Glasgow Math. J. **45** (2003) 263–267. © 2003 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089503001228. Printed in the United Kingdom

PRINCIPAL REES QUOTIENTS OF FREE INVERSE SEMIGROUPS

D. EASDOWN
School of Mathematics and Statistics, University of Sydney, Sydney, NSW 2006, Australia
e-mail: de@maths.usyd.edu.au

and L. M. SHNEERSON
Department of Mathematics and Statistics, Hunter College, The City University of New York,
695 Park Avenue, New York, NY 10021, U.S.A.
e-mail: lms6@nyu.edu

(Received 25 January, 2002; accepted 2 August, 2002)

Abstract. We prove that up to isomorphism $\langle a, b \mid ab = 0\rangle$ is the unique principal Rees quotient of a free inverse semigroup that is not trivial or monogenic with zero, satisfying a nontrivial identity in signature with involution.

2000 *Mathematics Subject Classification.* 20M05, 20M18, 20M99.

1. Introduction. In [**9**] the authors initiated the study of identities satisfied by finitely presented Rees quotients of free inverse semigroups. It was proved [**9**, Section 5] that a semigroup from that class satisfies a nontrivial semigroup identity if and only if its growth is polynomial. Key ingredients were a graphical technique due to Ufnarovsky [**10**], [**11**], Adjan's identity for the bicyclic semigroup [**1**], and the fact (easily deduced from [**8**]) that the bicyclic and free monogenic inverse semigroups satisfy the same semigroup identities. By contrast, however, an example was given in [**9**, Section 4] of a semigroup from that class that has exponential growth yet satisfies a nontrivial identity in signature with involution. In this paper we examine *principal* Rees quotients of free inverse semigroups; that is, Rees quotients by a principal ideal. We show that up to isomorphism

$$S = \langle a, b \mid ab = 0\rangle$$

(regarded as a presentation as an inverse semigroup with zero) is the unique member of this class that is not trivial or monogenic with zero yet satisfies a nontrivial identity in signature with involution. Of course S has exponential growth (since a and b^{-1} generate a noncyclic free subsemigroup), but S does not contain any nonmonogenic free inverse subsemigroups.

2. Preliminaries. We assume familiarity with the basic definitions and elementary results from the theory of inverse semigroups that can be found in any of [**2**], [**3**], [**4**] or [**6**]. We denote the free semigroup, free inverse semigroup and free group over an alphabet A by F_A, FI_A and G_A, respectively. Equality in free semigroups will be denoted by $\overset{\circ}{=}$. Recall that a word w is *reduced* if w does not contain xx^{-1} as a subword for any letter $x \in A \cup A^{-1}$. If $w \in F_{A\cup A^{-1}}$ then we denote by $\overline{w}$ the reduced word equivalent to w in G_A. Without causing confusion we shall identify elements of FI_A and G_A with words over the alphabet $A \cup A^{-1}$. Then the mapping $w \mapsto \overline{w}$ induces a homomorphism from FI_A to G_A. Denote by $[u, v]$ the commutator $uvu^{-1}v^{-1}$.

Recall that elements of FI_A may be regarded as birooted word trees (introduced in [**5**]), the terminology and theory of which are explained in [**3**]. (See also [**9**, Section 2].) If u and v are elements of FI_A, then the word tree of u is a subtree of the word tree of v if and only if v may be expressed as a product of elements, one of whose factors is u, in which case we say that u *divides* v. Any reference to Green's relation $\mathcal{J}$ will be with respect to FI_A. Note that two words over $A \cup A^{-1}$ are $\mathcal{J}$-related if and only if their word trees are identical.

The following lemma can be verified easily using Reilly's criterion [**7**, Theorem 2.2].

LEMMA 2.1. *If* $\epsilon, \delta, \nu, \eta \in \{\pm 1\}$ *and* $\epsilon \neq \nu$ *or* $\delta \neq \eta$ *then* $a^\epsilon b^\delta$ *and* $a^\nu b^\eta$ *freely generate an inverse subsemigroup of* $FI_{\{a,b\}}$.

Denote by $\langle u, v\rangle$ the inverse subsemigroup of FI_A generated by elements u and v. The next lemma follows easily by inspection of word trees.

LEMMA 2.2. *Suppose that* a *and* b *are letters from* A, *that* ab *divides a word* c *in* FI_A *and that* c *divides elements from each of* $\langle ab^{-1}, a^{-1}b\rangle$, $\langle ab, ab^{-1}\rangle$ *and* $\langle a^{-1}b, ab\rangle$. *Then* $c \mathcal{J} ab$.

