BIFURCATIONS AND SYMMETRY-BREAKING IN
SIMPLE MODELS OF NONLINEAR DYNAMOS

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Abstract. Low-order models of nonlinear dynamos can be used to investigate generic properties of more realistic mean field dynamos. Reducing the partial differential equations to a set of ordinary differential equations makes it possible to explore the bifurcation structure in considerable detail and to compute unstable solutions as well as ones that are stable. Complicated time-dependent behaviour is typically associated with a homoclinic or heteroclinic bifurcation. Destruction of periodic orbits at saddles or saddle-foci gives rise to Lorenz-like or Shil'nikov-like chaotic oscillations, while destruction of a quasiperiodic orbit leads to aperiodically modulated cycles. Changes in spatial symmetry can also be investigated. The interaction between solutions (steady or periodic) with dipole and quadrupole symmetry gives rise to a complicated bifurcation structure, with several recognizably different mixed-mode solutions; similar behaviour has also been found in spherical dynamo models. These results have implications for the expected behaviour of stellar dynamos.

1. Introduction

The remarkable advances in nonlinear dynamics over the past two decades have revealed various systematic patterns of behaviour that recur in many different nonlinear problems. My aim in this review is to show how detailed studies of simple truncated models can be used to clarify aspects of nonlinear behaviour in more realistic – and therefore more complicated – models of planetary and stellar dynamos. Thus I shall be concerned here with certain technical aspects of dynamo theory rather than with relating theory to observations.

The relationship between properties of toy models and those of the full partial differential equations is not always obvious. So why study such simple systems? The motivation for this approach is to demonstrate *generic* behaviour, although specific models lack *predictive* power. For example, chaos may appear as a consequence of an unjustifiable truncation – yet similar mechanisms do lead to chaos in accurate calculations. The discussion here will focus on two specific issues: first, the connection between chaotic behaviour and global (homoclinic or heteroclinic) bifurcations and, secondly, the breaking of spatio-temporal symmetries.

2. Temporal chaos

It is worth recalling that dynamo theory provided one of the earliest examples of chaotic behaviour in a dissipative system (Bullard 1978; Krause and Roberts 1981; Guckenheimer 1981). Allan (1958) noted that coupled disc dynamos exhibit aperiodic reversals. At that time it was widely believed that, for such a system, trajectories would be attracted to a fixed point, a limit cycle or a torus; the existence of structurally stable chaotic motion did not become generally accepted until the late '60s, after the key papers of Smale (1963, 1967) and Lorenz (1963) had appeared. Allan was fortunate in being advised by P. Swinnerton-Dyer, who was
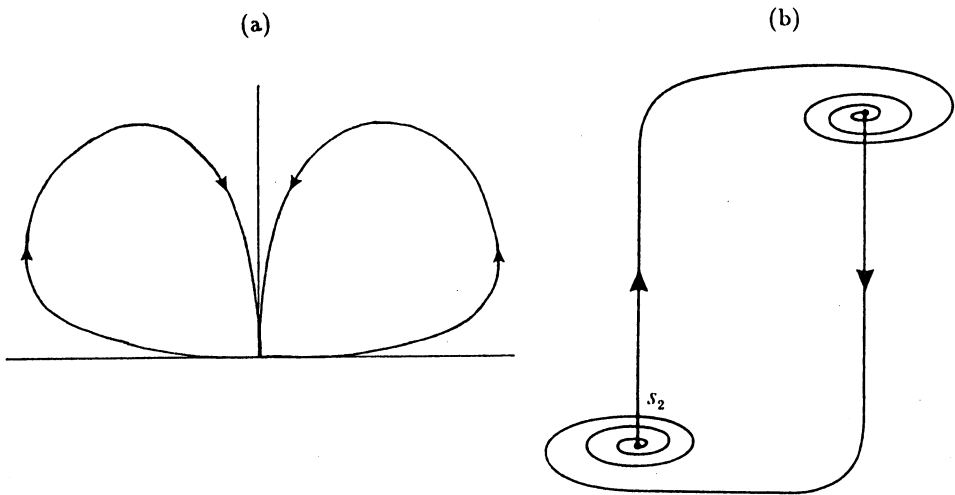
(a)   (b)

Fig. 1. Homoclinic and heteroclinic orbits. (a) Sketch showing a symmetrical pair of homoclinic connections to a saddle-point at the origin, as in the Lorenz system. (b) Sketch showing a heteroclinic connection between a symmetrical pair of saddle-foci.

familiar with Cartwright and Littlewood's (1945) study of the forced van der Pol equation, and the chaotic oscillations of coupled disc dynamos were therefore investigated in some detail (Allan 1962; Cook and Roberts 1970). Subsequently it was realized that similar behaviour occurred for a single dynamo with a shunt and series impedance (Malkus 1972). That model is actually described by the Lorenz equations (Robbins 1977; Knobloch 1981), which provide a paradigm for the study of temporal chaos.

