A THEORY OF CONVERGENCE

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The literature already contains several theories of limits which have great generality (**10**; **12**; **1**; **5**; **6**; **13**; **3**; **14**; **4**, p. 34). Nevertheless, the intrinsic importance and frequent use of the concept may justify the publication of another variant, provided that it has advantages in ease of application without sacrifice of generality. The theory of convergence studied in this note includes the other theories and their applications in a smooth way, without artifice.

1. Definition. If f is a function on a topological space D to a topological space R, and $a \in D$ and $b \in R$, the standard definition of

$$\lim_{x \to a} f(x) = b$$

can be phrased thus. Let $\mathfrak{N}$ be the family of all neighborhoods of a and $\mathfrak{Y}$ the family of all neighborhoods of b; to each $Y \in \mathfrak{Y}$ corresponds $N \in \mathfrak{N}$ such that if $x \in N$ then $f(x) \in Y$. Similarly, if the domain D of f is directed by $\succ$, so that $f, \succ$ is a net of points of S, the convergence of this net to $b \in S$ can be thus expressed. Each $n \in D$ defines a "final section" $D[\succ n]$ of D, where $D[\succ n]$ means the set $\{m \mid m \in D \,\&\, m \succ n\}$. Let $\mathfrak{N}$ consist of all final sections of D, and let $\mathfrak{Y}$ consist of all neighborhoods of b. Then

$$\lim_{n, \succ} f(n) = b$$

if and only if to each $Y \in \mathfrak{Y}$ corresponds a final section $D[\succ n] \in \mathfrak{N}$ such that if $m \in D[\succ n]$ then $f(m) \in \mathfrak{Y}$. These definitions and others too have the common form that we have given a function f, a family $\mathfrak{N}$ of subsets of the domain D_f of f, and another family $\mathfrak{Y}$ of sets, and the convergence of the function takes place if to each set $Y \in \mathfrak{Y}$ corresponds a set $N \in \mathfrak{N}$ such that if $x \in N$ then $f(x) \in Y$. Under these circumstances we shall say that "f converges over $\mathfrak{N}$ into $\mathfrak{Y}$," or "$f(x)$ converges into $\mathfrak{Y}$ as x converges into $\mathfrak{N}$."

Some theorems can be proved with no further assumptions. However, we prefer to devote most of our attention to families $\mathfrak{N}$ having properties which will enable us to prove a few of the most elementary theorems concerning limits of real-valued functions. Specifically, let $\mathfrak{N}$ be a family of subsets of some set D. We wish to find the weakest hypotheses on $\mathfrak{N}$ that will enable us to prove the following three statements.

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(i) *Some real-valued function on D converges over $\mathfrak{N}$ to some real number.*
(ii) *Some real-valued function on D does not converge over $\mathfrak{N}$ to every real number.*
(iii) *If f_1 and f_2 are real-valued functions on D, both of which converge over $\mathfrak{N}$ to* 1, *then $f_1 + f_2$ converges over $\mathfrak{N}$ to* 2.

It is easily seen that (i) is satisfied if and only if $\mathfrak{N}$ is non-empty, and (ii) is satisfied if and only if each $N \in \mathfrak{N}$ is non-empty. To investigate the effect of (iii), we let N_1 and N_2 be any members of $\mathfrak{N}$, and let f_1 and f_2 be their respective characteristic functions; then $f_1(x)$ and $f_2(x)$ converge over $\mathfrak{N}$ to 1. If (iii) holds, their sum converges to 2; therefore there exists N_3 in $\mathfrak{N}$ such that if $x \in N_3$, then

$$|f_1(x) + f_2(x) - 2| < 1 .$$

But this last statement holds on the intersection $N_1 \cap N_2$ and nowhere else. Hence $N_3 \subset N_1 \cap N_2$. Conversely, if $N_1 \cap N_2$ contains a member of $\mathfrak{N}$, $f_1 + f_2$ converges over $\mathfrak{N}$ to 2, and (iii) holds.

