrm{map}_*(BF_m,BT)_0
\end{tikzcd}$$

induced from the abelianization $F_m \to \mathbb{Z}^m$. Since $T$ is abelian, the left map is a homomorphism. Since the cofibre of the map $BF_m \to B\mathbb{Z}^m$ is simply-connected, the right map is a homotopy equivalence. Thus, the top map is a homotopy equivalence too, completing the proof. □

We consider the evaluation map $\omega\colon \mathrm{map}_*(B\mathbb{Z}^m, BT)_0 \times B\mathbb{Z}^m \to BT$ in cohomology. Since $B\mathbb{Z}^m$ is homotopy equivalent to the $m$-dimensional torus, we have

$$H^*(B\mathbb{Z}^m) = \Lambda(t_1, \dots, t_m), \quad |t_i| = 1.$$

For $I = \{i_1 < \cdots < i_k\} \subset [m]$, let:

$$t_I = t_{i_1} \cdots t_{i_k}.$$

https://doi.org/10.1017/prm.2023.112 Published online by Cambridge University Press

LEMMA 3.5. *For each $x_i \in H^*(BT)$, we have*

$$(\omega \circ (\Theta \times 1))^*(x_i) = y_i^1 \times t_1 + \cdots + y_i^m \times t_m.$$

*Proof.* For the evaluation map $\omega \colon \mathrm{map}_*(B\mathbb{Z}, BT)_0 \times B\mathbb{Z} \to BT$, we have

$$\omega^*(x_1) = y_1^1 \times t_1$$

as in [23], where we identify $\mathrm{map}_*(B\mathbb{Z}, BT)_0$ with $T$. By lemma 3.4, we may assume $\Theta^*(y_1^1) = y_1^1$. Let $\iota_i \colon B\mathbb{Z} \to B\mathbb{Z}^m$ and $\pi_i \colon B\mathbb{Z}^m \to B\mathbb{Z}$ denote the $i$-th inclusion and the $i$-th projection, respectively. Since $\omega \circ (\pi_i^* \times \iota_j)$ is trivial for $i \neq j$, $\omega^*(x_k)$ is a linear combination of $\pi_k^*(y_1^1) \times t_1, \ldots, \pi_k^*(y_1^m) \times t_m$. There is a commutative diagram:

$$\begin{tikzcd}
\mathrm{map}_*(B\mathbb{Z}, BT)_0 \times B\mathbb{Z} \arrow[r, "\omega"] \arrow[d, "\pi_i^* \times \iota_i"'] & BT \arrow[d, Rightarrow, no head] \\
\mathrm{map}_*(B\mathbb{Z}^m, BT)_0 \times B\mathbb{Z}^m \arrow[r, "\omega"] & BT.
\end{tikzcd}$$

Then, we get:

$$\begin{aligned}
(\Theta \times 1)^* \circ \omega^*(x_k) &= (\Theta \times 1)^*(\pi_k^*(y_1^1) \times t_1 + \cdots + \pi_k^*(y_1^m) \times t_m) \\
&= y_k^1 \times t_1 + \cdots + y_k^m \times t_m.
\end{aligned}$$

Thus, the proof is finished. □

Next, we consider the evaluation map $\omega \colon \mathrm{map}_*(B\mathbb{Z}^m, BG)_0 \times B\mathbb{Z}^m \to BG$ in cohomology. Recall that the rational cohomology of $BG$ is given by

$$H^*(BG) = \mathbb{Q}[z_1, \ldots, z_n].$$

We choose generators $z_1, \ldots, z_n$ as

$$j^*(z_i) = \begin{cases} x_1^i + \cdots + x_n^i & G = U(n) \\ x_1^{2i} + \cdots + x_n^{2i} & G = Sp(n), SO(2n+1) \end{cases}$$

and set $H^*(BSU(n)) = H^*(BU(n))/(z_1)$, where $j \colon BT \to BG$ denotes the natural map. For $i = 1, \ldots, n$ and $\emptyset \neq I \subset [m]$, we define $z_{i,I} \in H^*(\mathrm{map}_*(B\mathbb{Z}^m, BG)_0)$ by

$$\omega^*(z_i) = \sum_{\emptyset \neq I \subset [m]} z_{i,I} \times t_I$$

where $z_{i,I} = 1$ for $|z_i| = |I|$ and $z_{i,I} = 0$ for $|z_i| < |I|$.

PROPOSITION 3.6. *The rational cohomology of $\mathrm{map}_*(B\mathbb{Z}^m, BG)_0$ is a free commutative-graded algebra generated by*

$$\mathcal{S} = \{z_{i,I} \mid 1 \leqslant i \leqslant n,\, \emptyset \neq I \subset [m],\, |z_i| > |I|\}.$$

*Proof.* Since the rationalization of $BG$ is homotopy equivalent to a product of Eilenberg–MacLane spaces, so is the rationalization of $\mathrm{map}_*(B\mathbb{Z}^m, BG)_0$. Then, the cohomology of $\mathrm{map}_*(B\mathbb{Z}^m, BG)_0$ is a free commutative algebra. The rest can be proved quite similarly to [4, Proposition 2.20]. $\square$

We compute $\Theta^*(z_{i,I})$ for the classical group $G$ except for $SO(2n)$.