When chaotic behaviour appears in such a system it is important to distinguish between the mechanism that causes chaos, which is typically associated with a homoclinic or heteroclinic bifurcation, and the actual route to chaos, which often involves a cascade of period-doubling bifurcations. Consider, for example, the third-order Lorenz system

$$\dot{x} = \sigma(y - x),$$

$$\dot{y} = rx - y - xz,$$

$$\dot{z} = -\nu z + xy,$$

where x, y, z are real variables and r, ν, σ are real parameters. For $0 < \nu < \sigma - 1$ chaos is caused by a homoclinic explosion (Sparrow 1982), when a symmetrical pair of unstable periodic orbits become homoclinic to the origin, as sketched in Figure 1(a). Aperiodic motion on the Lorenz attractor has been used not only to represent reversals of the geomagnetic field but also to model stellar dynamos (Zel'dovich, Ruzmaikin and Sokoloff 1983; Schmalz and Stix 1991).

This is not the only mechanism that leads to chaos in the Lorenz system. In other applications (e.g. thermosolutal convection) the parameter σ may be negative;

for $r > 1$ there is a Hopf bifurcation from the trivial solution at $\sigma = -1$, giving rise to a periodic orbit which is destroyed in a heteroclinic bifurcation (where there is an orbit of infinite period connecting a symmetric pair of saddle-points). The saddles may develop into saddle-foci, so that the heteroclinic orbit takes the form shown in Figure 1(b). Behaviour then depends on the leading eigenvalues $\lambda, -\rho \pm i\omega$ at the saddle-foci: if $\lambda > \rho > 0$ the heteroclinic bifurcation leads to chaos (Shil'nikov 1965; Wiggins 1988). This mechanism, which is associated with cascades of period-doubling bifurcations, produces chaotic oscillations about the origin in the Lorenz system (e.g. Knobloch, Proctor and Weiss 1992).

Quasiperiodic solutions, with trajectories that lie on tori in phase space, exhibit more complicated behaviour. There are few unambiguous examples of chaos associated with the Ruelle-Takens mechanism, which relies on the structural instability of quasiperiodic orbits on n-tori with $n \geq 3$, although it is frequently invoked. Typically, the route to chaos proceeds via frequency-locking which gives rise to periodic orbits. Chaos is then caused by homoclinic or heteroclinic bifurcations such as those that have been described above. This pattern is illustrated by a complex generalization of the Lorenz equation, advanced as a model of nonlinear stellar dynamos (Jones, Weiss and Cattaneo 1985). The system

$$\dot{x} = \sigma(y - x),$$

$$\dot{y} = irx - y - x^*z,$$

$$\dot{z} = -\nu z + xy,$$

where the variables x, y, z are now complex, has a symmetry which allows it to be reduced to the fifth-order system

$$\dot{s} = s(u + u^*) - 2\sigma s,$$

$$\dot{u} = i(r - v) - u^2 + (\sigma - 1)u,$$

$$\dot{v} = -2isu - v(u - u^*) + \nu v,$$

where u, v are complex but s is real. For $\sigma = 1$, $\nu = 0.5$ there is a stationary bifurcation from the trivial solution at $r = 1$, which is followed by two Hopf bifurcations at $r = 2.07, 3.47$, giving rise to quasiperiodic motion on a two-torus. Frequency-locking leads to periodic solutions for finite intervals in r and, eventually, to a period-doubling cascade that is followed by chaotically modulated oscillations. Similar behaviour has been found in other simple dynamo models (Schmalz and Stix 1991; Feudel, Jansen and Kurths 1992).