In the terminology of H. Cartan [**5**], a family $\mathfrak{A}$ of sets is a *filter-base* if and only if $\mathfrak{A}$ is a non-empty family of non-empty sets such that if A and B are in $\mathfrak{A}$, there exists a set C of $\mathfrak{A}$ which is contained in A and in B. Therefore the results of the preceding paragraph may be summarized thus: *in order that* (i), (ii), *and* (iii) *be satisfied, it is necessary and sufficient that $\mathfrak{N}$ be a filter-base.*

If $\mathfrak{N}$ is a filter-base in the domain D_f of a function f, the range of f (structureless in itself) is organized or systematized by classifying its points into the images of the various N of $\mathfrak{N}$. Since σύνταξις means "putting together in order, arranging, . . . ; system, arrangement, organization, . . . " (Liddell and Scott, *Greek-English Lexicon*), we adopt the following definition.

(1) Definition. *A* syntax *is a system $\langle f; \mathfrak{N}\rangle$ in which f is a function and $\mathfrak{N}$ is a filter-base in the domain D_f of f. When $\langle f; \mathfrak{N}\rangle$ is a syntax and the values of f are in a set S, $\langle f; \mathfrak{N}\rangle$ is a* syntax of points of S.

Thus, for example, if $\mathfrak{N}$ consists of all neighborhoods of the number π, the syntax $\langle \sin; \mathfrak{N}\rangle$ is an appropriate tool in the study of the behavior of the sine-function near π. To study its behavior near some other point x_0 we would need to replace $\mathfrak{N}$ by the family of neighborhoods of x_0, producing a different syntax.

Not all our theorems require the full strength of definition (1). Definitions and theorems (2) to (12), except for (8) and with (12 ii) removed, are still valid if in (1) we replace the words "filter-base in" by "family of subsets of." Moreover, (8) and (12) in full hold if $\mathfrak{N}$ is assumed to be a family of subsets of D_f such that each pair of sets in N has a non-empty intersection. However, there does not appear to be enough advantage in this extra generality to warrant giving it any further attention.

(2) Definition. *If $\langle f; \mathfrak{N}\rangle$ is a syntax and $\mathfrak{Y}$ is a family of sets, $\langle f; \mathfrak{N}\rangle$* converges into $\mathfrak{Y}$ *if and only if to each set $Y \in \mathfrak{Y}$ corresponds a set $N \in \mathfrak{N}$ such that if $x \in N$, then $f(x) \in Y$.*

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Alternative wordings for "$\langle f;\, \mathfrak{N}\rangle$ converges into $\mathfrak{Y}$" are "f converges over $\mathfrak{N}$ into $\mathfrak{Y}$" and "$f(x)$ converges into $\mathfrak{Y}$ as x converges into $\mathfrak{N}$."

In many applications there is a unique point common to the sets Y in $\mathfrak{Y}$. This point, determined by $\mathfrak{Y}$, we could designate by $[\mathfrak{Y}]$. Even when the family $\mathfrak{Y}$ does not determine such a unique point, it still may be regarded as defining an ideal point $[\mathfrak{Y}]$ in some space of sets. A similar statement holds for $\mathfrak{N}$. Accordingly, the sentence "$\langle f;\, \mathfrak{N}\rangle$ converges into $\mathfrak{Y}$" can be symbolized in any one of the following three ways:

$$f(x) \to [\mathfrak{Y}] \text{ as } x \to [\mathfrak{N}], \qquad \langle f;\, \mathfrak{N}\rangle \to [\mathfrak{Y}],$$

$$\lim_{x \to [\mathfrak{N}]} f(x) = [\mathfrak{Y}].$$

(For this suggestion, I owe thanks to the referee.)

Remark 1. If we wish to discuss multiple-valued functions f, we could correspondingly change Definition 2, replacing the final "$f(x) \in Y$" by "all values of $f(x)$ are in Y." This would furnish a generalization of a mode of convergence of multiple-valued functions which has occasionally been used.

Remark 2. We can conveniently adapt Halmos's terminology, saying that a statement concerning $f(x)$ is "eventually" true if it is true of $f(x)$ for all x in some set $N \in \mathfrak{N}$. Thus (2) becomes "$\langle f;\, \mathfrak{N}\rangle \to [\mathfrak{Y}]$ if and only if for each $Y \in \mathfrak{Y}$, $f(x)$ is eventually in Y."

Remark 3. If a syntax $\langle f;\, \mathfrak{N}\rangle$ converges into $\mathfrak{Y}$, we may assume without loss of generality that $\mathfrak{Y}$ is a filter-base, since the set $\mathfrak{Y}'$ of all intersections of finite subfamilies of $\mathfrak{Y}$ is then a filter-base and $\langle f;\, \mathfrak{N}\rangle \to [\mathfrak{Y}']$.