Proposition 3.7. *For $i = 1, \dots, n$ and $\emptyset \neq I \subset [m]$, if $|z_i| > |I|$, then*

$$\Phi^* \circ \Theta^*(z_{i,I}) = \begin{cases} \frac{i!}{(i-|I|)!} z(i - |I| + 1, I) & G = U(n), SU(n) \\ \frac{(2i)!}{(2i-|I|)!} w(i - \frac{|I|+\epsilon(|I|)}{2} + 1, I) & G = Sp(n), SO(2n+1). \end{cases}$$

*Proof.* First, we prove the $G = U(n)$ case. By lemmas 2.3 and 3.2, we have

$$\phi^*(z_i) = \widehat{\phi}^*(j^*(z_i)) = \widehat{\phi}^*(x_1^i + \cdots + x_n^i) = \sum_{k=1}^n (x_k \times 1 + 1 \times x_k)^i.$$

By lemmas 2.4 and 2.5, there is a homotopy commutative diagram:

$$\begin{tikzcd} G/T \times T^m \times B\mathbb{Z}^m \arrow[r, "1\times\Theta\times 1"] \arrow[d, "\Phi\times 1"] & G/T \times \mathrm{map}_*(B\mathbb{Z}^m, BT)_0 \times B\mathbb{Z}^m \arrow[r, "1\times\omega"] \arrow[d, "\widehat{\Phi}\times 1"] & G/T \times BT \arrow[d, "\phi"] \\ \mathrm{Hom}(\mathbb{Z}^m, G) \times B\mathbb{Z}^m \arrow[r, "\Theta\times 1"] & \mathrm{map}_*(B\mathbb{Z}^m, BT)_0 \times B\mathbb{Z}^m \arrow[r, "\omega"] & BG. \end{tikzcd}$$

Then, by lemma 3.5, we get

$$\begin{aligned} &(\Phi \times 1)^* \circ (\Theta \times 1)^* \circ \omega^*(z_i) \\ &= (1 \times \Theta \times 1)^* \circ (\widehat{\Phi} \times 1)^* \circ \omega^*(z_i) \\ &= (1 \times \Theta \times 1)^* \circ (1 \times \omega)^* \circ \phi^*(z_i) \\ &= (1 \times \Theta \times 1)^* \circ (1 \times \omega)^* \left( \sum_{k=1}^n (x_k \times 1 + 1 \times x_k)^i \right) \\ &= \sum_{k=1}^n (x_k \times 1 + y_k^1 \times t_1 + \cdots + y_k^m \times t_m)^i \\ &= \sum_{\emptyset \neq I \subset [m]} \frac{i!}{(i-|I|)!} \left( \sum_{k=1}^n x_k^{i-|I|} \times y_k^I \right) \times t_I \\ &= \sum_{\emptyset \neq I \subset [m]} \frac{i!}{(i-|I|)!} \Phi^*(z(i - |I| + 1, I)) \times t_I. \end{aligned}$$

Thus, the $G = U(n)$ case is proved. The $G = SU(n)$ case follows immediately from the $G = U(n)$ case, and the $G = Sp(n), SO(2n+1)$ case can be proved verbatim. $\square$

https://doi.org/10.1017/prm.2023.112 Published online by Cambridge University Press

Now, we are ready to prove theorem 1.2.

*Proof of theorem* 1.2. Combine theorem 3.1 and proposition 3.7. □

### 3.3. Proof of theorem 1.3

We show a property of the rational cohomology of a nilpotent group that we are going to use. We refer to [**19**] for the localization of nilpotent groups. For a finitely generated group $\pi$, let $\overline{\mathrm{ab}}\colon \pi \to \mathbb{Z}^m$ denote the composite of the abelianization $\pi \to \pi^{\mathrm{ab}}$ and the projection $\pi^{\mathrm{ab}} \to \pi^{\mathrm{ab}}/\mathrm{Tor} \cong \mathbb{Z}^m$, where Tor is the torsion part of $\pi^{\mathrm{ab}}$.

