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Abstract

In this paper we show that any good toric contact manifold has a well-defined cylindrical contact homology, and describe how it can be combinatorially computed from the associated moment cone. As an application, we compute the cylindrical contact homology of a particularly nice family of examples that appear in the work of Gauntlett *et al.* on Sasaki–Einstein metrics. We show in particular that these give rise to a new infinite family of non-equivalent contact structures on $S^2 \times S^3$ in the unique homotopy class of almost contact structures with vanishing first Chern class.

1. Introduction

Contact homology is a powerful invariant of contact structures, introduced by Eliashberg *et al.* [EGH00] in the bigger framework of symplectic field theory. Its simplest version is called cylindrical contact homology and can be briefly described in the following way. Let (N, ξ) be a closed (i.e. compact without boundary) co-oriented contact manifold, $\alpha \in \Omega^1(N)$ a contact form ($\xi = \ker \alpha$), and $R_\alpha \in \mathcal{X}(N)$ the corresponding Reeb vector field ($\iota(R_\alpha)\, d\alpha \equiv 0$ and $\alpha(R_\alpha) \equiv 1$). Assume that α is non-degenerate, with the meaning that all contractible closed orbits of R_α are non-degenerate. Consider the graded $\mathbb{Q}$-vector space $C_*(N, \alpha)$ freely generated by the contractible closed orbits of R_α, where the grading is determined by an appropriate dimensional shift of the Conley–Zehnder index (when the first Chern class of the contact structure is zero this grading is integral, but otherwise it is jus a finite cyclic grading). One then uses suitable pseudo-holomorphic curves in the symplectization of (N, α) to define a linear map $\partial : C_*(N, \alpha) \to C_{*-1}(N, \alpha)$. Under suitable assumptions, one can prove that $\partial^2 = 0$ and the homology of $(C_*(N, \alpha), \partial)$ is independent of the choice of contact form α. This is the cylindrical contact homology $HC_*(N, \xi; \mathbb{Q})$, a graded $\mathbb{Q}$-vector space invariant of the contact manifold (N, ξ). (Note: for transversality reasons, the identity $\partial^2 = 0$ and the invariance property of contact homology are conditional on the completion of foundational work by Hofer *et al.* [HWZ07, HWZ09a, HWZ09b].)

The simplest example of a cylindrical contact homology computation, already described in [EGH00], is the case of the standard contact sphere $(S^{2n+1}, \xi_{\mathrm{st}})$, where

$$S^{2n+1} \cong \left\{ z \in \mathbb{C}^{n+1} : \sum_{j=1}^{n+1} |z_j|^2 = 1 \right\} \subset \mathbb{C}^{n+1}$$

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and

$$\xi_{\rm st} := TS^{2n+1} \cap iTS^{2n+1} = \text{hyperplane field of complex tangencies.}$$

This contact structure admits the natural contact form

$$\alpha_{\rm st} := \frac{i}{2} \sum_{j=1}^{n+1} (z_j \, d\bar{z}_j - \bar{z}_j \, dz_j)|_{S^{2n+1}}$$

with completely periodic Reeb flow given by

$$(R_{\rm st})_s(z_1, \ldots, z_{n+1}) \mapsto (e^{is} z_1, \ldots, e^{is} z_{n+1}), \quad s \in \mathbb{R}.$$

In the presence of a degenerate contact form, such as $\alpha_{\rm st}$, there are two approaches to compute cylindrical contact homology:

(i) the Morse–Bott approach of Bourgeois [Bou03a], which computes it directly from the spaces of contractible periodic orbits of the degenerate Reeb flow;

(ii) the non-degenerate approach above, which computes it from the countably many contractible periodic orbits of the Reeb flow associated to a non-degenerate perturbation of the original degenerate contact form.

Since approach (ii) is the approach we will use in this paper, let us give a description of how it can work in this $(S^{2n+1}, \xi_{\rm st})$ example. One can obtain a suitable perturbation of the contact form $\alpha_{\rm st}$ by perturbing the embedding of $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ via

$$S^{2n+1} \cong S^{2n+1}_a := \left\{ z \in \mathbb{C}^{n+1} : \sum_{j=1}^{n+1} a_j |z_j|^2 = 1 \right\} \subset \mathbb{C}^{n+1} \quad \text{with } a_j \in \mathbb{R}^+ \text{ for all } j = 1, \ldots, n+1,$$

and noting that

$$\xi_{\rm st} \cong \xi_a := TS^{2n+1}_a \cap iTS^{2n+1}_a.$$

The perturbed contact form α_a is then given by

$$\alpha_a := \frac{i}{2} \sum_{j=1}^{n+1} (z_j \, d\bar{z}_j - \bar{z}_j \, dz_j)|_{S^{2n+1}_a},$$

and the corresponding Reeb flow can be written as

$$(R_a)_s(z_1, \ldots, z_{n+1}) \mapsto (e^{i a_1 s} z_1, \ldots, e^{i a_{n+1} s} z_{n+1}), \quad s \in \mathbb{R}.$$