An *identity in signature with involution* is an equation of the form

$$P = Q \tag{1}$$

where P and Q are words (also called *terms*) from the free semigroup $F_{V \cup V^{-1}}$, where $V = \{x_1, \ldots, x_n\}$ is some finite alphabet (of *variables*). We say that an inverse semigroup S *satisfies* this identity if the equality (1) holds in S when each side is evaluated after substituting arbitrary elements of S for $x_1, \ldots, x_n$. We call the identity (1) *trivial* if it is satisfied by all inverse semigroups, *nontrivial* otherwise. By [**7**, Corollary 2.7] the free inverse semigroup FI_2 on two generators contains a copy of the free inverse semigroup on countably infinitely many generators, so that any identity satisfied by FI_2 must be satisfied by all inverse semigroups. Thus if FI_2 embeds in an inverse semigroup S, then S cannot satisfy any nontrivial identity in signature with involution.

3. Principal Rees quotients. In this section we prove a sequence of lemmas culminating in a theorem which gives an identity in signature with involution satisfied by the semigroup $S = \langle a, b \mid ab = 0\rangle$. We then prove that S is unique amongst the class of nontrivial nonmonogenic principal Rees quotients of free inverse semigroups that satisfy any such identity.

Throughout, denote by X the subsemigroup of S generated by a and b^{-1}. It is clear that multiplication in S of words from X is simply concatenation, so that a and b^{-1} are free generators of X.

LEMMA 3.1. *Let* w *be a nonempty reduced word that is nonzero in* S. *Then* $w \mathrel{\underline{\circ}} w_1^{-1}w_2$ *for some* $w_1, w_2 \in X^1$.

Proof. Observe that $ab = b^{-1}a^{-1} = 0$ in S. Let u be any reduced word that is nonzero in S. (i) If u begins with a or b^{-1}, then $u \in X$. (ii) If u ends with a^{-1} or b, then $u^{-1} \in X$. Thus we may suppose that w does not begin with a or b^{-1} and does not end with a^{-1} or b, for otherwise the conclusion of the lemma holds trivially. If w does not

https://doi.org/10.1017/S0017089503001228 Published online by Cambridge University Press

contain b^{-1}, then $w \circeq va^{\theta}$, for some positive integer θ and some word v ending with b, so that $a^{\theta}, v^{-1} \in X$ and the claim of the lemma follows. Suppose then that w contains b^{-1}, so that $w \circeq uv$, for some u not containing b^{-1} and v beginning with b^{-1}. Then $v \in X$. Also u does not end with b since w is reduced. Either u ends with a^{-1}, in which case $u^{-1} \in X$ and we are done, or u ends with a. In the latter case $u \circeq u_0 a^{\theta}$, for some $\theta > 0$ and u_0 ending with b, whence $u_0^{-1}, a^{\theta} v \in X$ and again the claim of the lemma follows, completing the proof.

LEMMA 3.2. *Let u, v be nonempty reduced words such that $u, v \in X$ and $uv^{-1} \neq 0$ in S. Then either u is a suffix of v or v is a suffix of u. In particular, if u and v have the same length, then $u \circeq v$.*

Proof. If u and v end with different letters, then without loss of generality $u \circeq u_0 a$, $v \circeq v_0 b^{-1}$, for some u_0, v_0, so that $uv^{-1} \circeq u_0 ab v_0 = 0$ in S, a contradiction. Hence u and v end with the same letter so that there is a nonempty maximal suffix $x \in X$ such that $u \circeq u_0 x$, $v \circeq v_0 x$, for some u_0 v_0 in X^1. Suppose that u_0 and v_0 are both nonempty. Then $u_0 v_0^{-1} \neq 0$ in S, because $u_0 v_0^{-1}$ divides uv^{-1}. Hence, from above, u_0 and v_0 end with the same letter, contradicting the maximality of x. Hence either u_0 or v_0 is empty, and the lemma is proved.

LEMMA 3.3. *Let u and v be nonempty reduced words in X such that the commutator $[u, v]$ is nonzero in S. Then $uv \circeq vu$.*

Proof. Observe that uv and vu are nonempty reduced words of the same length lying in X, and $(uv)(vu)^{-1} \neq 0$ in S. Hence $uv \circeq vu$ by Lemma 3.2.

LEMMA 3.4. *Let w be a reduced word such that $w^2 \neq 0$ in S. Then $w \circeq w_1^{-1} w_2 w_1$ or $w \circeq w_1^{-1} w_2^{-1} w_1$, for some $w_1, w_2 \in X$.*

Proof. By Lemma 3.1, $w \circeq u^{-1} v$, for some $u, v \in X$. Then vu^{-1} is a divisor of w^2 and so is nonzero in S. By Lemma 3.2, either u is a suffix of v or v is a suffix of u, and the conclusion of the lemma follows.