Lemma 3.8. *Let $\pi$ be a finitely generated nilpotent group. Then, the rationalization $\pi_{(0)}$ is abelian if and only if the map*

$$\overline{\mathrm{ab}}^*\colon H^2(B\mathbb{Z}^m) \to H^2(B\pi)$$

*is injective.*

*Proof.* By definition, the rationalization of $B\pi$ is rationally homotopy equivalent to an iterated principal $S^1$-bundles. Then, as in [**18**], the minimal model of $B\pi$ is given by $(\Lambda(x_1, \ldots, x_n), d)$ for $|x_i| = 1$ such that

$$\mathrm{d}x_1 = \cdots = \mathrm{d}x_m = 0, \quad \mathrm{d}x_k = \sum_{i,j<k} \alpha_{i,j} x_i x_j \neq 0 \quad (k > m).$$

Moreover, the minimal model of $B\mathbb{Z}^m$ is given by $(\Lambda(x_1, \ldots, x_m), d = 0)$ such that the map $\overline{\mathrm{ab}}\colon B\pi \to B\mathbb{Z}^m$ induces the inclusion $(\Lambda(x_1, \ldots, x_m), d = 0) \to (\Lambda(x_1, \ldots, x_n), d)$. Observe that $\pi_{(0)}$ is abelian if and only if the map $\overline{\mathrm{ab}}\colon B\pi \to B\mathbb{Z}^m$ is a rational homotopy equivalence. Then, $\pi_{(0)}$ is abelian if and only if $m = n$, which is equivalent to the map $\overline{\mathrm{ab}}^*\colon H^2(B\mathbb{Z}^m) \to H^2(B\pi)$ is injective. □

Now, we are ready to prove theorem 1.3.

*Proof of theorem* 1.3. By the naturality of the map $\Theta$, there is a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{Z}^m, G)_0 \arrow[r, "\Theta"] \arrow[d, "\overline{\mathrm{ab}}^*"'] & \mathrm{map}_*(B\mathbb{Z}^m, BG)_0 \arrow[d, "\overline{\mathrm{ab}}^*"] \\
\mathrm{Hom}(\pi, G)_0 \arrow[r, "\Theta"] & \mathrm{map}_*(B\pi, BG)_0.
\end{tikzcd}$$

Bergeron and Silberman [**9**] proved that the left map is a homotopy equivalence. Since the rationalization of $BG$ is a product of Eilenberg–MacLane spaces, there is

https://doi.org/10.1017/prm.2023.112 Published online by Cambridge University Press

a rational homotopy equivalence

$$\mathrm{map}_*(X, BG)_0 \simeq_{(0)} \prod_{n-i\geqslant 1}^{\infty} K(H^i(X)\otimes \pi_n(BG), n-i) \tag{3.1}$$

for any connected CW complex $X$, which is natural with respect to $X$ and $G$. In particular, since $\pi_4(BG)\cong\mathbb{Z}$, there is a monomorphism $\iota\colon H_2(X)\to QH^2(\mathrm{map}_*(X, BG)_0)$ which is natural with respect to $X$, where $QA$ denotes the module of indecomposables of an augmented algebra $A$. Then, there is a commutative diagram:

$$\begin{tikzcd}
H_2(B\pi) \arrow[r, "\iota"] \arrow[d, "\overline{\mathrm{ab}}_*"] & QH^2(\mathrm{map}_*(B\pi, BG)) \arrow[r, "\Theta^*"] \arrow[d, "(\overline{\mathrm{ab}}^*)^*"] & QH^2(\mathrm{Hom}(B\pi, G)_0) \arrow[d, "(\overline{\mathrm{ab}}^*)^*", "\cong"'] \\
H_2(B\mathbb{Z}^m) \arrow[r, "\iota"] & QH^2(\mathrm{map}_*(B\mathbb{Z}^m, BG)) \arrow[r, "\Theta^*"] & QH^2(\mathrm{Hom}(B\mathbb{Z}^m, G)_0).
\end{tikzcd}$$

By theorem 3.1 and propositions 3.6 and 3.7, the composite of the bottom maps is an isomorphism. Thus, by lemma 3.8, the statement is proved. □

*Proof of corollary* 1.4. It is well known that a nilmanifold $M$ is homotopy equivalent to the classifying space of a finitely generated torsion-free nilpotent group. Thus, by theorem 1.3, the proof is finished. □

### 3.4. Proof of theorem 1.5

Before we begin the proof of theorem 1.5, we consider the case of $SO(2n)$ for $n = 2, 3$. We need the following lemma.

LEMMA 3.9. *Let $G, H$ be compact-connected Lie groups. If there is a covering $G\to H$, then there is a commutative diagram:*

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{Z}^m, BG)_0 \arrow[r, "\Theta"] \arrow[d] & \mathrm{map}_*(B\mathbb{Z}^m, BG)_0 \arrow[d] \\
\mathrm{Hom}(\mathbb{Z}^m, BH)_0 \arrow[r, "\Theta"] & \mathrm{map}_*(B\mathbb{Z}^m, BH)_0,
\end{tikzcd}$$

*where the vertical maps are isomorphisms in rational cohomology and rational homotopy groups.*