If the a_j are $\mathbb{Q}$-independent, the contact form α_a is non-degenerate and the Reeb flow has exactly $n+1$ simple closed orbits $\gamma_1, \ldots, \gamma_{n+1}$, where each γ_ℓ corresponds to the orbit of the Reeb flow through the point $p_\ell \in S^{2n+1}_a$ with coordinates

$$z_j = \begin{cases} 1/\sqrt{a_\ell} & \text{if } j = \ell, \\ 0 & \text{if } j \neq \ell. \end{cases}$$

As it turns out, in this example any closed Reeb orbit γ_ℓ^N has even contact homology degree, which implies that the boundary operator is zero and

$$HC_*(S^{2n+1}, \xi_{\rm st}; \mathbb{Q}) = C_*(\alpha_a).$$

After some simple Conley–Zehnder index computations one concludes that

$$HC_*(S^{2n+1}, \xi_{\rm st}; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } * \geqslant 2n \text{ and even,} \\ 0 & \text{otherwise.} \end{cases}$$

The standard contact sphere $(S^{2n+1}, \xi_{\mathrm{st}})$ is the most basic example of a good toric contact manifold and, as we show in this paper, this contact homology calculation has a toric description that generalizes to any good toric contact manifold.

Closed toric contact manifolds are the odd-dimensional analogues of closed toric symplectic manifolds. They can be defined as contact manifolds of dimension $2n+1$ equipped with an effective Hamiltonian action of a torus of dimension $n+1$, and have been classified by Banyaga and Molino [BM93, BM96, Ban99], Boyer and Galicki [BG00], and Lerman [Ler03a].

Good toric contact manifolds of dimension three are (S^3, ξ_{st}) and its finite quotients. Good toric contact manifolds of dimension greater than three are closed toric contact manifolds whose torus action is not free. These form the most important class of closed toric contact manifolds, and can be classified by the associated moment cones, in the same way that Delzant's theorem classifies closed toric symplectic manifolds by the associated moment polytopes.

In this paper we show that any good toric contact manifold has a well-defined cylindrical contact homology, and describe how it can be combinatorially computed from the associated moment cone.

To be more precise, consider the following definition.

Definition 1.1. A non-degenerate contact form is called *nice* if its Reeb flow has no closed contractible orbit of contact homology degree equal to -1, 0 or 1. A non-degenerate contact form is called *even* if all closed contractible orbits of its Reeb flow have even contact homology degree.

The following proposition is a direct generalization to even contact forms of a well-known result for nice contact forms.

Proposition 1.2. *Let (N, ξ) be a contact manifold with an even or nice non-degenerate contact form α. Then the boundary operator $\partial : C_*(N, \alpha) \to C_{*-1}(N, \alpha)$ satisfies $\partial^2 = 0$, and the homology of $(C_*(N, \alpha), \partial)$ is independent of the choice of even or nice non-degenerate contact form α. Hence, the cylindrical contact homology $HC_*(N, \xi; \mathbb{Q})$ is a well-defined invariant of the contact manifold (N, ξ).*

Our first main result is the following theorem.

Theorem 1.3. *Any good toric contact manifold admits even non-degenerate toric contact forms. The corresponding cylindrical contact homology, isomorphic to the chain complex associated to any such contact form, is a well-defined invariant that can be combinatorially computed from the associated good moment cone.*

By applying this theorem to a particularly nice family of examples, originally considered by the mathematical physicists Gauntlett *et al.* in the context of their work on Sasaki–Einstein metrics [GMSW04a] (see also [Abr10, MSY06]), we obtain the second main result of this paper.

Theorem 1.4. *There are infinitely many non-equivalent contact structures ξ_k on $S^2 \times S^3$, $k \in \mathbb{N}_0$, in the unique homotopy class determined by the vanishing of the first Chern class. These contact structures are toric and can be distinguished by the degree zero cylindrical contact*

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homology. More precisely,

$$\operatorname{rank} HC_*(S^2 \times S^3, \xi_k; \mathbb{Q}) = \begin{cases} k & \text{if } * = 0, \\ 2k+1 & \text{if } * = 2, \\ 2k+2 & \text{if } * > 2 \text{ and even}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.5. All even non-degenerate toric contact forms that we consider for this family of examples have exactly four simple closed Reeb orbits. As we will see in § 6, by a suitable choice of such contact forms it is possible to concentrate all relevant contact homology information in the multiples of a single simple closed Reeb orbit: the one with minimal action or, equivalently, minimal period.

Remark 1.6. Van Koert constructs in [van08] an infinite family of non-equivalent contact structures on $S^2 \times S^3$ with vanishing first Chern class. Since his contact structures have vanishing degree zero contact homology, they are necessarily different from the ones given by the $k > 0$ cases of Theorem 1.4. We will see that $(S^2 \times S^3, \xi_0)$ is contactomorphic to the unit cosphere bundle of S^3.