The destruction of tori at heteroclinic bifurcations turns out to be a very complicated process. Kirk (1991, 1993) has studied the third-order system

$$\dot{s} = \lambda s + asz + cs^2 z \cos \phi,$$

$$\dot{\phi} = \omega - csz \sin \phi,$$

$$\dot{z} = \mu - z^2 - s^2 + bz^3,$$

where (s, ϕ, z) may be regarded as cylindrical polar co-ordinates, λ and μ are control parameters, and ω, a, b, c, are constants. When $c = 0$ the system is axisymmetric and we may restrict our attention to the dynamics in meridional planes. In the parameter range of interest there are two saddle-points on the z-axis, together with a symmetrical pair of foci, each of which undergoes a Hopf bifurcation at $\lambda = 0$, shedding a limit cycle, as sketched in Figure 2a. Each limit cycle swells until it forms a heteroclinic connection between a pair of saddle-points on the z-axis and is destroyed in a heteroclinic bifurcation, as shown in Figure 2b (Guckenheimer and Holmes 1983). In three dimensions the focus becomes a limit cycle and the periodic orbit becomes a torus but the dynamics is unaltered. Breaking axial symmetry by setting $c \neq 0$ introduces richer dynamical behaviour (cf. Langford 1983). The degeneracy of the heteroclinic bifurcations is broken: now there is a region in parameter space that is bounded by curves on which there are heteroclinic tangencies, as indicated in Figure 2c. Within this region there is a heteroclinic tangle that leads to chaotic behaviour, associated with frequency-locking in many overlapping tongues.

It is tempting to relate these simple models to the observed modulation of the solar cycle (Weiss, Cattaneo and Jones 1984). Recurrent grand minima in the ^{14}C record have a characteristic form and a well-defined mean period of around 200 yr, though they are not strictly periodic. Chaotic modulation of this kind can be ascribed to the "ghost" of a torus that influences the behaviour of a system. In this way, studying a toy model can clarify the dynamics of a much more complicated process.

3. Symmetry breaking

In models of planetary and stellar dynamos the azimuthally averaged magnetic field may possess a symmetry with respect to reflection about the equatorial plane. For fields with *dipole* symmetry the azimuthal (toroidal) component of the field is *antisymmetric* about the equator, while the azimuthal component of the vector potential is symmetric; for fields with *quadrupole* symmetry the azimuthal component of the field is *symmetric* and the vector potential is antisymmetric. The solar magnetic field has approximate dipole symmetry but significant deviations persist for many cycles. At the end of the Maunder minimum the lack of symmetry was much more conspicuous: of 54 sunspots recorded between 1671 and 1713 only two were in the northern hemisphere.