*Proof.* Let $K$ be the fibre of the covering $G\to H$. Then, $K$ is a finite subgroup of $G$ contained in the centre. In particular, the map $BG\to BH$ is a rational homotopy equivalence, implying that the right map is a rational homotopy equivalence. As is shown in [**15**], the left map is a covering map with fibre $K^m$, so it is an isomorphism in rational homotopy groups because the fundamental groups of $\mathrm{Hom}(\mathbb{Z}^m, G)_0$ and $\mathrm{Hom}(\mathbb{Z}^m, H)_0$ are abelian as in [**16**]. It is also proved in [**21**] that the left map is an isomorphism in rational cohomology, completing the proof. □

https://doi.org/10.1017/prm.2023.112 Published online by Cambridge University Press

COROLLARY 3.10. *For $n = 2, 3$, the map*

$$\Theta\colon \mathrm{Hom}(\mathbb{Z}^m, SO(2n))_0 \to \mathrm{map}_*(B\mathbb{Z}^m, BSO(2n))_0$$

*is surjective in rational cohomology.*

*Proof.* By lemma 3.9, it is sufficient to prove the statement for $Spin(2n)$, instead of $SO(2n)$. Then, since $Spin(4) \cong SU(2) \times SU(2)$ and $Spin(6) \cong SU(4)$, the proof is finished by theorem 1.2. □

Now, we begin the proof of theorem 1.5. For a monomial $z = x_1^{i_1} \cdots x_n^{i_n} y_1^{I_1} \cdots y_n^{I_n}$ in $H^*(BT \times T^m)$, let

$$d(z) = (i_1 + |I_1|, \dots, i_n + |I_n|)$$

where $I_1, \dots, I_n \subset [m]$. If all entries of $d(z)$ are even (resp. odd), then we call a monomial $z$ even (resp. odd).

LEMMA 3.11. *If $G = SO(2n)$, then every element of $H^*(BT \times T^m)^W$ is a linear combination of even and odd monomials.*

*Proof.* Given $1 \leqslant i < j \leqslant n$, there is $w \in W$ such that:

$$w(x_k) = \begin{cases} -x_k & k = i, j \\ x_k & k \neq i, j \end{cases} \quad w(y_k) = \begin{cases} -y_k & k = i, j \\ y_k & k \neq i, j. \end{cases}$$

Then, every monomial $z$ in $H^*(BT \times T^m)$ satisfies $w(z) = (-1)^{d_i + d_j} z$, where $d(z) = (d_1, \dots, d_n)$. So if $z$ is contained in some element of $H^*(BT \times T^m)^W$, $d_1 + d_2, d_2 + d_3, \dots, d_{n-1} + d_n$ are even. Thus, $z$ is even for $d_1$ even, and $z$ is odd for $d_1$ odd, completing the proof. □

We define a map

$$\pi\colon H^*(BT \times T^m) \to H^*(BT \times T^m)^W, \quad x \mapsto \sum_{w \in W} w(x).$$

For $m \geqslant 3$ and $G = SO(2n)$ with $n \geqslant 4$, let

$$\bar{a} = x_1 \dots x_{n-4} y_{n-3}^1 y_{n-2}^2 y_{n-1}^3 y_n^1 y_n^2 y_n^3 \in H^*(BT \times T^m)$$

and let $a = \pi(\bar{a})$.

LEMMA 3.12. *The element $(\alpha \times 1)^*(a)$ of $H^*(SO(2n)/T \times T^m)^W$ is indecomposable, where $\alpha\colon G/T \to BT$ is as in § 2.*

*Proof.* It is easy to see that $\alpha^*(x_1 \dots x_{n-4}) \neq 0$ in $H^*(SO(2n)/T)$ because

$$H^*(SO(2n)/T) = \mathbb{Q}[x_1, \dots, x_n]/(p_1, \dots, p_{i-1}, e)$$

where $p_i$ is the $i$-th elementary symmetric polynomial in $x_1^2, \dots, x_n^2$ and $e = x_1 \dots x_n$. Then, $(\alpha \times 1)^*(\bar{a}) \neq 0$ in $H^*(SO(2n)/T \times T^m)$. So, since $a$ includes the term $2^{n-1}(n-4)!\bar{a}$, we have $(\alpha \times 1)^*(a) \neq 0$ in $H^*(SO(2n)/T \times T^m)^W$.

https://doi.org/10.1017/prm.2023.112 Published online by Cambridge University Press