It is easy to see that the usual theorems on limits of sums, products, etc., of real functions can be established in the customary manner. If S is a set partially ordered by $\succ$, let $D[\succ a]$ denote the set $\{s \mid s \in S \,\&\, s \succ a\}$. Such sets will be called "final sections" of S. If we wish to avoid adjoining ∞ to the reals, we can notice that "$f(x) \to \infty$" is the same as "$f(x)$ converges into the final sections of the real number system." The final sections of S form a filter-base if and only if S is directed by $\succ$, so convergence of nets is a special case of convergence of syntaxes. The convergence defined by Bennett **(1)** is similarly covered. If f is on a topological space D_f to the extended real numbers, it is lower semi-continuous at a if and only if as $x \to a$, $f(x)$ converges into the family $\mathfrak{Y}$ of open sets containing $\{y \mid y \geqslant f(a)\}$.

2. Subsyntaxes. In order to define the concept of subsyntax it is convenient first to define the relation "finer than."

(3) Definition. *Let $\mathfrak{M}$, $\mathfrak{N}$ be families of sets. Then $\mathfrak{M}$ is* finer than *$\mathfrak{N}$ if and only if each $N \in \mathfrak{N}$ contains a set $M \in \mathfrak{M}$.*

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(4) DEFINITION. *Let f be a function, and let $\mathfrak{M}$, $\mathfrak{N}$ be filter-bases in the domain of f. Then $\langle f; \mathfrak{M}\rangle$ is a* subsyntax *of $\langle f; \mathfrak{N}\rangle$ if and only if $\mathfrak{M}$ is finer than $\mathfrak{N}$.*

(5) COROLLARY. *If $\langle f; \mathfrak{N}''\rangle$ is a subsyntax of $\langle f; \mathfrak{N}'\rangle$, and the latter is a subsyntax of $\langle f; \mathfrak{N}\rangle$, then $\langle f; \mathfrak{N}''\rangle$ is a subsyntax of $\langle f; \mathfrak{N}\rangle$.*

(6) COROLLARY. *If $\langle f; \mathfrak{M}\rangle$ is a subsyntax of $\langle f; \mathfrak{N}\rangle$, and $\mathfrak{Y}$ is a family of sets, and f converges over $\mathfrak{N}$ into $\mathfrak{Y}$, then f also converges over $\mathfrak{M}$ into $\mathfrak{Y}$.*

As a special case, suppose that $\langle f; \mathfrak{N}\rangle$ is a syntax and A is a subset of D_f such that $A \cap N$ is non-empty for every $N \in \mathfrak{N}$. We define $\mathfrak{M}$ to be the family of all such intersections $A \cap N$; then $\mathfrak{M}$ is easily seen to be a filter-base which is finer than $\mathfrak{N}$. Hence $\langle f; \mathfrak{M}\rangle$ is a subsyntax of $\langle f; \mathfrak{N}\rangle$. When D_f is the set of positive integers, and A is an infinite subset $\{n_1, n_2 \ldots\}$ of D_f, the subsyntax $\langle f; \mathfrak{M}\rangle$ does not differ in any essential way from the traditional subsequence

$$\{f(n_1), f(n_2), \ldots\}.$$

A similar statement holds if D_f is directed by a relation $\succ$ and A is a cofinal subset. But this method of constructing subsyntaxes is by no means the only one possible, as we shall see in several later theorems (e.g., Theorems (11) and (14)).

3. Cluster-points. Our next definition generalizes the concept of cluster-point of a sequence.

(7) DEFINITION. *Let $\langle f; \mathfrak{N}\rangle$ be a syntax and $\mathfrak{Y}$ a family of sets. Then $\langle f; \mathfrak{N}\rangle$* clusters at $[\mathfrak{Y}]$ *if and only if for each $N \in \mathfrak{N}$ and each $Y \in \mathfrak{Y}$ there exists $x \in N$ such that $f(x) \in Y$.*

In particular, when $\mathfrak{Y}$ is the family of neighborhoods of a point p in a topological space and $\langle f; \mathfrak{N}\rangle$ clusters at $[\mathfrak{Y}]$ we say that p is a cluster-point of $\langle f; \mathfrak{N}\rangle$.