LEMMA 3.5. *Suppose that $u \in X$, $v \circeq x_1^{-1} x_2 x_1$, for some $x_1, x_2 \in X$, and $vu^2vu^{-1}v^{-1}$ is nonzero in S. Then $\overline{uv} \circeq \overline{vu}$.*

Proof. Certainly ux_1^{-1} is nonzero in S since it divides $vu^2vu^{-1}v^{-1}$. Hence, by Lemma 3.2, either u is a suffix of x_1 or x_1 is a suffix of u. Suppose first that $x_1 \circeq yu$, for some $y \in X$. Certainly yuy^{-1} is nonzero in S since it also divides $vu^2vu^{-1}v^{-1}$. Hence, by Lemma 3.2 again, y is a suffix of yu and so $\overline{yuy^{-1}} \in X$. Further $vu^2vu^{-1}v^{-1}$ is divided by $yu^2y^{-1}x_2yu^{-1}y^{-1}x_2^{-1}$, which in turn is divided by $yuy^{-1}x_2yu^{-1}y^{-1}x_2^{-1}$, since y is a suffix of yu. Hence $[\overline{yuy^{-1}}, x_2]$ is nonzero in S, so that $\overline{yuy^{-1}}x_2 \circeq x_2\overline{yuy^{-1}}$, by Lemma 3.3. It follows that $\overline{uv} \circeq \overline{vu}$. Suppose now that $u \circeq u_0 x_1$, for some $u_0 \in X$. Then $u^2vu^{-1}v^{-1}$ is divided by $[x_1u_0, x_2]$, so that the latter is nonzero in S. Hence $x_1u_0x_2 \circeq x_2x_1u_0$, by Lemma 3.3. It follows again that $\overline{uv} \circeq \overline{vu}$, and the lemma is proved.

LEMMA 3.6. *Let $u, v \in S$ and suppose that each of*

$$vu^2v(uv^2u)^{-1}, \quad (vu^2v)^{-1}uv^2u, \quad v^{-1}u^2v^{-1}(uv^{-2}u)^{-1}, \quad (v^{-1}u^2v^{-1})^{-1}uv^{-2}u$$

is nonzero in S. Then $\overline{uv} \circeq \overline{vu}$.

Proof. Without loss of generality we may suppose that u and v are reduced. Certainly u^2 and v^2 are nonzero in S so that, by Lemma 3.4, either u or $u^{-1} \circeq w_1^{-1} w_2 w_1$

https://doi.org/10.1017/S0017089503001228 Published online by Cambridge University Press

and either v or $v^{-1} \,\underline{\circ}\, w_3^{-1}w_4w_3$, for some $w_1, w_2, w_3, w_4 \in X$. The four expressions listed in the hypothesis of the lemma are permuted by interchanging u with u^{-1} or v with v^{-1}. Hence, without loss of generality we may suppose that $u \,\underline{\circ}\, w_1^{-1}w_2w_1$ and $v \,\underline{\circ}\, w_3^{-1}w_4w_3$. Certainly $uv \neq 0$ in S, and so $w_1w_3^{-1}$, being a divisor of uv, is nonzero in S. Hence, by Lemma 3.2, w_1 is a suffix of w_3 or w_3 is a suffix of w_1. Suppose that $w_3 \,\underline{\circ}\, w_5w_1$, for some $w_5 \in X$. Put $u_0 \,\underline{\circ}\, w_2$ and $v_0 \,\underline{\circ}\, w_5^{-1}w_4w_5$. Then $v_0u_0^2v_0u_0^{-1}v_0^{-1} \neq 0$ in S since it divides $vu^2v(uv^2u)^{-1}$. Hence, by Lemma 3.5, $\overline{u_0v_0} \,\underline{\circ}\, \overline{v_0u_0}$, so that

$$\overline{uv} \;\underline{\circ}\; \overline{w_1^{-1}u_0v_0w_1} \;\underline{\circ}\; \overline{w_1^{-1}v_0u_0w_1} \;\underline{\circ}\; \overline{vu},$$

and we are done. The case in which w_3 is a suffix of w_1 follows from the previous case because $vu^2v(uv^2u)^{-1}$ is inverted by interchanging u and v, and the proof is complete.