Now, we suppose that $(\alpha\times 1)^*(a)$ is decomposable. Then, there are $b, c \in \widetilde{H}^*(BT\times T^m)$ such that $\pi(b)\pi(c)$ includes the monomial $\bar{a}$, and so we may assume $\bar{a} = bc$. Note that

$$(1,\dots,1,3) = d(\bar{a}) = d(bc) = d(b) + d(c).$$

Then, since $d(b)\neq 0$ and $d(c)\neq 0$, it follows from lemma 3.11 that we may assume $d(b) = (1,\dots,1)$, implying $b = x_1\dots x_{n-4}y^1_{n-3}y^2_{n-2}y^3_{n-1}y^i_n$ for some $i = 1, 2, 3$. Let $\sigma$ be the transposition of $n$ and $k$, where $k = n-3, n-2, n-1$ for $i = 1, 2, 3$, respectively. Then, $\sigma$ belongs to $W$, and $\sigma(b) = b$. Let $W = V\sqcup V\sigma$ be the coset decomposition. Then, we have

$$\pi(b) = \sum_{v\in V} v(b+\sigma(b)) = \sum_{v\in V} v(b-b) = 0$$

and so we get $(\alpha\times 1)^*(a) = 0$, which is a contradiction. Thus, we obtain that $(\alpha\times 1)^*(a)$ is indecomposable, as stated. □

Proposition 3.13. *If $m\geqslant 3$ and $n\geqslant 4$, then $(\alpha\times 1)^*(a)\in H^*(SO(2n)/T\times T^m)^W$ does not belong to the image of the composite*

$$SO(2n)/T\times_W T^m \xrightarrow{\Phi} \mathrm{Hom}(\mathbb{Z}^m, SO(2n))_0 \xrightarrow{\Theta} \mathrm{map}_*(B\mathbb{Z}^m, BSO(2n))_0$$

*in rational cohomology.*

*Proof.* First, we consider the $m = 3$ case. Suppose that there is $\hat{a}\in H^*(\mathrm{map}_*(B\mathbb{Z}^3, BSO(2n))_0)$ such that $(\alpha\times 1)^*(a) = \Phi^*(\Theta^*(\hat{a}))$. Then, by lemma 3.12, $\Phi^*(\Theta^*(\hat{a}))$ is indecomposable. On the contrary, by lemma 2.4, we have $\Phi^*(\Theta^*(\hat{a})) = \Theta^*(\widehat{\Phi}^*(\hat{a})) = \widehat{\Phi}(\hat{a})$, and by proposition 3.6, every indecomposable element of the image of $\widehat{\Phi}^*$ cannot contain a monomial $x_1^{i_1}\dots x_n^{i_n}y_1^{I_1}\cdots y_n^{I_n}$ with $|I_1|+\cdots+|I_n| > 4$. Thus, we obtain a contradiction, so $(\alpha\times 1)^*(a)$ does not belong to the image of $\Phi^*\circ\Theta^*$.

Next, we consider the case $m > 3$. Since $\mathbb{Z}^3$ is a direct summand of $\mathbb{Z}^m$, the maps $\widehat{\Phi}$ and $\Theta$ for $m = 3$ are homotopy retracts of the maps $\widehat{\Phi}$ and $\Theta$ for $m > 3$, respectively. Thus, the $m = 3$ case above implies the $m > 3$ case, completing the proof. □

Now, we are ready to prove theorem 1.5.

*Proof of theorem* 1.5. Combine theorem 2.1 and proposition 3.13. □

## 4. Rational homotopy groups

This section proves theorem 1.6. We begin with a simple lemma. Let $\mathrm{hur}^*\colon H^*(X)\to \mathrm{Hom}(\pi_*(X), \mathbb{Q})$ denote the dual Hurewicz map. As in the proof of theorem 1.3, let $QA$ denote the module of indecomposables of an augmented algebra $A$. We refer to [**13**] for rational homotopy theory.

Lemma 4.1. *Let $X$ be a simply-connected space such that there is a map:*

$$X \to \prod_{i=2}^{n} K(V_i, i)$$

*which is a rational equivalence in dimension $\leqslant n$, where $V_i$ is a $\mathbb{Q}$-vector space of finite dimension. Then, for $i \leqslant n+2$, the map*

$$\mathrm{hur}^* \colon QH^i(X) \to \mathrm{Hom}(\pi_i(X), \mathbb{Q})$$

*is injective.*

*Proof.* The minimal model of $X$ in dimension $\leqslant n$ is given by

$$(\Lambda(V_2 \oplus \cdots \oplus V_n), d = 0)$$

where $\Lambda V$ denotes the free commutative-graded algebra generated by a graded vector space $V$ and each $V_i$ is of degree $i$. Then, there is no element of degree one in the minimal model of $X$, so any element of $QH^i(X)$ for $i \leqslant n+2$ is represented by an indecomposable element of the minimal model of $X$. Since the module of indecomposables of the minimal model of $X$ is isomorphic to $\mathrm{Hom}(\pi_*(X), \mathbb{Q})$ through the dual Hurewicz map, the proof is finished. □

We recall a property of the minimal generating set $\mathcal{S}(m, G)$ that we are going to use. Let:

$$d(m, G) = \begin{cases} 2n - m & G = U(n), SU(n) \\ 2n + 1 & G = Sp(n), SO(2n+1). \end{cases}$$

Let $\mathbb{Q}\{S\}$ denote the graded $\mathbb{Q}$-vector space generated by a graded set $S$. We consider a map:

$$\lambda = \prod_{x \in \mathcal{S}(m,G)} x \colon \mathrm{Hom}(\mathbb{Z}^m, G)_0 \to \prod_{x \in \mathcal{S}(m,G)} K(\mathbb{Q}, |x|).$$

The following is proved in [**21**].