(8) COROLLARY. *If $\mathfrak{Y}$ is a family of sets and $\langle f; \mathfrak{N}\rangle$ a syntax which converges into $\mathfrak{Y}$, then $\langle f; \mathfrak{N}\rangle$ clusters at $[\mathfrak{Y}]$.*

For if $N \in \mathfrak{N}$ and $Y \in \mathfrak{Y}$, there exists $N' \in \mathfrak{N}$ such that $f(N') \subset Y$; N and N' have a point x in common, for which $x \in N$ and $f(x) \in Y$.

(9) COROLLARY. If *$\langle f; \mathfrak{M}\rangle$ is a subsyntax of $\langle f; \mathfrak{N}\rangle$, and $\langle f; \mathfrak{M}\rangle$ clusters at $[\mathfrak{Y}]$, so does $\langle f; \mathfrak{N}\rangle$.*

For let $Y \in \mathfrak{Y}$ and $N \in \mathfrak{N}$. Then N contains a set M of the family $\mathfrak{M}$, and there exists $x \in M$ such that $f(x) \in Y$.

The following rather obvious statement is needed in several proofs.

(10) LEMMA. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be filter-bases such that for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ the intersection $A \cap B$ is non-empty. Let $\mathfrak{C}$ consist of all sets of the form $A \cap B$, $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. Then $\mathfrak{C}$, $\mathfrak{A} \cup \mathfrak{C}$ and $\mathfrak{A} \cup \mathfrak{B} \cup \mathfrak{C}$ are filter-bases.*

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(11) THEOREM. *If $\langle f; \mathfrak{N}\rangle$ is a syntax and $\mathfrak{Y}$ is a filter-base and $\langle f; \mathfrak{N}\rangle$ clusters at $[\mathfrak{Y}]$, there exists a subsyntax $\langle f; \mathfrak{M}\rangle$ of $\langle f; \mathfrak{N}\rangle$ which converges into $\mathfrak{Y}$.*

Proof. Let $\mathfrak{M}$ consist of all intersections $N \cap f^{-1}(Y)$ with $N \in \mathfrak{N}$ and $Y \in \mathfrak{Y}$. By (10) this is a filter-base, and it is obviously finer than $\mathfrak{N}$; and for each Y in $\mathfrak{Y}$, for all x in $N \cap f^{-1}(Y)$ (N an arbitrary fixed member of $\mathfrak{N}$) we find $f(x)$ in Y.

Remark 1. Under the hypotheses of (11) there exists a net $(g(\alpha) \mid \alpha \in A)$, $\succ$ of points of D_f such that $g, \succ$ converges into $\mathfrak{N}$ and $(f(g(\alpha)) \mid \alpha \in A)$, $\succ$ converges into $\mathfrak{Y}$. For we can choose A to be the family $\mathfrak{M}$ of intersections and for each $\alpha \in A$ choose $g(\alpha) \in \alpha$.

Remark 2. Still under the hypotheses of (11), if $\mathfrak{N}$ and $\mathfrak{Y}$ have countable subsets $\{N_i \mid i = 1, 2, \ldots \}$ and $\{ Y_i \mid i = 1, 2, \ldots \}$ which are finer than $\mathfrak{N}$ and $\mathfrak{Y}$ respectively, there is a sequence $(g(m) \mid m = 1, 2, \ldots)$ of points D_f such that as $m \to \infty$, $g(m)$ converges into $\mathfrak{N}$ and $f(g(m))$ into $\mathfrak{Y}$. For we can choose

$$g(m) \in X_m \cap f^{-1}(Y_m), \qquad m = 1, 2, \ldots .$$

4. Uniqueness. Let us first dispose of the somewhat trivial question of the convergence of syntaxes $\langle f; \mathfrak{N}\rangle$ such that f has a constant value p on some set N of the family $\mathfrak{N}$. It is evident that such a syntax converges into the sets of a family $\mathfrak{Y}$ if and only if p is in every set Y of $\mathfrak{Y}$. In particular, if $\mathfrak{Y}$ is the family of all neighborhoods of a point q in a topological space S, this syntax $\langle f; \mathfrak{N}\rangle$ converges to q if and only if p is in every neighborhood of q. Thus if p is in a topological space S, and $\langle f; \mathfrak{N}\rangle$ is a syntax, and f is constantly equal to p on some set N of $\mathfrak{N}$, then $\langle f; \mathfrak{N}\rangle$ converges to p; and it converges to p alone if S is a T_1-space.