A recent preprint by Pati [Pat09] discusses a generalization of the Morse–Bott approach of Bourgeois to compute the contact homology of S^1-bundles over certain symplectic orbifolds and applies it to toric contact manifolds. His explicit examples do not overlap with the ones in this paper.

Note also the recent preprint by Hamilton [Ham10] discussing inequivalent contact structures on simply-connected 5-manifolds which arise as S^1-bundles over simply-connected 4-manifolds. His contact structures have non-zero first Chern class.

The paper is organized as follows. Section 2 contains the necessary introduction to toric contact manifolds, their classification and main properties. The Conley–Zehnder index is described in § 3, where we also give a proof of its invariance property under symplectic reduction by a circle action (Lemma 3.4). This result, which plays an important role in the paper and could also be of independent interest, is known to experts but we could not find a reference. Section 4 gives a more detailed description of cylindrical contact homology and contains a proof of Proposition 1.2 (restated there as Proposition 4.2). Section 5 contains the proof of Theorem 1.3, while the examples and cylindrical contact homology computations relevant for Theorem 1.4 are the subject of § 6.

Notation. In this paper, unless explicitly stated otherwise, closed Reeb orbit means contractible closed Reeb orbit.

2. Toric contact manifolds

In this section we introduce toric contact manifolds via toric symplectic cones, and describe their classification and explicit construction via the associated moment cones. We will also describe the fundamental group and the first Chern class of a toric symplectic cone, as well as the space of toric contact forms and Reeb vector fields that are relevant in this context. For further details see Lerman's papers [Ler03a, Ler03b, Ler04].

2.1 Symplectic cones

Definition 2.1. A *symplectic cone* is a triple (W, ω, X), where (W, ω) is a connected symplectic manifold, i.e. $\omega \in \Omega^2(W)$ is a closed and non-degenerate 2-form, and $X \in \mathcal{X}(W)$ is a vector field generating a proper $\mathbb{R}$-action $\rho_t : W \to W$, $t \in \mathbb{R}$, such that $\rho_t^*(\omega) = e^{2t}\omega$. Note that the *Liouville vector field* X satisfies $\mathcal{L}_X\omega = 2\omega$, or equivalently

$$\omega = \tfrac{1}{2} d(\iota(X)\omega).$$

A *closed* symplectic cone is a symplectic cone (W, ω, X) for which the quotient $W/\mathbb{R}$ is closed.

Definition 2.2. A *co-orientable contact manifold* is a pair (N, ξ), where N is a connected odd-dimensional manifold, and $\xi \subset TN$ is a hyperplane distribution globally defined by $\xi = \ker \alpha$ for some $\alpha \in \Omega^1(N)$ such that $d\alpha|_\xi$ is non-degenerate. Such a 1-form α is called a *contact form* for ξ, and the non-degeneracy condition is equivalent to ξ being *maximally non-integrable*, i.e. its integral submanifolds have at most half of its dimension.

A *co-oriented* contact manifold is a triple $(N, \xi, [\alpha])$, where (N, ξ) is a co-orientable contact manifold and $[\alpha]$ is the conformal class of some contact form α, i.e.

$$[\alpha] = \{e^h \alpha \mid h \in C^\infty(N)\}.$$

Given a co-oriented contact manifold $(N, \xi, [\alpha])$, with contact form α, let

$$W := N \times \mathbb{R}, \quad \omega := d(e^t \alpha) \quad \text{and} \quad X := 2\frac{\partial}{\partial t},$$

where t is the $\mathbb{R}$-coordinate. Then (W, ω, X) is a symplectic cone, usually called the *symplectization* of $(N, \xi, [\alpha])$.

Conversely, given a symplectic cone (W, ω, X) let

$$N := W/\mathbb{R}, \quad \xi := \pi_*(\ker(\iota(X)\omega)) \quad \text{and} \quad \alpha := s^*(\iota(X)\omega),$$

where $\pi : W \to N$ is the natural principal $\mathbb{R}$-bundle quotient projection and $s : N \to W$ is any global section (note that such global sections always exist, since any principal $\mathbb{R}$-bundle is trivial). Then $(N, \xi, [\alpha])$ is a co-oriented contact manifold whose symplectization is the symplectic cone (W, ω, X).

In fact,

$$\text{co-oriented contact manifolds} \xleftrightarrow{1:1} \text{symplectic cones}$$

(see [Ler03b, ch. 2] for details). Under this bijection, closed contact manifolds correspond to closed symplectic cones and toric contact manifolds correspond to toric symplectic cones (see below). Moreover, the following are equivalent:

(i) choice of a contact form for $(N, \xi, [\alpha])$;

(ii) choice of a global section of $\pi : W \to N$;

(iii) choice of an $\mathbb{R}$-equivariant splitting $W \cong N \times \mathbb{R}$.