Theorem 4.2. *Let $G$ be the classical group except for $SO(2n)$. Then, the map*

$$\lambda^* \colon \Lambda(\mathbb{Q}\{\mathcal{S}(m, G)\}) \to H^*(\mathrm{Hom}(\mathbb{Z}^m, G)_0)$$

*is an isomorphism in dimension $\leqslant d(m, G)$.*

We define a map $\mathrm{hur}^* \colon \mathcal{S}(m, G) \to \mathrm{Hom}(\pi_*(\mathrm{Hom}(\mathbb{Z}^m, G)_0), \mathbb{Q})$ by the linear part of the map $\lambda$ in the minimal models.

Lemma 4.3. *If $G$ is the classical group except for $SO(2n)$, then the map*

$$\mathrm{hur}^* \colon \mathbb{Q}\{\mathcal{S}(m, G)\} \to \mathrm{Hom}(\pi_*(\mathrm{Hom}(\mathbb{Z}^m, G)_0), \mathbb{Q})$$

*is injective in dimension $\leqslant d(m, G) + 2$.*

*Proof.* By [**16**], $\mathrm{Hom}(\mathbb{Z}^m, G)_0$ is simply-connected whenever $G$ is simply-connected. Then, by lemma 3.9, we may assume $\mathrm{Hom}(\mathbb{Z}^m, G)_0$ is simply-connected for $G = SU(n), Sp(n), SO(2n+1)$ as long as we consider rational cohomology and rational homotopy groups. By theorem 4.2, the map $\lambda$ is an isomorphism in rational cohomology in dimension $\leqslant d(m, G)$. Then, the statement for $G = SU(n), Sp(n), SO(2n+1)$ is proved by the J.H.C. Whitehead theorem and lemma 4.1. For $G = U(n)$, we may consider $S^1 \times SU(n)$ by lemma 3.9, instead of $U(n)$. In this case, the dual Hurewicz map for $G = U(n)$ is identified with the map:

$$1 \times \mathrm{hur}^* \colon \mathbb{Q}^m \times \mathbb{Q}\{\mathcal{S}(m, SU(n))\} \to \mathbb{Q}^m \times \mathrm{Hom}(\pi_*(\mathrm{Hom}(\mathbb{Z}^m, SU(n))_0), \mathbb{Q})$$

because $\mathrm{Hom}(\mathbb{Z}^m, S^1 \times SU(n))_0 = (S^1)^m \times \mathrm{Hom}(\mathbb{Z}^m, SU(n))_0$. Thus, the statement follows from the $G = SU(n)$ case. □

Lemma 4.4. *For $G = U(n), SU(n)$, the map*

$$\mathrm{hur}^* \colon \mathbb{Q}\{\mathcal{S}(m, G)\} \to \mathrm{Hom}(\pi_*(\mathrm{Hom}(\mathbb{Z}^m, G)_0), \mathbb{Q})$$

*is injective.*

*Proof.* Let $G = U(n), SU(n)$. We induct on $m$. If $m = 1$, then the statement is obvious. Assume that the statement holds less than $m$. Take any $\emptyset \neq I \subset [m]$. Then, there are the obvious inclusion $\iota_I \colon \mathbb{Z}^{|I|} \to \mathbb{Z}^m$ and the obvious projection $\pi_I \colon \mathbb{Z}^m \to \mathbb{Z}^{|I|}$ such that $\pi_I \circ \iota_I = 1$. In particular, we get maps $\iota_I^* \colon \mathrm{Hom}(\mathbb{Z}^m, G)_0 \to \mathrm{Hom}(\mathbb{Z}^{|I|}, G)_0$ and $\pi_I^* \colon \mathrm{Hom}(\mathbb{Z}^{|I|}, G)_0 \to \mathrm{Hom}(\mathbb{Z}^m, G)_0$ such that $\iota_I^* \circ \pi_I^* = 1$. Note that the map $\pi_I^*$ induces a map $(\pi_I^*)^* \colon \mathcal{S}(m, G) \to \mathcal{S}(|I|, G)$ such that:

$$(\pi_I^*)^*(z(d, J)) = \begin{cases} z(d, J) & J \subset I \\ 0 & J \not\subset I. \end{cases} \tag{4.1}$$

Then, there is a commutative diagram:

$$\begin{tikzcd} \mathbb{Q}\{\mathcal{S}(m, G)\} \arrow[r, "(\pi_I^*)^*"] \arrow[d, "\mathrm{hur}^*"'] & \mathbb{Q}\{\mathcal{S}(|I|, G)\} \arrow[d, "\mathrm{hur}^*"] \\ \mathrm{Hom}(\pi_*(\mathrm{Hom}(\mathbb{Z}^m, G)_0), \mathbb{Q}) \arrow[r, "((\pi_I^*)_*)^*"] & \mathrm{Hom}(\pi_*(\mathrm{Hom}(\mathbb{Z}^{|I|}, G)_0), \mathbb{Q}). \end{tikzcd} \tag{4.2}$$