In the next theorem and several others we need to make use of the identity-function, which at each x in the universe of discourse has the functional value x. For this function we shall use the name id; thus id $x = x$ for all x.

(12) THEOREM. *Let $\mathfrak{Y}, \mathfrak{Y}'$ be filter-bases. The following conditions are equivalent.*
(i) *For every $Y \in \mathfrak{Y}$ and every $Y' \in \mathfrak{Y}$, $Y \cap Y'$ is non-empty.*
(ii) *There exists a syntax $\langle f; \mathfrak{N}\rangle$ which converges into $\mathfrak{Y}$ and into $\mathfrak{Y}'$.*
(iii) *There exists a syntax $\langle f; \mathfrak{N}\rangle$ which converges into $\mathfrak{Y}$ and clusters at* $[\mathfrak{Y}']$.

Proof. By (8), (ii) implies (iii). Let (iii) hold, and let $Y \in \mathfrak{Y}$ and $Y \in \mathfrak{Y}'$. For some $N \in \mathfrak{N}$, $f(N) \subset Y$, and there exists $x \in N$ such that $f(x) \in Y'$, so $f(x) \in Y \cap Y'$, and (i) holds. If (i) holds, let $\mathfrak{N}$ be the set of all intersections $Y \cap Y'$ with $Y \in \mathfrak{Y}$ and $Y' \in \mathfrak{Y}'$. By (10), this is a filter-base; it is finer than $\mathfrak{Y}$ and $\mathfrak{Y}'$, so $\langle \mathrm{id}; \mathfrak{N}\rangle$ is a subsyntax of $\langle \mathrm{id}; \mathfrak{Y}\rangle$ and $\langle \mathrm{id}; \mathfrak{Y}'\rangle$. Since it is evident that these syntaxes converge into $\mathfrak{Y}$ and $\mathfrak{Y}'$ respectively, by (6) $\langle \mathrm{id}; \mathfrak{N}\rangle$ converges into $\mathfrak{Y}$ and into $\mathfrak{Y}'$. Hence (i) implies (ii).

In particular, if $\mathfrak{Y}$ and $\mathfrak{Y}'$ are the families of neighborhoods of points p, p' in a topological space, there exists a syntax $\langle f; \mathfrak{N}\rangle$ converging to both p and p' (or converging to one and having the other as cluster-point) if and only if

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every neighborhood of p meets every neighborhood of p'. This implies $p = p'$ if and only if the space is a Hausdorff space.

5. Decided Syntaxes. We adapt to syntaxes an expressive phrase of Tukey's **(14)**. If $\langle f; \mathfrak{N}\rangle$ is a syntax and Y a set, the syntax *decides for* Y if $f(x)$ is eventually in Y, and *decides against* Y if $f(x)$ is eventually not in Y; in either case, $\langle f; \mathfrak{N}\rangle$ *decides about* Y. A syntax will be called a *decided syntax* if it decides about every set Y.

(13) DEFINITION. *A syntax $\langle f; \mathfrak{N}\rangle$ is* decided *if to every set Y corresponds $N \in \mathfrak{N}$ having one of the following properties*:

(i) *for all* $x \in N$, $f(x) \in Y$; *or*

(ii) *for all* $x \in N$, $f(x) \notin Y$.

The "decided syntaxes" are the analogues of the "ultra-filters" of Cartan **(4)** and the "universal nets" of Kelley **(7)**.

Remark 1. Clearly (13) is unaltered if we consider only sets Y contained in the range of f.

Remark 2. The syntax $\langle f; \mathfrak{N}\rangle$ is decided if and only if the images $f(N)$, $N \in \mathfrak{N}$ form an ultra-filter.