Axisymmetric mean-field dynamos have an α-effect that is antisymmetric, while the angular velocity is symmetric. The linear (kinematic) problem yields eigenfunctions corresponding either to dipole or to quadrupole solutions, with eigenvalues that may be real (associated with monotonically growing or decaying solutions) or complex (associated with oscillatory solutions). In the nonlinear regime there are steady or periodic solutions with pure dipole or quadrupole symmetry, lying on branches that bifurcate from the trivial (field-free) solution. Numerical studies have also revealed stable mixed-mode solutions on branches that emerge from symmetry-breaking bifurcations (e.g. Brandenburg *et al.* 1989; Schmitt and Schüssler 1989; Moss, Tuominen and Brandenburg 1990; Rädler *et al.* 1990). These asymmetric
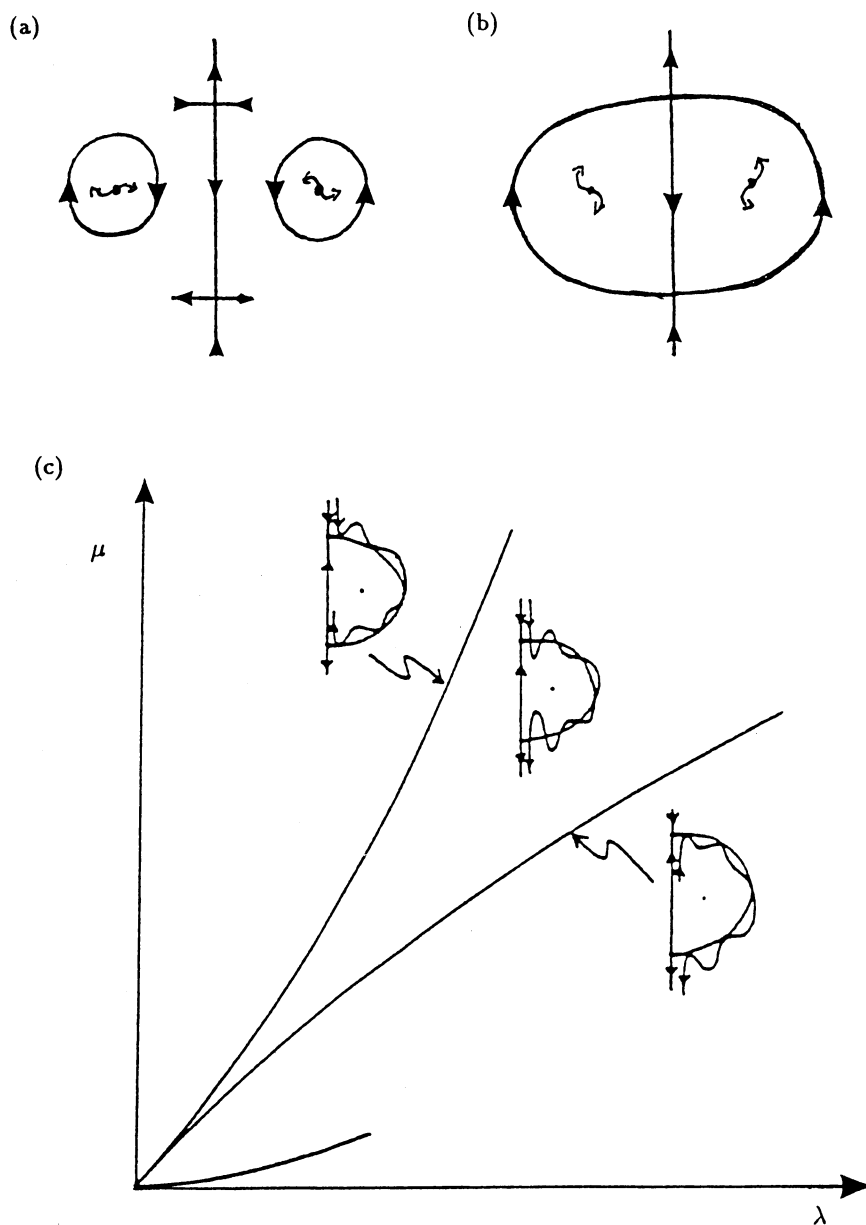
Fig. 2. Destruction of tori at heteroclinic bifurcations. (a) Limit cycles enclosing a symmetrical pair of unstable foci in the sz-phase plane. (b) Symmetrical pair of heteroclinic connections. (c) Bifurcations in the $\lambda\mu$-parameter space: within the tongue bounded by curves of first and last heteroclinic tangencies there are at least two heteroclinic orbits (after Kirk 1991).

mixed-mode solutions may persist indefinitely as the dynamo number is increased but they also provide a means of transferring stability from one branch of pure solutions to another. In order to investigate the underlying bifurcation structure it is essential to follow branches with unstable solutions as well as those with solutions that are stable. This, however, is only feasible for a low-order system. Hence it becomes necessary to construct a simplified model that captures the essential features of the full problem.

An obvious approach is to begin with an axisymmetric dynamo operating in a spherical shell (Schmitt and Schüssler 1989) and to reduce it to a plane model, with a toroidal field $B(x,t)$ and a vector potential $A(x,t)$ for the poloidal field such that $A = B = 0$ at the poles, where the colatitude $x = 0, \pi$ (cf. Stix 1972). Now consider the particular choice of nonlinear saturation mechanisms given by the dimensionless equations

$$\frac{\partial A}{\partial t} = \frac{D \cos x}{1 + \tau B^2} + \frac{\partial^2 A}{\partial x^2},$$

$$\frac{\partial B}{\partial t} = \frac{\sin x}{1 + \kappa B^2} + \frac{\partial^2 B}{\partial x^2} - \lambda B^3$$

(Jennings 1991; Jennings and Weiss 1991). Here D is the dynamo number, and α - quenching, ω-quenching and buoyancy losses are parametrized by τ, κ and λ, respectively.

This *system* possesses the dipole symmetry

$$d : (x, t) \rightarrow (\pi - x, t), \quad (A, B) \rightarrow (A, -B)$$

together with the quadrupole symmetry

$$q : (x, t) \rightarrow (\pi - x, t), \quad (A, B) \rightarrow (-A, B)$$

and these two symmetries generate the group D_2 of order four (which describes the symmetries of a rectangle). *Solutions* may possess one or more of these symmetries. The trivial solution possesses the full D_2 symmetry but dipole solutions, with $\partial A/\partial x = B = 0$ at $x = \pi/2$ (the equator), have the symmetry d only, while quadrupole solutions, with $A = \partial B/\partial x = 0$ at $x = \pi/2$, have the symmetry q.