Now, we assume

$$\sum_{z(d,J) \in \mathcal{S}(m,G)} a_{d,J} \mathrm{hur}^*(z(d, J)) = 0$$

https://doi.org/10.1017/prm.2023.112 Published online by Cambridge University Press

for $a_{d,J} \in \mathbb{Q}$. Then, by (4.1) and (4.2), we have

$$\begin{aligned} 0 &= ((\pi_I^*)_*)^* \left( \sum_{z(d,J)\in\mathcal{S}(m,G)} a_{d,J} \mathrm{hur}^*(z(d,J)) \right) \\ &= \sum_{z(d,J)\in\mathcal{S}(m,G)} a_{d,J} \mathrm{hur}^*(\pi_I^*)^*(z(d,J))) \\ &= \sum_{\substack{z(d,J)\in\mathcal{S}(m,G) \\ J\subset I}} a_{d,J} \mathrm{hur}^*(z(d,J)) \\ &= \sum_{z(d,J)\in\mathcal{S}(|I|,G)} a_{d,\iota_I(J)} \mathrm{hur}^*(z(d,J)). \end{aligned}$$

So, since the right map of (4.2) is injective for $I \neq [m]$ by the induction hypothesis, we get $a_{d,J} = 0$ for $J \neq [m]$, implying:

$$\sum_{z(d,[m])\in\mathcal{S}(m,G)} a_{d,[m]} \mathrm{hur}^*(z(d,[m])) = 0.$$

Note that every $z(d,[m]) \in \mathcal{S}(m,G)$ is of degree $\leqslant 2n - m + 1$. Then, by lemma 4.3, we get $a_{d,[m]} = 0$, completing the proof. □

Now, we prove theorem 1.6.

*Proof of theorem* 1.6. Let $\mathcal{S}_i$ and $\mathcal{S}_i(m,G)$ denote the degree $i$ parts of $\mathcal{S}$ and $\mathcal{S}(m,G)$, respectively, where $\mathcal{S}$ is as in proposition 3.6. Then, by proposition 3.6 and theorem 4.2, there is a commutative diagram:

$$\begin{tikzcd} \mathbb{Q}\{\mathcal{S}_i\} \arrow[r, "\Theta^*"] \arrow[d, "\mathrm{hur}^*"'] & \mathbb{Q}\{\mathcal{S}_i(m,G)\} \arrow[d, "\mathrm{hur}^*"] \\ \mathrm{Hom}(\pi_i(\mathrm{map}_*(B\mathbb{Z}^m, BG)_0), \mathbb{Q}) \arrow[r, "(\Theta_*)^*"] & \mathrm{Hom}(\pi_i(\mathrm{Hom}(\mathbb{Z}^m, G)_0), \mathbb{Q}). \end{tikzcd}$$

Let $K_i$ denote the kernel of the bottom map. Clearly, the dimension of $K_i$ coincides with

$$\dim \mathrm{Coker}\{\Theta_* \colon \pi_i(\mathrm{Hom}(\mathbb{Z}^m, G)_0) \otimes \mathbb{Q} \to \pi_*(\mathrm{map}_i(B\mathbb{Z}^m, BG)_0) \otimes \mathbb{Q}\}$$

and so we compute $\dim K_i$. By proposition 3.6, the left map is an isomorphism. Then, we get:

$$\dim K_i \geqslant \dim \mathbb{Q}\{\mathcal{S}_i\} - \dim \mathbb{Q}\{\mathcal{S}_i(m,G)\}.$$

By lemma 4.3, the equality holds for $G = Sp(n), SO(2n+1)$ and $i \leqslant d(m,G) + 2$, and by lemma 4.4, the equality holds for $G = U(n), SU(n)$ and all $i$. We can easily compute:

$$\dim \mathbb{Q}\{\mathcal{S}_i\} - \dim \mathbb{Q}\{\mathcal{S}_i(m,G)\} = \begin{cases} \sum_{i<k\leqslant n} \binom{m}{2k-i} & G = U(n), SU(n) \\ \sum_{i/3<k\leqslant n} \binom{m}{4k-i} & G = Sp(n), SO(2n+1) \end{cases}$$

and thus the proof is finished. □

## Acknowledgements

The authors were supported in part by JSPS KAKENHI JP17K05248 and JP19K03473 (Kishimoto), JP21J10117 (Takeda), and JP19K14535 (Tsutaya).