The choice of a contact form α for a contact manifold (N, ξ) gives rise to the *Reeb vector field* $R_\alpha \in \mathcal{X}(N)$, uniquely defined by

$$\iota(R_\alpha)\, d\alpha \equiv 0 \quad \text{and} \quad \alpha(R_\alpha) \equiv 1,$$

and corresponding *Reeb flow* $(R_\alpha)_s : N \to N$ satisfying

$$(R_\alpha)_s^*(\alpha) = \alpha, \quad \forall s \in \mathbb{R}.$$

The obvious horizontal lift of R_α to the symplectic cone $(W = N \times \mathbb{R}, \omega = d(e^t\alpha), X = 2(\partial/\partial t))$ will also be denoted by R_α. It satisfies

$$[R_\alpha, X] = 0 \quad \text{and} \quad \iota(R_\alpha)\omega = -d(e^t).$$

In other words, the lift of the Reeb flow is X-preserving and Hamiltonian.

Remark 2.3. On a symplectic cone (M, ω, X), any X-preserving symplectic action of a Lie group G is Hamiltonian. In fact, the map $\mu : M \to \mathfrak{g}^*$ defined by

$$\langle \mu, Y \rangle = \omega(X, Y_M), \quad \forall Y \in \mathfrak{g},$$

where Y_M is the vector field on M induced by Y via the G-action, is a moment map [Ler03b].

Remark 2.4. Any co-oriented contact manifold $(N, \xi, [\alpha])$ has well-defined Chern classes

$$c_k(\xi) \in H^{2k}(N; \mathbb{Z}), \quad k = 1, \ldots, n,$$

given by the Chern classes of the conformal symplectic vector bundle

$$(\xi, [d\alpha|_\xi]) \longrightarrow N.$$

Under the canonical isomorphism $\pi^* : H^*(N; \mathbb{Z}) \to H^*(W, \mathbb{Z})$, induced by the natural principal $\mathbb{R}$-bundle projection $\pi : W \to N$, these Chern classes coincide with the Chern classes of the tangent bundle of the symplectization (W, ω, X). In fact

$$(TW, \omega) \cong \varepsilon^2 \oplus \pi^*(\xi, [d\alpha|_\xi]),$$

where ε^2 is a trivial rank-2 symplectic vector bundle. The choice of a contact form α gives rise to an explicit isomorphism

$$\varepsilon^2 \cong \text{span}\{X, R_\alpha\} \quad \text{and} \quad \pi^*(\xi) \cong (\text{span}\{X, R_\alpha\})^\omega.$$

Example 2.5. The most basic example of a symplectic cone is $\mathbb{R}^{2(n+1)} \backslash \{0\}$ with linear coordinates

$$(u_1, \ldots, u_{n+1}, v_1, \ldots, v_{n+1}),$$

symplectic form

$$\omega_{\text{st}} = du \wedge dv := \sum_{j=1}^{n+1} du_j \wedge dv_j$$

and Liouville vector field

$$X_{\text{st}} = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} := \sum_{j=1}^{n+1} \left(u_j \frac{\partial}{\partial u_j} + v_j \frac{\partial}{\partial v_j} \right).$$

The associated co-oriented contact manifold is isomorphic to $(S^{2n+1}, \xi_{\text{st}})$, where $S^{2n+1} \subset \mathbb{C}^{n+1}$ is the unit sphere and ξ_{st} is the hyperplane distribution of complex tangencies, i.e.

$$\xi_{\text{st}} = TS^{2n+1} \cap i\, TS^{2n+1}.$$

The restriction of $\alpha_{\text{st}} := \iota(X_{\text{st}})\omega_{\text{st}}$ to S^{2n+1} is a contact form for ξ_{st}. Its Reeb flow $(R_{\text{st}})_s$ is the restriction to S^{2n+1} of the diagonal flow on $\mathbb{C}^{n+1}$ given by

$$(R_{\text{st}})_s \cdot (z_1, \ldots, z_{n+1}) = (e^{is} z_1, \ldots, e^{is} z_{n+1}),$$

where

$$z_j = u_j + iv_j, \quad j = 1, \ldots, n+1,$$

give the usual identification $\mathbb{R}^{2(n+1)} \cong \mathbb{C}^{n+1}$.

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Example 2.6. Let (M,ω) be a symplectic manifold such that the cohomology class

$$\frac{1}{2\pi}[\omega] \in H^2(M,\mathbb{R}) \text{ is integral, i.e. in the image of the natural map } H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R}).$$

Suppose that $H^2(M,\mathbb{Z})$ has no torsion, so that the above natural map is injective and we can consider $H^2(M,\mathbb{Z}) \subset H^2(M,\mathbb{R})$. Denote by $\pi : N \to M$ the principal circle bundle with first Chern class

$$c_1(N) = \frac{1}{2\pi}[\omega].$$

A theorem of Boothby and Wang [BW58] asserts that there is a connection 1-form α on N with $d\alpha = \pi^*\omega$ and, consequently, α is a contact form. We will call $(N, \xi := \ker(\alpha))$ the *Boothby–Wang* manifold of (M,ω). The associated symplectic cone is the total space of the corresponding line bundle $L \to M$ with the zero section deleted. The Reeb vector field R_α generates the natural S^1-action of N, associated to its circle bundle structure.

