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THE QUANTITATIVE DISTRIBUTION OF HECKE EIGENVALUES

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Abstract

In this paper, we prove that the Sato–Tate conjecture for primitive Maass forms holds on average. We also investigate the rate of convergence in the Sato–Tate conjecture and establish some estimates of the discrepancy with respect to the Sato–Tate measure on the average of primitive Maass forms.

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1. Introduction

Let H_k be the set of all the normalised primitive holomorphic cusp forms of even integral weight k for the modular group $\Gamma = SL_2(\mathbb{Z})$. The generalised Ramanujan conjecture for primitive holomorphic cusp forms was proved by Deligne [5] in 1974 and implies that for any $f \in H_k$, its normalised Hecke eigenvalues $\lambda_f(p)$, with p running through all the primes, lie in the interval $[-2, 2]$.

Given $f \in H_k$, the asymptotic distribution of the Hecke eigenvalues $\lambda_f(p)$, as the primes p vary, is an interesting and difficult problem. In the 1960s, inspired by the Sato–Tate conjecture, Serre conjectured that for any $f \in H_k$, $\lambda_f(p)$ for $p \le x$ distribute nicely: as $x \to \infty$ they are equidistributed in $[-2, 2]$ with respect to the Sato–Tate measure

$$d\mu_\infty(x) = \begin{cases} \dfrac{1}{\pi}\sqrt{1 - \dfrac{x^2}{4}}\,dx & x \in [-2, 2], \\ 0 & \text{otherwise.} \end{cases}$$

This is also called the Sato–Tate conjecture. It has significant implications in number theory. For example, it is a consequence of Langlands' functoriality conjecture, yielding the analytic properties of symmetric power L-functions. In 2006, Nagoshi [8, Theorem 1] proved that the Sato–Tate conjecture holds on average of the normalised primitive holomorphic cusp forms $f \in H_k$. In 2011, the Sato–Tate conjecture was proved by Barnet-Lamb *et al.* [2].

Naturally, one may ask what the rate of convergence to the Sato–Tate conjecture is. In fact, a good estimate on the rate of convergence has significant implications in number theory. For example, Akiyama and Tanigawa [1] conjectured that the discrepancy with respect to the Sato–Tate measure is $O(x^{-1/2+\epsilon})$ for any $\epsilon > 0$. Moreover, they proved that their conjecture implies the generalised Riemann hypothesis for the L-functions associated to elliptic curves over $\mathbb{Q}$ which have no complex multiplication. In this direction, we get an estimate on the rate of convergence on average of the normalised primitive cusp forms $f \in H_k$ which implies that Akiyama and Tanigawa's conjecture is true on average when the weight k is sufficiently large. More precisely, we prove the following theorem.

THEOREM 1.1. *Suppose that* $k = k(x)$ *satisfies* $\log k/\log x \to \infty$ *as* $x \to \infty$. *Let* $\pi(x)$ *denote the number of primes up to x. For any interval* $[\alpha, \beta] \subset [-2, 2]$,

$$\frac{1}{|H_k|\pi(x)}\#\{(f,p) : f \in H_k,\ p \le x \text{ and } \lambda_f(p) \in [\alpha,\beta]\}$$
$$= \int_\alpha^\beta d\mu_\infty + O\bigg(\frac{\log x}{\log k} + \frac{(\log x)\log_2 x}{x}\bigg)$$

where $d\mu_\infty$ *is the Sato–Tate measure,* $\log_r$ *is the r-fold iterated logarithm and the implied constant is absolute.*

REMARK 1.2. This is a quantitative version of Nagoshi's result [8, Theorem 1]. The proof of Theorem 1.1 is very similar to the proof of Theorem 1.3 and hence we shall omit it.

In the context of primitive Maass forms, we also have the Sato–Tate conjecture, but the conjecture is open. In this paper, we prove that Theorem 1.1 also holds for primitive Maass forms, which implies that the Sato–Tate conjecture for primitive Maass forms holds on average. One major obstacle is that the generalised Ramanujan conjecture is still unknown for primitive Maass forms and the 'exceptional' eigenvalues (whose absolute values are bigger than 2) raise extra difficulties compared with the case of primitive holomorphic cusp forms.