This description can be extended to cover both steady and periodic solutions (Jennings and Weiss 1991). We consider solutions of period P (where P is arbitrary for steady solutions) and inspect them after an interval $\frac{1}{2}P$. Then steady solutions have the symmetry

$$t_e : (x, t) \rightarrow (x, t + \tfrac{1}{2}P), \quad (A, B) \rightarrow (A, B).$$

The symmetries $\{d, q, t_e\}$ generate the group D_{2h} of order eight (which describes the symmetries of a cuboid). This group includes three more spatiotemporal symmetries. Hopf bifurcations from the trivial solution yield oscillations with the symmetry

$$t_i : (x, t) \rightarrow (x, t + \tfrac{1}{2}P), \quad (A, B) \rightarrow (-A, -B);$$

hence oscillatory dipole solutions possess the symmetry

$$t_q = dt_i : (x, t) \rightarrow (\pi - x, t + \tfrac{1}{2}P), \quad (A, B) \rightarrow (-A, B),$$
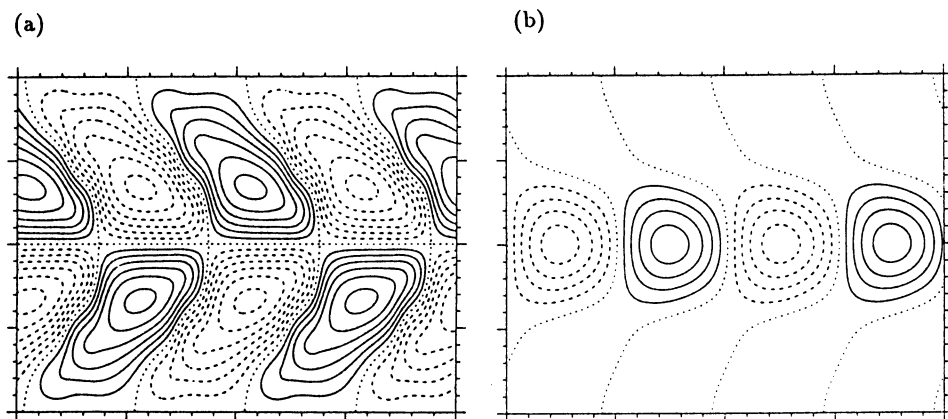
Fig. 3. Butterfly diagrams for (a) pure dipole (d) and (b) pure quadrupole (q) solutions. The toroidal field B is contoured as a function of colatitude and time.

while oscillatory quadrupole solutions possess the symmetry

$$t_d = q t_i : (x, t) \rightarrow (\pi - x, t + \tfrac{1}{2}P), \quad (A, B) \rightarrow (A, -B).$$

The trivial solution has the symmetry of the full group D_{2h}, whose invariant subgroups describe the symmetries of different nontrivial solutions. Pure dipole and quadrupole solutions have D_2 symmetry. Thus steady dipole (ds) solutions possess the symmetries $\{d, t_e, t_d\}$, while steady quadrupole (qs) solutions possess the symmetries $\{q, t_e, t_q\}$. However, steady mixed-mode (ms) solutions only have the Z_2 symmetry t_e. Oscillatory dipole (d) and quadrupole (q) solutions have symmetries $\{d, t_e, t_q\}$ and $\{q, t_i, t_d\}$ respectively. The symmetries of the toroidal field are illustrated in Figure 3, which shows examples of butterfly diagrams for pure dipole and quadrupole solutions. Breaking one of these symmetries yields a mixed-mode periodic solution, of which there are three different types. The first, of type mi with symmetry t_i, is illustrated in Figure 4a; the second, of type mq with symmetry t_q, is illustrated in Figure 4b; the third is of type md with symmetry t_d. The next symmetry-breaking bifurcation leads to periodic solutions of type e with no non-trivial symmetry, as illustrated in Figure 4c.