(14) THEOREM. *Every syntax has a decided subsyntax.*

Let $\langle f; \mathfrak{N}\rangle$ be a syntax. The collection Φ of all filter-bases in D_f which contain $\mathfrak{N}$ is partially ordered by $\supset$. If Φ_0 is a linearly ordered subset of Φ, the union $\mathfrak{U}$ of all the members of Φ_0 is itself a member of Φ. By the Hausdorff maximal principle (Zorn's lemma) there exists a maximal member $\mathfrak{M}$ of Φ. Then $\langle f; \mathfrak{M}\rangle$ is a syntax, and since each $N \in \mathfrak{N}$ contains (in fact, is) a member of M, $\langle f; \mathfrak{M}\rangle$ is a subsyntax of $\langle f; \mathfrak{N}\rangle$. To prove it a decided syntax let Y be any set. If there is no $M \in \mathfrak{M}$ such that $f(M) \cap Y$ is empty, let $\mathfrak{B}$ consist of $f^{-1}(Y)$ alone. This is a filter-base, and for all $M \in \mathfrak{M}$ and all $B \in \mathfrak{B}$ we know that $M \cap B$ is not empty. The set $\mathfrak{C}$ of all such intersections is a filter-base by (10), and so is $\mathfrak{M} \cup \mathfrak{B} \cup \mathfrak{C}$. This contains the maximal filter-base $\mathfrak{M}$, so it is identical with $\mathfrak{M}$, and $\mathfrak{B} \subset \mathfrak{M}$; that is, $f^{-1}(Y)$ is a member M' of $\mathfrak{M}$. Hence $f(M') \subset Y$, which completes the proof.

(15) THEOREM. *If $\mathfrak{Y}$ is a family of sets, and a decided syntax $\langle f; \mathfrak{N}\rangle$ clusters at* $[\mathfrak{Y}]$, *it converges into* $\mathfrak{Y}$.

For each $Y \in \mathfrak{Y}$, $f(N) \cap Y$ is non-empty for all $N \in \mathfrak{N}$, so by (13) there exists $N \in \mathfrak{N}$ such $f(N) \subset Y$.

6. Characterizations of Compactness. Syntaxes serve as well as nets or filters in characterizing compactness.

(16) THEOREM. *Let S be a topological space. Then the following statements are equivalent.*

(i) *S is compact.*

(ii) *Every syntax of points of S has a cluster point in S.*

(iii) *For every function f on S to S and every filter-base $\mathfrak{N}$ in S, the syntax $\langle f; \mathfrak{N}\rangle$ has a cluster point in S.*

(iv) *For every filter-base $\mathfrak{N}$ in S, the syntax $\langle \mathrm{id}; \mathfrak{N}\rangle$ has a cluster-point in S.*

(v) *For every basis $\mathfrak{B}$ of closed sets in S and every filter-base $\mathfrak{N} \subset \mathfrak{B}$, the syntax $\langle \mathrm{id}; \mathfrak{N}\rangle$ has a cluster-point in S.*

(vi) *There exists a basis $\mathfrak{B}$ of closed sets in S such that for every filter-base $\mathfrak{N} \subset \mathfrak{B}$, the syntax $\langle \mathrm{id}; \mathfrak{N}\rangle$ has a cluster-point in S.*

(vii) *Every syntax $\langle f; \mathfrak{N}\rangle$ of points of S has a subsyntax which converges to a point of S.*

(viii) *Every decided syntax of points of S converges to a point of S.*

Proof. (i) $\to$ (ii). Let $\langle f; \mathfrak{N}\rangle$ be a syntax of points of a compact space S. The set of all images $f(N)$, $N \in \mathfrak{N}$ has the finite intersection property, and therefore so has the set of closures $\overline{f(N)}$, $N \in \mathfrak{N}$. Since S is compact, there exists a point p common to all these closures. For each $N \in \mathfrak{N}$, $p \in \overline{f(N)}$, so for each neighborhood U of p the intersection $U \cap f(N)$ is non-empty, and p is a cluster-point of $\langle f; \mathfrak{N}\rangle$.

(ii) $\to$ (iii) $\to$ (iv) $\to$ (v) $\to$ (vi). Obvious.

(vi) $\to$ (i). Let $\mathfrak{A}$ be a family of closed subsets of S having the finite intersection property. Define $\mathfrak{A}'$ to be the family of all sets which are intersections of finitely many members of $\mathfrak{A}$; this too has the finite intersection property. Let $\mathfrak{N}$ consist of all members of the basis $\mathfrak{B}$ which contain a set $A' \in \mathfrak{A}'$; this is a filter-base, and by (vi) the syntax $\langle \mathrm{id}; \mathfrak{N}\rangle$ has a cluster-point p in S. For each N in $\mathfrak{N}$, every neighborhood of p meets id $N = N$, so $p \in \bar{N} = N$. Thus p is in all sets $N \in \mathfrak{N}$. If $A \in \mathfrak{A}$, it is the intersection of all sets of $\mathfrak{B}$ which contain A. But all these sets are in $\mathfrak{N}$, hence contain p, and thus $p \in A$.