To start with, we briefly give the setting of Maass forms. Let $\mathbb{H}$ be the open upper plane in $\mathbb{C}$. The non-Euclidean Laplace operator on $\mathbb{H}$ is given by

$$\Delta = -y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big).$$

Denote the space spanned by the Maass cusp forms for Γ by $C(\Gamma\backslash\mathbb{H})$. Let $\{u_j : j \ge 0\}$ be a complete orthonormal basis for $C(\Gamma\backslash\mathbb{H})$ consisting of the common eigenfunctions of the Laplacian Δ and the Hecke operators T_n, $n = 1, 2, \dots$, where u_0 is a constant function. Then

$$\Delta u_j = (\tfrac{1}{4} + t_j^2)u_j, \quad T_n u_j = \lambda_j(n)u_j$$

and we have the Fourier expansion

$$u_j(z) = \sqrt{y}\rho_j(1)\sum_{n\neq 0}\lambda_j(n)K_{it_j}(2\pi|n|y)e(nx) \qquad (z = x + iy \in \mathbb{H})$$

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where $0 < t_1 \le t_2 \le \cdots$, $\lambda_j(n) \in \mathbb{R}$, $\rho_j(1) \ne 0$ and K_ν is the K-Bessel function of order ν. Moreover, we know that

$$r(T) = \#\{j : 0 < t_j \le T\} = \frac{1}{12}T^2 + O(T\log T) \tag{1.1}$$

(Weyl's law). Like primitive holomorphic cusp forms, we have the generalised Ramanujan conjecture

$$|\lambda_j(p)| \le 2 \quad \text{for all primes } p. \tag{1.2}$$

Unfortunately, (1.2) is far out of reach and the best result is due to Kim and Sarnak [6], who proved that for all primes p,

$$|\lambda_j(p)| \le p^\theta + p^{-\theta} \tag{1.3}$$

where $\theta = 7/64$.

By a primitive Maass form we mean $\rho_j(1)^{-1}u_j(z)$. Our main result is as follows.

THEOREM 1.3. *Suppose that* $T = T(x)$ *satisfies* $\log T/\log x \to \infty$ *as* $x \to \infty$. *For any* $[\alpha,\beta] \subset (-\infty,\infty)$,

$$\begin{aligned}&\frac{1}{r(T)\pi(x)}\#\{(j,p) : 1 \le j \le r(T),\ p \le x \text{ and } \lambda_j(p) \in [\alpha,\beta]\}\\ &\quad = \int_\alpha^\beta d\mu_\infty + O\Big(\frac{\log x}{\log T} + \frac{(\log x)\log_2 x}{x}\Big),\end{aligned}$$

where the implied constant is absolute.

2. Preliminary lemmas

We first cite some results in [3] and [9], modified to fit our situation. Let $\varphi_{u,v} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be the normalised characteristic functions defined as

$$\varphi_{u,v}(x) = \begin{cases} 1 & \text{if } u < x - n < v \text{ for some } n \in \mathbb{Z},\\ \frac{1}{2} & \text{if } u - x \in \mathbb{Z} \text{ or if } v - x \in \mathbb{Z},\\ 0 & \text{otherwise,}\end{cases}$$

where $u < v < u + 1$. For our purposes, we take $0 \le u < v \le 1/2$, and define

$$\widetilde{\varphi}_{u,v}(x) = \varphi_{u,v}(x) + \varphi_{-v,-u}(x) \in [0,1]$$

for any $x \in \mathbb{R}$, since the two intervals (u, v) and $(-v,-u)$ do not overlap in $\mathbb{R}/\mathbb{Z}$. Note that $\varphi_{1-v,1-u}(x) = \varphi_{-v,-u}(x) = \varphi_{u,v}(-x)$.

Moreover, we set

$$k_M(x) = \sum_{|\ell|\le M}\Big(1 - \frac{|\ell|}{M+1}\Big)e(\ell x) = \frac{1}{M+1}\Big(\frac{\sin\pi(M+1)x}{\sin\pi x}\Big)^2 \tag{2.1}$$

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and

$$j_M(x) = \sum_{|\ell|\le M} \widehat{J}\Big(\frac{\ell}{M+1}\Big) e(\ell x)$$

where $\widehat{J}(0) = 1$ and $\widehat{J}(t) = \pi t(1-|t|)\cot \pi t + |t|$ for $0 < |t| < 1$ and M is a positive integer whose specification is at our disposal.