When $M = \mathbb{CP}^n$, with its standard Fubini–Study symplectic form, we recover Example 2.5, i.e. $(N,\xi) \cong (S^{2n+1}, \xi_{\mathrm{st}})$ and $\pi : S^{2n+1} \to \mathbb{CP}^n$ is the Hopf map.

2.2 Toric symplectic cones

Definition 2.7. A *toric symplectic cone* is a symplectic cone (W,ω,X) of dimension $2(n+1)$ equipped with an effective X-preserving symplectic $\mathbb{T}^{n+1}$-action, with moment map $\mu : W \to \mathfrak{t}^* \cong \mathbb{R}^{n+1}$ such that $\mu(\rho_t(w)) = e^{2t}\rho_t(w)$, for all $w \in W, t \in \mathbb{R}$. Its *moment cone* is defined to be the set

$$C := \mu(W) \cup \{0\} \subset \mathbb{R}^{n+1}.$$

Example 2.8. Consider the usual identification $\mathbb{R}^{2(n+1)} \cong \mathbb{C}^{n+1}$ given by

$$z_j = u_j + iv_j, \quad j = 1, \ldots, n+1,$$

and the standard $\mathbb{T}^{n+1}$-action defined by

$$(y_1, \ldots, y_{n+1}) \cdot (z_1, \ldots, z_{n+1}) = (e^{iy_1} z_1, \ldots, e^{iy_{n+1}} z_{n+1}).$$

The symplectic cone $(\mathbb{R}^{2(n+1)} \backslash \{0\}, \omega_{\mathrm{st}}, X_{\mathrm{st}})$ of Example 2.5 equipped with this $\mathbb{T}^{n+1}$-action is a toric symplectic cone. The moment map $\mu_{\mathrm{st}} : \mathbb{R}^{2(n+1)} \backslash \{0\} \to \mathbb{R}^{n+1}$ is given by

$$\mu_{\mathrm{st}}(u_1, \ldots, u_{n+1}, v_1, \ldots, v_{n+1}) = \tfrac{1}{2}(u_1^2 + v_1^2, \ldots, u_{n+1}^2 + v_{n+1}^2),$$

and the moment cone is $C = (\mathbb{R}_0^+)^{n+1} \subset \mathbb{R}^{n+1}$.

In [Ler03a] Lerman completed the classification of closed toric symplectic cones, initiated by Banyaga and Molino [Ban99, BM93, BM96], and continued by Boyer and Galicki [BG00]. The ones that are relevant for toric Kähler–Sasaki geometry are characterized by having good moment cones.

Definition 2.9 (Lerman). A cone $C \subset \mathbb{R}^{n+1}$ is *good* if there exists a minimal set of primitive vectors $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$, with $d \geqslant n+1$, such that the following hold.

(i) $C = \bigcap_{j=1}^d \{x \in \mathbb{R}^{n+1} : \ell_j(x) := \langle x, \nu_j \rangle \geqslant 0\}$.

(ii) Any codimension-k face of C, $1 \leqslant k \leqslant n$, is the intersection of exactly k facets whose set of normals can be completed to an integral base of $\mathbb{Z}^{n+1}$.

Theorem 2.10 (Banyaga–Molino, Boyer–Galicki, Lerman). *For each good cone $C \subset \mathbb{R}^{n+1}$ there exists a unique closed toric symplectic cone $(W_C, \omega_C, X_C, \mu_C)$ with moment cone C.*

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Definition 2.11. The closed toric symplectic cones (respectively closed toric contact manifolds) characterized by Theorem 2.10 will be called *good* toric symplectic cones (respectively *good* toric contact manifolds).

Remark 2.12. According to Lerman's classification (see [Ler03a, Theorem 2.18]), the list of closed toric contact manifolds that are *not good* is the following:

(i) certain overtwisted contact structures on three-dimensional lens spaces (including $S^1 \times S^2$);

(ii) the tight contact structures ξ_n, $n \geqslant 1$, on $\mathbb{T}^3 = S^1 \times \mathbb{T}^2$, defined as

$$\xi_n = \ker(\cos(n\theta)\, dy_1 + \sin(n\theta)\, dy_2), \quad (\theta, y_1, y_2) \in S^1 \times \mathbb{T}^2$$

(Giroux [Gir94] and Kanda [Kan97] proved independently that these are all inequivalent);

(iii) a unique toric contact structure on each principal $\mathbb{T}^{n+1}$-bundle over the sphere S^n, with $n \geqslant 2$.