From (15), (ii) $\to$ (viii); from (14), (viii) $\to$ (vii). From (8) and (9), (vii) $\to$ (ii). This completes the proof.

Remark 1. While (ii) is a useful consequence of compactness, it is undesirable as a test for compactness, since the family of all syntaxes of points of S is as numerous as the class of all sets. This is the principal reason for including (iii), (iv), (v), and (vi); the cardinal number of syntaxes involved in these criteria can be estimated in terms of the cardinality of S, the last being the least. (Criteria (v) and (vi) were suggested by the referee.)

7. Tychonoff's Theorem. Let A be a non-empty set, and for each $\alpha \in A$ let S_α be a topological space. The cartesian product $\prod_\alpha S_\alpha$ of these S_α is by definition the set of all functions ϕ on A such that for each $\alpha \in A$, $\phi(\alpha) \in S_\alpha$. For each element ϕ of $\prod_\alpha S_\alpha$ and each $\alpha \in A$, the value of ϕ at α is called the *component* of ϕ in S_α and denoted by ϕ_α. The product is topologized by defining neighborhoods to be cartesian products $\prod_\alpha U_\alpha$ in which for finitely many α, U_α is a neighborhood in S_α, and for all other $\alpha \in A$, $U_\alpha = S_\alpha$; the product thus topologized is the "topological product" of the S_α.

If $\langle f; \mathfrak{N}\rangle$ is a syntax of points of $\prod_\alpha S_\alpha$ and for each α in A the component f_α of f is understood to be the function $(f_\alpha(x) \mid x \in D_f)$, then $\langle f_\alpha; \mathfrak{N}\rangle$ is a syntax of points of S_α. If $\langle f; \mathfrak{N}\rangle$ is a decided syntax so is $\langle f_\alpha; \mathfrak{N}\rangle$, as we now show. Let Y_α be any set in the range of f_α, and let Y be the cartesian product $\prod_\beta Z_\beta$, where $Z_\alpha = Y_\alpha$ and $Z_\beta = S_\beta$ for $\beta \in A$, $\beta \neq \alpha$. Then $f(x) \in Y$ if and only if $f_\alpha(x) \in Y_\alpha$. There exists an $N \in \mathfrak{N}$ on which the statement $f(x) \in Y$ is invariably true or invariably false; correspondingly the statement $f_\alpha(x) \in Y_\alpha$ is true for all $x \in X$ or false for all $x \in X$, and so $\langle f_\alpha; \mathfrak{N}\rangle$ is a decided syntax.

From this we obtain a proof of Tychonoff's theorem which is essentially a copy of Cartan's **(5)**.

(17) THEOREM. *If for each $\alpha \in A$ the set S_α is a compact topological space, the topological product of the S_α is also compact.*

Let $\langle f; \mathfrak{N}\rangle$ be a decided syntax of points of the product space. For each $\alpha \in A$, $\langle f_\alpha; \mathfrak{N}\rangle$ is a decided syntax of points of S_α, so by (16) it converges to a point p_α of S_α. Let $p = (p_\alpha \mid \alpha \in A)$; this is a point of $\prod_\alpha S_\alpha$. Now let U be a neighborhood $\prod_\alpha U_\alpha$ of p, where for all α in a finite set $B \subset A$ the set U_α is a neighborhood of p_α in S_α, and for all $\alpha \in A - B$, $U_\alpha = S_\alpha$. For $\alpha \in B$, there exists $N_\alpha \in \mathfrak{N}$ such that $f_\alpha(N_\alpha) \subset U_\alpha$. Since $\mathfrak{N}$ is a filter-base, there exists $N \in \mathfrak{N}$ which is contained in all the N_α, $\alpha \in B$. Then $f_\alpha(N) \subset U_\alpha$, $\alpha \in B$, so $f(N) \subset U$. This proves $\langle f; \mathfrak{N}\rangle$ converges to p, and by (16) the topological product is compact.

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