Furthermore, define

$$\begin{aligned} \widetilde{\alpha}_{u,v}(x) &= \widehat{\alpha}_{u,v}(0) + \sum_{1\le|\ell|\le M} \widehat{\alpha}_{u,v}(\ell)\cos(2\pi\ell x), \\ \widetilde{\beta}_{u,v}(x) &= (2M+2)^{-1} \sum_{|\ell|\le M} \widehat{\beta}_{u,v}(\ell)\cos(2\pi\ell x) \end{aligned} \tag{2.2}$$

where $\widehat{\alpha}_{u,v}(0) = 2(v-u)$, $\widehat{\beta}_{u,v}(0) = 4$ and, for $\ell \ne 0$,

$$\begin{aligned} \widehat{\alpha}_{u,v}(\ell) &= (\pi i\ell)^{-1}\widehat{J}\Big(\frac{\ell}{M+1}\Big)(e(-\ell u) - e(-\ell v)), \\ \widehat{\beta}_{u,v}(\ell) &= 2\Big(1 - \frac{|\ell|}{M+1}\Big)(e(-\ell u) + e(-\ell v)). \end{aligned}$$

Taking $N = 1$ in [7, Proposition 1],

$$|\widehat{\alpha}_n(\ell)| \le 2\widehat{k}_M(\ell), \quad |\widehat{\beta}_n(m)| \le 4\widehat{k}_M(m) \tag{2.3}$$

where $\widehat{k}_M(\ell) = (1 - |\ell|/(M+1))$.

The following lemma is proved in [7, (2.8)] and [7, Proposition 1] with $N = 1$, which is a modified version of [3, Theorem 7].

Lemma 2.1. *For any $x \in \mathbb{R}/\mathbb{Z}$,*

$$|\widetilde{\varphi}_{u,v}(x) - \widetilde{\alpha}_{u,v}(x)| \le \widetilde{\beta}_{u,v}(x).$$

Moreover,

$$0 \le \widetilde{\alpha}_{u,v}(x) \le 1, \quad 0 \le \widetilde{\beta}_{u,v}(x) \le 2. \tag{2.4}$$

The following unweighted Kuznetsov trace formula is proved in [7, Lemma 3.3].

Lemma 2.2. *Let $\kappa_0 = 11/155$, $\eta_0 = 43/620$ and m, n be any positive integers. For arbitrarily small $\epsilon > 0$,*

$$\sum_{t_j\le T} \lambda_j(m)\lambda_j(n) = \frac{1}{12}T^2\delta_{mn=\square}\frac{\sigma((m,n))}{\sqrt{mn}} + O_\epsilon(T^{2-\kappa_0+\epsilon}(mn)^{\eta_0+\epsilon}),$$

where $\sigma(\ell) = \sum_{d|\ell} d$ and $\delta_{\ell=\square} = 1$ if ℓ is a square and $\delta_{\ell=\square} = 0$ otherwise.

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3. Proof of Theorem 1.3

The Hecke eigenvalue $\lambda_j(p)$ can be expressed in terms of the Satake parameters of an automorphic representation. Consequently, $\lambda_j(p) = \alpha_{u_j}(p) + \beta_{u_j}(p)$ with $\alpha_{u_j}(p),\ \beta_{u_j}(p) \in \mathbb{C}$ and $\alpha_{u_j}(p)\beta_{u_j}(p) = 1$. For any prime p, we write $\alpha_{u_j}(p) = e^{i\theta_j(p)}$ so that

$$\lambda_j(p) = 2\cos\theta_j(p)$$

where $\theta_j(p) \in [0,\pi] \cup i(0,\theta\log p] \cup (\pi + i(0,\theta\log p])$ with $\theta = 7/64$. (Recall that $\lambda_j(p) \in \mathbb{R}$ and (1.3).) We also have (see [4, Lemma 3])

$$\lambda_j(p^n) = \frac{\sin(n+1)\theta_j(p)}{\sin\theta_j(p)} =: X_n(2\cos\theta_j(p)), \tag{3.1}$$

that is, X_n is the nth Chebyshev polynomial. (We adopt the notation in [7].)