Item (iii) classifies all closed toric contact manifolds of dimension $2n+1$, $n \geqslant 2$, and free $\mathbb{T}^{n+1}$-action [Ler03a]. Hence, a closed toric contact manifold of dimension greater than three is good if and only if the corresponding torus action is not free.

Example 2.13. Let $P \subset \mathbb{R}^n$ be an *integral Delzant polytope*, i.e. a Delzant polytope with integral vertices or, equivalently, the moment polytope of a closed toric symplectic manifold (M_P, ω_P, μ_P) such that $(1/2\pi)[\omega] \in H^2(M_P, \mathbb{Z})$. Then, its *standard cone*

$$C := \{z(x, 1) \in \mathbb{R}^n \times \mathbb{R} : x \in P, z \geqslant 0\} \subset \mathbb{R}^{n+1}$$

is a good cone. Moreover, we have the following.

(i) The toric symplectic manifold (M_P, ω_P, μ_P) is the $S^1 \cong \{\mathbf{1}\} \times S^1 \subset \mathbb{T}^{n+1}$ *symplectic reduction* of the toric symplectic cone $(W_C, \omega_C, X_C, \mu_C)$ (at level one).

(ii) The contact manifold $(N_C := \mu_C^{-1}(\mathbb{R}^n \times \{1\}), \alpha_C := (\iota(X_C)\omega_C)|_{N_C})$ is the *Boothby–Wang* manifold of (M_P, ω_P). The restricted $\mathbb{T}^{n+1}$-action makes it a *toric contact manifold*.

(iii) The toric symplectic cone (W_C, ω_C, X_C) is the *symplectization* of (N_C, α_C).

See [Ler03c, Lemma 3.7] for a proof of these facts.

If $P \subset \mathbb{R}^n$ is the standard simplex, i.e. $M_P = \mathbb{CP}^n$, then its standard cone $C \subset \mathbb{R}^{n+1}$ is the moment cone of $(W_C = \mathbb{C}^{n+1} \backslash \{0\}, \omega_{\mathrm{st}}, X_{\mathrm{st}})$ equipped with the $\mathbb{T}^{n+1}$-action given by

$$\begin{aligned}(y_1, \ldots, y_n, y_{n+1}) &\cdot (z_1, \ldots, z_n, z_{n+1}) \\ &= (e^{i(y_1+y_{n+1})} z_1, \ldots, e^{i(y_n+y_{n+1})} z_n, e^{iy_{n+1}} z_{n+1}).\end{aligned}$$

The moment map $\mu_C : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{R}^{n+1}$ is given by

$$\mu_C(z) = \tfrac{1}{2}(|z_1|^2, \ldots, |z_n|^2, |z_1|^2 + \cdots + |z_n|^2 + |z_{n+1}|^2)$$

and

$$N_C := \mu_C^{-1}(\mathbb{R}^n \times \{1\}) = \{z \in \mathbb{C}^{n+1} : \|z\|^2 = 2\} \cong S^{2n+1}.$$

Remark 2.14. Up to a possible twist of the action by an automorphism of the torus $\mathbb{T}^{n+1}$, any good toric symplectic cone can be obtained via an orbifold version of the Boothby–Wang construction of Example 2.6, where the base is a toric symplectic orbifold.

2.3 Explicit models

Like the case of closed toric symplectic manifolds, the existence part of Theorem 2.10 follows from an explicit symplectic reduction construction, starting from a standard $(\mathbb{R}^{2d}\backslash\{0\}, \omega_{\mathrm{st}}, X_{\mathrm{st}})$ (cf. Example 2.8). Since it will be needed later, we will briefly describe it here. Complete details can be found, for example, in [Ler03a] (proof of Lemma 6.3).

Let $C \subset (\mathbb{R}^{n+1})^*$ be a good cone defined by

$$C = \bigcap_{j=1}^{d} \{x \in (\mathbb{R}^{n+1})^* : \ell_j(x) := \langle x, \nu_j \rangle \geqslant 0\}$$

where $d \geqslant n+1$ is the number of facets and each ν_j is a primitive element of the lattice $\mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1}$ (the inward-pointing normal to the jth facet of C).

Let $(e_1, \ldots, e_d)$ denote the standard basis of $\mathbb{R}^d$, and define a linear map $\beta : \mathbb{R}^d \to \mathbb{R}^{n+1}$ by

$$\beta(e_j) = \nu_j, \quad j = 1, \ldots, d. \tag{1}$$

The conditions of Definition 2.9 imply that β is surjective. Denoting by $\mathfrak{k}$ its kernel, we have short exact sequences

$$0 \to \mathfrak{k} \xrightarrow{\iota} \mathbb{R}^d \xrightarrow{\beta} \mathbb{R}^{n+1} \to 0 \text{ and its dual, } 0 \to (\mathbb{R}^{n+1})^* \xrightarrow{\beta^*} (\mathbb{R}^d)^* \xrightarrow{\iota^*} \mathfrak{k}^* \to 0.$$

Let K denote the kernel of the map from $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ to $\mathbb{T}^{n+1} = \mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1}$ induced by β. More precisely,

$$K = \left\{ [y] \in \mathbb{T}^d : \sum_{j=1}^{d} y_j \nu_j \in 2\pi\mathbb{Z}^n \right\}. \tag{2}$$

It is a compact abelian subgroup of $\mathbb{T}^d$ with Lie algebra $\mathfrak{k} = \ker(\beta)$. Note that K need not be connected (this will be relevant in the proof of Proposition 2.15).