To solve the nonlinear equations it is convenient to represent A and B as finite Fourier series of the form
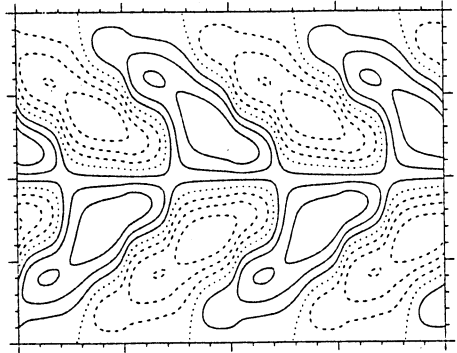
$$A = \sum_{n=1}^{N} A_n(t) \sin nx, \quad B = \sum_{n=1}^{N} B_n(t) \sin nx.$$

Then solutions with dipole symmetry have $A_n = 0$ (n even) and $B_n = 0$ (n odd), while those with quadrupole symmetry have $A_n = 0$ (n odd) and $B_n = 0$ (n even).
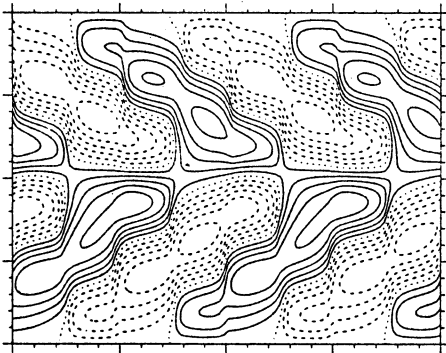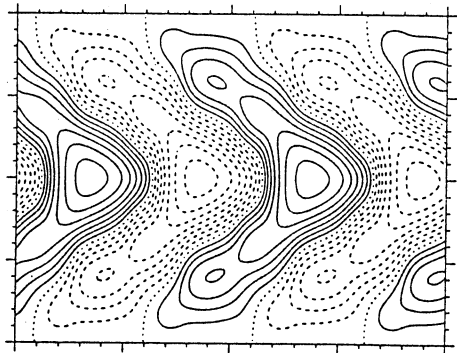
(a)
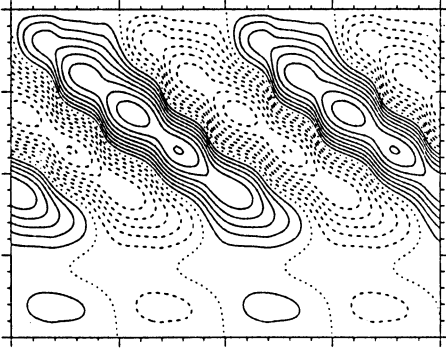
(b)

(c)

(d)

Fig. 4. Butterfly diagrams for mixed-mode periodic solutions: (a) type mi with symmetry t_i; (b) type mq with symmetry t_q; (c) type e with no non-trivial symmetry. (d) Development of spatial structure at large $|D|$ in a solution of type q. (After Jennings and Weiss 1991.)

In order to follow both stable and unstable solution branches and so to construct a bifurcation diagram, these series were severely truncated by taking $N = 7$. For $D < 0$ linear theory then gives bifurcations from the trivial solution at $|D| = 9, 102, 246, 273, 474, 1152, \ldots$ to solutions of types qs, d, qs, qs and d, respectively.

The nonlinear results obtained for $\kappa = \lambda = 1, \tau = 0$ are summarized in Figure 5. Details of this bifurcation diagram are of course sensitive to the many simplifying assumptions that were made in order to produce it. Nevertheless, it exhibits many qualitative features that are generic and might appear in any mean-field dynamo. Although the first mode to become unstable is of type qs the steady quadrupole solutions undergo a Hopf bifurcation and stability is transferred to oscillatory dipole solutions via a branch of mixed-mode solutions of type mq. Note that these solutions possess the symmetry t_q, which is the only non-trivial symmetry shared by solutions of types qs and d. Similarly, the transfer of symmetry from pure dipole to pure quadrupole oscillations has to involve a branch of mixed-mode oscillations of type mi with the symmetry t_i.

The bifurcation diagram displays several typical features of nonlinear systems. The preferred solution does not necessarily lie on the first branch to bifurcate from the trivial solution. Indeed, there are several stable solutions for certain parameter ranges. In this system, on the other hand, most of the structure involves interactions between the two principal branches of periodic dipole and quadrupole solutions, corresponding to the linear modes with highest growth-rates. The periods of the nonlinear solutions decrease as the magnitude of the dynamo number is increased, while they gradually acquire more complicated spatial structures : for example, the quadrupole solution for $|D| = 225$, shown in Figure 4d, is more complex than that for $|D| = 3500$ in Figure 3b. These features are all likely to appear in any axisymmetric spherical dynamo. Moreover, the same approach can be extended to non-axisymmetric ($m = 1$) solutions (Gubbins and Zhang 1992). A systematic procedure for describing non-axisymmetric instabilities in rotating systems is outlined by Knobloch (1993).