The value of $\theta_j(p)$ is uniquely determined. Consider

$$\lambda_j(p) = 2\cos\theta_j(p) \in (a,b) \subset [-2,2].$$

Then $\lambda_j(p) \in (a,b)$ is equivalent to $\theta_j(p)/2\pi \in (u(b), v(a)) \subset [0,1/2]$, or equivalently, $\widetilde{\varphi}_{u(b),v(a)}(\theta_j(p)/(2\pi)) = 1$, where

$$u(b) = \frac{\arccos(\frac{b}{2})}{2\pi} \quad \text{and} \quad v(a) = \frac{\arccos(\frac{a}{2})}{2\pi}.$$

Therefore,

$$\#\{1 \le j \le r(T) : \lambda_j(p) \in [a,b]\} \sim \sum_{\substack{1\le j\le r(T)\\ \theta_j(p)\in[0,\pi]}} \widetilde{\varphi}_{u(b),v(a)}\Big(\frac{\theta_j(p)}{2\pi}\Big)$$

or more precisely,

$$\sum_{\substack{1\le j\le r(T)\\ \lambda_j(p)\in(a,b)}} 1 \le \sum_{\substack{1\le j\le r(T)\\ \theta_j(p)\in[0,\pi]}} \widetilde{\varphi}_{u(b),v(a)}\Big(\frac{\theta_j(p)}{2\pi}\Big) \le \sum_{\substack{1\le j\le r(T)\\ \lambda_j(p)\in[a,b]}} 1.$$

Proposition 3.1. *Let $\theta_j(p)$ be as defined above. Then, for $0 \le u < v \le 1/2$,*

$$\begin{aligned}\frac{1}{r(T)\pi(x)} &\sum_{\substack{1\le j\le r(T),p\le x\\ \theta_j(p)\in[0,\pi]}} \widetilde{\varphi}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big)\\ &= 2\int_u^v (1-\cos 2\pi t)\,dt + O\Big(\frac{\log x}{\log T} + \frac{(\log x)\log_2 x}{x}\Big).\end{aligned} \tag{3.2}$$

Proof. By Lemma 2.1,

$$\sum_{\substack{1\le j\le r(T),p\le x\\ \theta_j(p)\in[0,\pi]}} \Big|\widetilde{\varphi}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big) - \widetilde{\alpha}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big)\Big| \le \sum_{\substack{1\le j\le r(T),p\le x\\ \theta_j(p)\in[0,\pi]}} \widetilde{\beta}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big).$$

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Hence, writing $\widetilde{\varphi}_{u,v}(x) = \widetilde{\alpha}_{u,v}(x) + \widetilde{\varphi}_{u,v}(x) - \widetilde{\alpha}_{u,v}(x)$,

$$\sum_{\substack{1\le j\le r(T),p\le x\\ \theta_j(p)\in[0,\pi]}} \widetilde{\varphi}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big) = \sum_{\substack{1\le j\le r(T),p\le x\\ \theta_j(p)\in[0,\pi]}} \widetilde{\alpha}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big) + O\Big(\sum_{\substack{1\le j\le r(T),p\le x\\ \theta_j(p)\in[0,\pi]}} \widetilde{\beta}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big)\Big). \tag{3.3}$$

The incomplete sums in (3.3) lead to the problem of controlling the 'exceptional' eigenvalues (that is, eigenvalues with $\theta_j(p) \notin [0,\pi]$).

When $\theta_j(p) = i\vartheta_j(p)$ or $\pi + i\vartheta_j(p)$ for some real $\vartheta_j(p)$, we do not have the inequalities (2.4). However, by (2.2), (2.3) and (2.1),

$$\begin{aligned}\Big|\widetilde{\alpha}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big)\Big| &\le 2\sum_{|\ell|\le M} \widehat{k}_M(\ell)\cosh(\ell\vartheta_j(p))\\ &\le 2\sum_{|\ell|\le M} \widehat{k}_M(\ell)\cosh(2\ell\vartheta_j(p))\end{aligned}$$

as $\cosh(\phi) \le \cosh(2\phi)$ for real ϕ. Thus by (2.1) the last line gives

$$\Big|\widetilde{\alpha}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big)\Big| \le 2k_M\Big(\frac{\theta_j(p)}{\pi}\Big) = \frac{2}{M+1}X_M(2\cos(\theta_j(p)))^2 = \frac{2}{M+1}\lambda_j(p^M)^2,$$

by (3.1). Similarly,

$$\Big|\widetilde{\beta}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big)\Big| \le \frac{4}{(M+1)^2}\lambda_j(p^M)^2.$$