Consider $\mathbb{R}^{2d}$ with its standard symplectic form

$$\omega_{\mathrm{st}} = du \wedge dv = \sum_{j=1}^{d} du_j \wedge dv_j,$$

and identify $\mathbb{R}^{2d}$ with $\mathbb{C}^d$ via $z_j = u_j + iv_j$, $j = 1, \ldots, d$. The standard action of $\mathbb{T}^d$ on $\mathbb{R}^{2d} \cong \mathbb{C}^d$ is given by

$$y \cdot z = (e^{iy_1} z_1, \ldots, e^{iy_d} z_d)$$

and has a moment map given by

$$\phi_{\mathbb{T}^d}(z_1, \ldots, z_d) = \sum_{j=1}^{d} \frac{|z_j|^2}{2} e_j^* \in (\mathbb{R}^d)^*.$$

Since K is a subgroup of $\mathbb{T}^d$, K acts on $\mathbb{C}^d$ with moment map

$$\phi_K = \iota^* \circ \phi_{\mathbb{T}^d} = \sum_{j=1}^{d} \frac{|z_j|^2}{2} \iota^*(e_j^*) \in \mathfrak{k}^*. \tag{3}$$

The toric symplectic cone (W_C, ω_C, X_C) associated to the good cone C is the symplectic reduction of $(\mathbb{R}^{2d}\backslash\{0\}, \omega_{\mathrm{st}}, X_{\mathrm{st}})$ with respect to the K-action, i.e.

$$W_C = Z/K \text{ where } Z = \phi_K^{-1}(0)\backslash\{0\} \equiv \text{zero level set of moment map in } \mathbb{R}^{2d}\backslash\{0\},$$

the symplectic form ω_C comes from ω_{st} via symplectic reduction, while the $\mathbb{R}$-action of the Liouville vector field X_C and the action of $\mathbb{T}^{n+1} \cong \mathbb{T}^d/K$ are induced by the actions of X_{st} and $\mathbb{T}^d$ on Z.

2.4 Fundamental group and first Chern class

Lerman showed in [Ler04] how to compute the fundamental group of a good toric symplectic cone, which is canonically isomorphic to the fundamental group of the associated good toric contact manifold.

Proposition 2.15 (See [Ler04]). *Let W_C be the good toric symplectic cone determined by a good cone $C \subset \mathbb{R}^{n+1}$. Let $\mathcal{N} := \mathcal{N}\{\nu_1, \ldots, \nu_d\}$ denote the sublattice of $\mathbb{Z}^{n+1}$ generated by the primitive integral normal vectors to the facets of C. The fundamental group of W_C is the finite abelian group*

$$\mathbb{Z}^{n+1}/\mathcal{N}.$$

Proof (Outline). (i) We know that

$$W_C = Z/K,$$

where $K \subset \mathbb{T}^d$ acts on $\mathbb{C}^d$ with moment map $\phi_K : \mathbb{C}^d \to \mathfrak{k}^*$ defined by (3) and $Z = \phi_K^{-1}(0) \backslash \{0\}$.

(ii) The set Z has the homotopy type of

$$\mathbb{C}^d \backslash (V_1 \cup \cdots \cup V_r),$$

where each $V_j \subset \mathbb{C}^d$ is a linear subspace of complex codimension at least two. In particular,

$$\pi_0(Z) = \pi_1(Z) = \pi_2(Z) = 1.$$

(iii) The torus K acts freely on Z, and the long exact sequence of homotopy groups for the fibration

$$K \to Z \to W_C$$

implies that

$$\pi_1(W_C) = \pi_0(K).$$

(iv) The fact that $K = \ker \beta$, with $\beta : \mathbb{T}^d \to \mathbb{T}^{n+1}$ defined by (1), implies that

$$\pi_0(K) = \mathbb{Z}^{n+1}/\mathcal{N}.$$

Recall from Remark 2.4 that the Chern classes of the tangent bundle of a symplectic cone can be canonically identified with the Chern classes of the associated co-oriented contact manifold. The following proposition gives a combinatorial characterization of the vanishing of the first Chern class of good toric symplectic cones.