THEOREM 3.7. *$S = \langle a, b \mid ab = 0\rangle$ satisfies the identity*

$$PQ = QP \tag{2}$$

where

$$P = P(x, y) = [yx^2y, x^{-1}y^{-2}x^{-1}]$$

and

$$Q = Q(x, y) = [y^{-1}x^2y^{-1}, x^{-1}y^2x^{-1}].$$

Proof. Let $u, v \in S$. If $P(u, v)$ or $Q(u, v)$ is zero in S then (2) holds trivially when u, v are substituted for x, y respectively. Suppose that $P(u, v)$ and $Q(u, v)$ are nonzero in S. In particular

$$vu^2v(uv^2u)^{-1}, \quad (vu^2v)^{-1}uv^2u, \quad v^{-1}u^2v^{-1}(uv^{-2}u)^{-1}, \quad (v^{-1}u^2v^{-1})^{-1}uv^{-2}u$$

are all nonzero in S, so that $\overline{uv} \,\underline{\circ}\, \overline{vu}$, by Lemma 3.6. Hence $\overline{P(u, v)}$ and $\overline{Q(u, v)}$ are empty so that $P(u, v)$ and $Q(u, v)$ are idempotents and thus commute. Hence (2) holds when u, v are substituted for x, y respectively, and the theorem is proved.

THEOREM 3.8. *Let $T = \langle A \mid c = 0\rangle$, for some word c over $A \cup A^{-1}$. Then T satisfies a nontrivial identity in signature with involution if and only if T is trivial, monogenic with zero or isomorphic to $\langle a, b \mid ab = 0\rangle$.*

Proof. The 'if part' is immediate by Theorem 3.7. Suppose that T satisfies a nontrivial identity in signature with involution. In particular T contains no nonmonogenic free inverse subsemigroup. Suppose A contains at least 3 distinct letters a_1, a_2, a_3. Then each of $\langle a_1, a_2\rangle$, $\langle a_1, a_3\rangle$ and $\langle a_2, a_3\rangle$ is a nonmonogenic free inverse subsemigroup of FI_A so that each contains an element divided by c. But by inspection this is impossible. Hence A has at most 2 letters. If A has 1 letter then S is trivial or monogenic with zero. Suppose finally that $A = \{a, b\}$. If c is $\mathcal{J}$-equivalent to a single letter, then S is monogenic with zero. Suppose that c is not $\mathcal{J}$-equivalent to a single letter. Without loss of generality we may suppose that a^2 or ab divides c. By Lemma 2.1, the inverse subsemigroups $\langle ab^{-1}, a^{-1}b\rangle$, $\langle ab, ab^{-1}\rangle$ and $\langle a^{-1}b, ab\rangle$ are free of rank 2, so that c must divide elements from each of them. It is impossible however that a^2 divides an element of $\langle ab, ab^{-1}\rangle$. Hence ab divides c, and so, by Lemma 2.2, $c \;\mathcal{J}\; ab$, and T is isomorphic to $\langle a, b \mid ab = 0\rangle$, completing the proof.

https://doi.org/10.1017/S0017089503001228 Published online by Cambridge University Press

ACKNOWLEDGEMENT. The authors gratefully acknowledge the support of the Australian Research Council and the School of Mathematics and Statistics at the University of Sydney, which made possible this collaborative work.

REFERENCES

1. S. I. Adjan, Identities of special semigroups, *Dokl. Acad. Nauk.* **143**, No. 3 (1962), 499–502 (in Russian).

2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys, Vol. I and Vol. II (Amer. Math. Soc., Providence, Rhode Island, 1961 and 1967).

3. P. M. Higgins, *Techniques of semigroup theory* (Oxford University Press, 1992).

4. J. M. Howie, *An introduction to semigroup theory* (Academic Press, 1976).

5. W. D. Munn, Free inverse semigroups, *Proc. Lond. Math. Soc. (3)* **29** (1974), 385–404.

6. M. Petrich, *Inverse semigroups* (Wiley & Sons, 1984).

7. N. R. Reilly, Free generators of free inverse semigroups, *Bull. Austral. Math. Soc.* **7** (1972), 407–24; Corrigenda, *ibid.* **9** (1973), 479.

8. H. E. Scheiblich, A characterization of a free elementary inverse semigroup, *Semigroup Forum* **2** (1971), 76–79.

9. L. M. Shneerson and D. Easdown, Growth and satisfiability of identities in a class of finitely presented inverse semigroups with zero, *Internat. J. Algebra Comput.* **6** (1996), 105–121.

10. V. A. Ufnarovsky, Growth criterion for graphs and algebras given by words, *Mat. Zam.* **31**, No. 3 (1982), 465–472 (in Russian).

11. V. A. Ufnarovsky, *Combinatorial and asymptotic methods in algebra*, VINITI Modern Problems of Mathematics, Fundamental Directions, Volume 57 (Moscow, 1990) (in Russian).