Therefore, we conclude that

$$\sum_{\substack{1\le j\le r(T),p\le x\\ \theta_j(p)\notin[0,\pi]}} \Big|\widetilde{\alpha}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big)\Big| \le \sum_{\substack{1\le j\le r(T),p\le x\\ \theta_j(p)\notin[0,\pi]}} \frac{2\lambda_j(p^M)^2}{M+1} \le \sum_{1\le j\le r(T),p\le x} \frac{2\lambda_j(p^M)^2}{M+1}.$$

By Lemma 2.2,

$$\sum_{\substack{1\le j\le r(T),p\le x\\ \theta_j(p)\notin[0,\pi]}} \Big|\widetilde{\alpha}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big)\Big| \ll_\epsilon \frac{1}{M+1}(r(T)\pi(x) + T^{2-\kappa_0+\epsilon}x^{2M(\eta_0+\epsilon)+1}), \tag{3.4}$$

where κ_0 and η_0 are defined as in Lemma 2.2. Similarly,

$$\sum_{\substack{1\le j\le r(T),p\le x\\ \theta_j(p)\notin[0,\pi]}} \Big|\widetilde{\beta}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big)\Big| \ll_\epsilon \frac{1}{M+1}(r(T)\pi(x) + T^{2-\kappa_0+\epsilon}x^{2M(\eta_0+\epsilon)+1}). \tag{3.5}$$

Combining (3.3)–(3.5),

$$\begin{aligned}\sum_{\substack{1\le j\le r(T),p\le x\\ \theta_j(p)\in[0,\pi]}} \widetilde{\varphi}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big) &= \sum_{1\le j\le r(T),p\le x} \widetilde{\alpha}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big)\\ &\quad + O_\epsilon\Big(\sum_{1\le j\le r(T),p\le x} \widetilde{\beta}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big)\\ &\quad + \frac{1}{M+1}(r(T)\pi(x) + T^{2-\kappa_0+\epsilon}x^{2M(\eta_0+\epsilon)+1})\Big).\end{aligned} \tag{3.6}$$

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On the other hand, since, for $\ell \geq 2$,

$$2\cos \ell\theta = X_\ell(2\cos\theta) - X_{\ell-2}(2\cos\theta),$$

we obtain, for $\ell \geq 2$,

$$\sum_{j=1}^{r(T)} 2\cos \ell\theta_j(p) = \sum_{j=1}^{r(T)} \lambda_j(p^\ell) - \sum_{j=1}^{r(T)} \lambda_j(p^{\ell-2}).$$

By Lemma 2.2 and the prime number theorem, for $\ell \geq 2$,

$$\sum_{p\leq x}\sum_{j=1}^{r(T)} \cos \ell\theta_j(p) = \delta_{p^\ell=\square} \sum_{p\leq x} \frac{T^2}{24}(p^{-\ell/2} - p^{-\ell/2+1}) + O_\epsilon(T^{2-\kappa_0+\epsilon} x^{\ell(\eta_0+\epsilon)+1}),$$

where $\delta_{p^\ell=\square}$, κ_0 and η_0 are defined as in Lemma 2.2. Therefore, by (2.3) and Weyl's law (1.1),

$$\begin{aligned}
\sum_{p\leq x}\sum_{j=1}^{r(T)} \widetilde{\alpha}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big) &= \sum_{\ell=-M}^{M} \widehat{\alpha}_{u,v}(\ell) \sum_{p\leq x}\sum_{j=1}^{r(T)} \cos(\ell\theta_j(p)) \\
&= \widehat{\alpha}_{u,v}(0) r(T)\pi(x) + (\widehat{\alpha}_{u,v}(-2) + \widehat{\alpha}_{u,v}(2)) \sum_{p\leq x} \frac{T^2}{24}(p^{-1} - 1) \\
&\quad + O_\epsilon(T^2 \log_2 x + T^{2-\kappa_0+\epsilon} x^{M(\eta_0+\epsilon)+1}) \\
&= 2r(T)\pi(x) \int_u^v (1 - \cos 4\pi t)\, dt \\
&\quad + O_\epsilon\Big(\frac{r(T)\pi(x)}{(M+1)} + T^2 \log_2 x + T^{2-\kappa_0+\epsilon} x^{M(\eta_0+\epsilon)+1}\Big).
\end{aligned}$$