## References

1 A. Adem and F. R. Cohen. Commuting elements and spaces of homomorphisms. *Math. Ann.* **338** (2007), 587–626.
2 A. Adem, F. R. Cohen and E. Torres-Giese. Commuting elements, simplicial spaces, and filtrations of classifying spaces. *Math. Proc. Cambridge Philos. Soc.* **152** (2012), 91–114.
3 A. Adem, J. M. Gómez and S. Gritschacher. On the second homotopy group of spaces of commuting elements in Lie groups. *IMRN* **2022** (2022), 19617–19689.
4 M. F. Atiyah and R. Bott. The Yang–Mills equations over Riemann surfaces. *Philos. Trans. R. Soc. London, Ser. A* **308** (1983), 523–615.
5 T. J. Baird. Cohomology of the space of commuting $n$-tuples in a compact Lie group. *Algebra Geom. Topol.* **7** (2007), 737–754.
6 T. Baird, L. C. Jeffrey and P. Selick. The space of commuting $n$-tuples in $SU(2)$. *Ill. J. Math.* **55** (2011), 805–813.
7 T. Baird and D. A. Ramras. Smoothing maps into algebraic sets and spaces of flat connections. *Geom. Dedicata* **174** (2015), 359–374.
8 M. Bergeron. The topology of nilpotent representations in reductive groups and their maximal compact subgroups. *Geom. Topol.* **19** (2015), 1383–1407.
9 M. Bergeron and L. Silberman. A note on nilpotent representations. *J. Group Theory* **19** (2016), 125–135.
10 R. Bott. The space of loops on a Lie group. *Mich. Math. J.* **5** (1958), 35–61.
11 C. Broto, R. Levi and B. Oliver. *The theory of p-local groups: a survey, homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*. Contemporary Mathematics, vol. 346, pp. 51–84 (Providence, RI: American Mathematical Society, 2004).
12 M. C. Crabb. Spaces of commuting elements in $SU(2)$. *Proc. Edinburgh Math. Soc.* **54** (2011), 67–75.
13 Y. Félix, S. Halperin and J.-C. Thomas. *Rational homotopy theory*. Graduate Texts in Mathematics, vol. 205 (New York: Springer-Verlag, 2001).
14 M. Fuchs. Verallgemeinerte homotopie-homomorphismen und klassifizierende Raüme. *Math. Ann.* **161** (1965), 197–230.
15 W. M. Goldman. Topological components of spaces of representations. *Invent. Math.* **93** (1988), 557–607.
16 J. M. Gómez, A. Pettet and J. Souto. On the fundamental group of $\mathrm{Hom}(\mathbb{Z}^k, G)$. *Math. Z.* **271** (2012), 33–44.
17 J. Grodal. *The classification of p-compact groups and homotopical group theory. Proceedings of the International Congress of Mathematicians*, vol. II, pp. 973–1001 (New Delhi: Hindustan Book Agency, 2010).
18 K. Hasegawa. Minimal models of nilmanifolds. *Proc. Am. Math. Soc.* **106** (1989), 65–71.
19 P. Hilton, G. Mislin and J. Roitberg. *Localization of nilpotent groups and spaces.* In *North-Holland Mathematics Studies, vol. 15, Notas de Matemática (Notes on Mathematics)*, vol. 55 (Amsterdam, Oxford/New York: North-Holland Publishing Co./American Elsevier Publishing Co., Inc., 1975).

20 S. Jackowski, J. McClure and B. Oliver. *Maps between classifying spaces revisited. The Čech centennial (Boston, MA, 1993), 263–298*. Contemporary Mathematics, vol. 181 (Providence, RI: American Mathematical Society, 1995).

21 D. Kishimoto and M. Takeda. Spaces of commuting elements in the classical groups. *Adv. Math.* **386** (2021), 107809.

22 D. Kishimoto and M. Takeda. *Torsion in the space of commuting elements in a Lie group*. Can. J. Math. (accepted).

23 D. Kishimoto and A. Kono. On the cohomology of free and twisted loop spaces. *J. Pure Appl. Algebra* **214** (2010), 646–653.

24 J. Milnor. Construction of universal bundles, II. *Ann. Math.* **63** (1956), 430–436.

25 D. A. Ramras and M. Stafa. Hilbert–Poincaré series for spaces of commuting elements in Lie groups. *Math. Z.* **292** (2019), 591–610.

26 D. A. Ramras and M. Stafa. Homological stability for spaces of commuting elements in Lie groups. *Int. Math. Res. Not.* **2021** (2021), 3927–4002.

27 J. D. Stasheff. Homotopy associativity of H-spaces, I & II. *Trans. Am. Math. Soc.* **108** (1963), 275–292. 293–312.

28 D. Sullivan. *Geometric topology, part I: localization, periodicity and Galois symmetry*. K-Monographs in Math (Dordrecht: Kluwer Academic Publishers, 2004).

29 M. Tsutaya. Mapping spaces from projective spaces. *Homol. Homotopy Appl.* **18** (2016), 173–203.