4. Conclusion

The examples presented here show how highly simplified model problems can illuminate aspects of nonlinear behaviour in more realistic configurations. The art lies in formulating an appropriate model. In studies of thermal convection this can be achieved by following a systematic procedure which leads to normal form equations that are valid in some asymptotic limit. Sometimes the most interesting behaviour occurs outside this domain of validity – yet even if the bifurcations are a consequence of severe truncation they may be relevant to the behaviour of the full system in some more extreme parameter regime. The Lorenz equations, for instance, are significant, not as an approximate representation of Rayleigh-Bénard convection but because of the rich behaviour they exhibit.

The difficulty with stellar dynamos is that there is no reliable procedure for deriving model systems. Without some separation of scales it is impossible to justify mean field dynamo theory in a star. So we are reduced to constructing *ad hoc* model systems, guided by the observed behaviour of the large-scale field in the Sun.
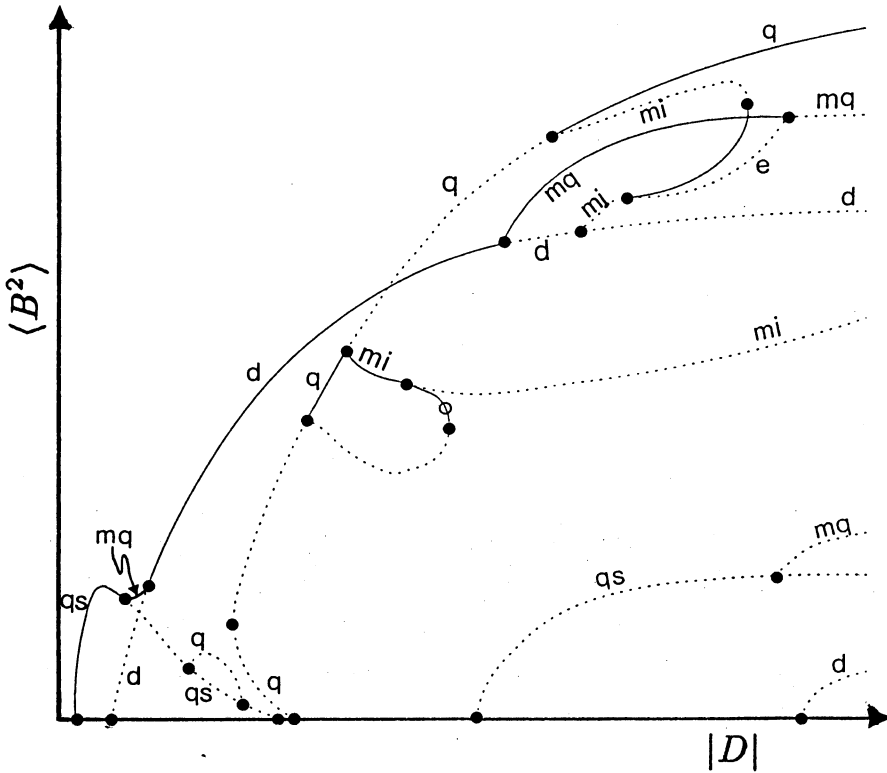
Fig. 5. Bifurcation diagram for the truncated model problem. $\langle B^2 \rangle$ is plotted as a function of $|D|$ (not to scale) along different solution branches, labelled according to the solution type. Stable (unstable) solutions are denoted by full (broken) lines and full (hollow) circles indicate local (global) bifurcations. (After Jennings and Weiss 1992.)

We infer from these models that cyclic dynamos are likely to have shorter mean periods and to generate magnetic fields with more complicated spatial structures in rapidly rotating stars. However, it is not clear how far we can use toy models to probe nonlinear saturation processes e.g. by calculating the variation of the cycle frequency with dynamo number (Noyes, Weiss and Vaughan 1984; Jennings and Weiss 1991). At present we can only hope that such semi-quantitative calculations will yield reliable predictions; otherwise we shall have to wait until it becomes feasible to compute a fully nonlinear three-dimensional model of a stellar dynamo.

Acknowledgements

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