Here we have also used the fact that

$$\begin{aligned}
\widehat{\alpha}_{u,v}(-2) + \widehat{\alpha}_{u,v}(2) &= \frac{1}{\pi} \widehat{J}\Big(\frac{2}{M+1}\Big)(\sin 4\pi v - \sin 4\pi u) \\
&= \frac{1}{\pi}\Big(\frac{2\pi}{M+1}\Big(1 - \frac{2}{M+1}\Big)\cot\frac{2\pi}{M+1} + \frac{2}{M+1}\Big)(\sin 4\pi v - \sin 4\pi u) \\
&= \frac{1}{\pi}(\sin 4\pi v - \sin 4\pi u) + O\Big(\frac{1}{M+1}\Big).
\end{aligned}$$

Similarly,

$$\sum_{p\leq x}\sum_{j=1}^{r(T)} \widetilde{\beta}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big) \ll_\epsilon \frac{1}{M+1}(r(T)\pi(x) + T^2 \log_2 x + T^{2-\kappa_0+\epsilon} x^{M(\eta_0+\epsilon)+1}).$$

Combining these with (3.6),

$$\begin{aligned}
\sum_{\substack{1\leq j\leq r(T), p\leq x \\ \theta_j(p)\in[0,\pi]}} \widetilde{\varphi}_{u,v}\Big(\frac{\theta_j(p)}{2\pi}\Big) &= 2r(T)\pi(x) \int_u^v (1 - \cos 4\pi t)\, dt \\
&\quad + O_\epsilon\Big(\frac{r(T)\pi(x)}{(M+1)} + T^2 \log_2 x + T^{2-\kappa_0+\epsilon} x^{M(\eta_0+\epsilon)+1}\Big).
\end{aligned}$$

Hence the statement follows by taking $M = \lfloor \kappa_0 \log T / 10\eta_0 \log x \rfloor$. $\square$

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Now we are ready to complete the proof. By the definition of $d\mu_\infty$, it is sufficient to prove that the statement holds for any interval $I = [a, b] \subset [-2, 2]$. Let x be sufficiently large, and write $I = [a, b] \subset (-2, 2)$. We choose $[u, v] \subset [u', v']$ $(\subset [0, 1/2])$ such that $\mathrm{u}(M + 1), \mathrm{v}(M + 1) \in \mathbb{Z}$ for $(\mathrm{u}, \mathrm{v}) = (u, v)$ and (u', v'), the complement has a small measure

$$\big|[u', v'] \setminus [u, v]\big| \ll \frac{1}{M}$$

where $M = \lfloor \kappa_0 \log T/10\eta_0 \log x \rfloor$, and also, for $\theta \in [0, \pi]$,

$$\widetilde{\varphi}_{u,v}\Big(\frac{\theta}{2\pi}\Big) \le \chi_{[a,b]}(2\cos\theta) \le \widetilde{\varphi}_{u',v'}\Big(\frac{\theta}{2\pi}\Big),$$

where $\chi_{[a,b]}$ denotes the characteristic function over $[a, b]$. Applying Proposition 3.1 to $\widetilde{\varphi}_{u,v}$ and $\widetilde{\varphi}_{u',v'}$, we obtain lower and upper bounds of the form in the right-hand side of (3.2) for

$$\frac{1}{r(T)\pi(x)} \#\{(j, p) : 1 \le j \le r(T),\ p \le x \text{ and } \lambda_j(p) \in I\}.$$

Then it remains to show that

$$\int_{\mathrm{u}}^{\mathrm{v}} 2(1 - \cos 4\pi t)\, dt = \int_I d\mu_\infty + O\Big(\frac{1}{M}\Big) \tag{3.7}$$

for $(\mathrm{u}, \mathrm{v}) = (u, v)$ and (u', v'). By a change of variable $y = 2\cos 2\pi t$,

$$\int_{\mathrm{u}}^{\mathrm{v}} 2(1 - \cos 4\pi t)\, dt = \int_{\mathrm{u}}^{\mathrm{v}} 4\sin^2 2\pi t\, dt = \int_{2\cos 2\pi \mathrm{v}}^{2\cos 2\pi \mathrm{u}} \frac{\sqrt{4 - y^2}}{2\pi}\, dy.$$

As $[2\cos 2\pi v, 2\cos 2\pi u] \subset [a, b] \subset [2\cos 2\pi v', 2\cos 2\pi u']$, (3.7) follows. Finally, we relax the condition $[a, b] \subset (-2, 2)$ to $[-2, 2]$ with (3.7). The proof of Theorem 1.3 is complete.

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