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A NEW COHOMOLOGICAL CRITERION FOR THE p-NILPOTENCE OF GROUPS

MAURIZIO BRUNETTI

ABSTRACT. Let G be a finite group, H a copy of its p-Sylow subgroup, and $K(n)^*(-)$ the n-th Morava K-theory at p. In this paper we prove that the existence of an isomorphism between $K(n)^*(BG)$ and $K(n)^*(BH)$ is a sufficient condition for G to be p-nilpotent.

1**. Introduction and statement of results.** Let p be any prime number. A finite group G is said to be p-*nilpotent* if the elements of order prime to p form a (normal) subgroup N. In this case the quotient G/N is obviously isomorphic to a p-Sylow subgroup H of G. Let $h^*(-)$ be any mod p or p-local cohomology theory. For any group G the restriction homomorphism

$$h^*(Bi)\colon h^*(BG) \longrightarrow h^*(BH)$$

is injective, and it is additionally surjective if G is p-nilpotent.

In the past, several people were interested in results going in the other direction. Tate proved in [11] that if $H^1(BG;\mathbb{Z}/p)$ and $H^1(BH;\mathbb{Z}/p)$ are isomorphic, then G is p-nilpotent. On the other hand a theorem by Atiyah whose proof is sketched in [8] states that the existence of an isomorphism between $H^i(BG;\mathbb{Z}/p)$ and $H^i(BH;\mathbb{Z}/p)$ for all sufficiently large i is also a sufficient condition for G to be p-nilpotent. Finally, by arguments related to the celebrated Atiyah's description of $K^*(BG)$, the complex K-theory of the classifying space of a group G in terms of its complex representation ring [1], it is not hard to prove that G is p-nilpotent if and only if $K^*(BG) \cong K^*(BH)$.

Let n be a positive integer, and $K(n)^*(-)$ the n-th Morava K-theory at p. It is now well known that the rank of $K(n)^*(BG)$ as $K(n)^*$-module is finite [9], and it is possible to introduce a $K(n)^*$-Euler characteristic for BG. By its group theoretical significance we prove the following theorem.

THEOREM 1.1. *A finite group G is p-nilpotent if and only if, for some n, the restriction map*

$$K(n)^*(BG) \longrightarrow K(n)^*(BH)$$

is an isomorphism, where H is a p-Sylow subgroup of G.

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This result confirms the special role played by Morava K-theories among all complex oriented cohomology theories, and induces to guess the answer to the following natural question. Let $f\colon G_1 \longrightarrow G_2$ be a homomorphism between two finite groups. It is known that if $K(n)^*(Bf)$ is an isomorphism for all $n \geq 0$ then BG_1 and BG_2 are stably p-homotopy equivalent. This result follows by [10], once you know that $\sum^\infty BG$ is *harmonic* (see Lemma 5.5 in [7]); you can also use the fact that a map between spaces which induces an isomorphism in all Morava K-theories is a homology equivalence (see [2]). Is it possible in the statement above to replace "*for all n*" with "*at least one n*"? A positive answer should allow us to include Morava K-theories in a family of functors which is the p-local analogue of that one introduced in [6], where the author finds sufficient conditions for a functor F from the category of finite groups to the category of (graded) abelian groups to satisfy the following property: any homomorphism of finite groups inducing an isomorphism of F is itself an isomorphism.

The author would like to thank Nick Kuhn who turned author's attention on [6], and on the *harmonicity* of classifying spaces of finite groups in the sense explained in [10].

2**. Proof of the theorem.** Throughout all this section, groups will be finite. Following notations introduced in the previous section, we start to recall that the difference between the ranks respectively of $K(n)^{\text{even}}(BG)$ and $K(n)^{\text{odd}}(BG)$ as $K(n)^*$-modules is called $K(n)^*$*-Euler* characteristic of BG, and it is denoted by $\chi_{n,p}(G)$.

LEMMA 2.1. *The number $\chi_{n,p}(G)$ is equal to the cardinality of the set $G_{n,p}$ of conjugacy classes of n-tuples of commuting elements of G whose order is a power of the prime p.*

PROOF. See [3]. ∎

PROPOSITION 2.2. *Denoting by H a p-Sylow subgroup of a group G, $K(1)^*(BG)$ and $K(1)^*(BH)$ are isomorphic if and only if the group G is p-nilpotent.*

PROOF. If G is p-nilpotent, $h^*(BG)$ and $h^*(BH)$ are isomorphic for any cohomology theory $h^*(-)$ whose coefficients ring is p-local (*i.e.* a local ring with residual characteristic p) or mod p (*i.e.* when $h^t(pt)$ is an F_p-vector space for every integer t).

Suppose now $K(1)^*(BG)$ and $K(1)^*(BH)$ isomorphic; then $\chi_{1,p}(G) = \chi_{1,p}(H)$. In other words, using the group theoretical significance of this number found for the first time in [5], and one of Sylow's elementary theorems, if two elements of H are conjugate in G, then they are conjugated in particular by an element of H. This fact actually implies the p-nilpotence of G (see [4, IV, 4.9]). ∎

We can finally approach what Proposition 2.2 leaves still to prove of Theorem 1.1.

Given an integer $n > 1$, suppose $K(n)^*(BG)$ and $K(n)^*(BH)$ isomorphic, and nevertheless G is not p-nilpotent. In this case, by Proposition 2.2, $\chi_{1,p}(G)$ is strictly less than $\chi_{1,p}(H)$. Therefore there exist at least two elements in H having order a power of p, say h and k, such that they are conjugate in G but not in H. Notice now that every element in the set $G_{n,p}$ has the form

$$[(g_1, \ldots, g_n)],$$

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and, by definition, all of the elements $g_1, \ldots, g_n$ are contained in the same p-Sylow subgroup, therefore each class in $G_{n,p}$ can be represented by an n-tuple

$$(h_1, \ldots, h_n)$$

where all the h_i's are in H. It follows that the n-tuples

$$(h, h, \ldots, h) \quad \text{and} \quad (k, k, \ldots, k)$$

represent the same class in $G_{n,p}$ but not in $H_{n,p}$, hence $\chi_{n,p}(G) < \chi_{n,p}(H)$, and an isomorphism between two free modules with different ranks cannot exist.

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Dipartimento di Matematica e applicazioni
Università di Napoli
via Claudio 21
I-80125 Napoli
Italy

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