Proposition 2.16. *Let (W_C, ω_C, X_C) be the good toric symplectic cone determined by the good cone $C \in \mathbb{R}^{n+1}$ via the explicit symplectic reduction construction of the previous subsection. Let $K \in \mathbb{T}^d$ be defined by (2), and denote by $\chi_1, \ldots, \chi_d$ the characters that determine its natural representation on $\mathbb{C}^d$. Then*

$$c_1(TW_C) = 0 \Leftrightarrow \chi_1 + \cdots + \chi_d = 0.$$

Proof. It follows from the symplectic reduction description of W_C as Z/K, where $K \subset \mathbb{T}^d$ acts freely on $Z \subset \mathbb{C}^d$, that the quotient map $Z \to W_C$ is a principal K-bundle. We then have the following classifying diagram.

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & EK \arrow[d] \\ W_C \arrow[r, "f"] & BK \end{tikzcd}$$

Consider the induced map between the homotopy long exact sequences of these two principal fibrations with the same fiber K. Note that EK is contractible. As we pointed out in the proof of the previous proposition, Lerman showed in [Ler04] that $\pi_0(Z) = \pi_1(Z) = \pi_2(Z) = 1$. This implies that

$$f_* : \pi_i(W_C) \longrightarrow \pi_i(BK) \text{ is an isomorphism for } i = 0, 1, 2.$$

Since $\pi_3(BK) \cong \pi_2(K) = 1$, we also know that

$$f_* : \pi_3(W_C) \longrightarrow \pi_3(BK) \text{ is surjective.}$$

This means that the map $f : W_C \to BK$ is 3-connected and so induces an isomorphism in homology, and also in cohomology, in degree less than or equal to two. In particular,

$$f^* : H^2(BK; \mathbb{Z}) \longrightarrow H^2(W_C; \mathbb{Z}) \text{ is an isomorphism.}$$

The natural representation of $K \subset \mathbb{T}^d$ on $\mathbb{C}^d$ and this principal K-bundle $Z \to W_C$ give rise to a vector bundle $Z \times_K \mathbb{C}^d \to W_C$ with the following classifying diagram.

$$\begin{tikzcd} Z \times_K \mathbb{C}^d \arrow[r] \arrow[d] & EK \times_K \mathbb{C}^d \arrow[d] \\ W_C \arrow[r, "f"] & BK \end{tikzcd}$$

One can also think of this vector bundle as the quotient by K of the trivial K-equivariant vector bundle $Z \times \mathbb{C}^d \to Z$ that one gets by restricting the tangent bundle of $\mathbb{C}^d$ to Z. Let $\mathfrak{k}_{\mathbb{C}}$ denote the complexified Lie algebra of K. The trivial vector bundle $W_C \times \mathfrak{k}_{\mathbb{C}} \to W_C$ can be seen as a sub-bundle of $Z \times_K \mathbb{C}^d \to W_C$ via the map

$$\begin{aligned} W_C \times \mathfrak{k}_{\mathbb{C}} &\longrightarrow Z \times_K \mathbb{C}^d \\ ([z], v) &\longmapsto [z, X_v] \end{aligned}$$

where we use the description of W_C as Z/K and $X_v \in T_z\mathbb{C}^d \cong \mathbb{C}^d$ is induced by the free action of K on Z. The quotient bundle $(Z \times_K \mathbb{C}^d)/(W_C \times \mathfrak{k}_{\mathbb{C}})$ is naturally isomorphic to TW_C, and this shows that

$$Z \times_K \mathbb{C}^d \cong TW_C \oplus (W_C \times \mathfrak{k}_{\mathbb{C}}).$$

Hence

$$c_1(TW_C) = c_1(Z \times_K \mathbb{C}^d) = f^* c_1(EK \times_K \mathbb{C}^d) \quad \text{and} \quad c_1(TW_C) = 0 \Leftrightarrow c_1(EK \times_K \mathbb{C}^d) = 0.$$

Since

$$H^2(BK; \mathbb{Z}) \cong \text{character group of } K$$

and $c_1(EK \times_K \mathbb{C}^d) \in H^2(BK; \mathbb{Z})$ is given by

$$c_1(EK \times_K \mathbb{C}^d) = \chi_1 + \cdots + \chi_d,$$

the result follows.

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Remark 2.17. Let $k_1, \dots, k_{d-n-1} \in \mathbb{Z}^d \subset \mathbb{R}^d$ be an integral basis for the Lie algebra of $K \subset \mathbb{T}^d$. Proposition 2.16 states that

$$c_1(TW_C) = 0 \Leftrightarrow \sum_{j=1}^{d} (k_i)_j = 0, \quad \text{for all } i = 1, \dots, d-n-1.$$

2.5 Sasaki contact forms and Reeb vectors

Let (W, ω, X) be a good toric symplectic cone of dimension $2(n+1)$, with corresponding closed toric manifold (N, ξ). Denote by $\mathcal{X}_X(W, \omega)$ the set of X-preserving symplectic vector fields on W and by $\mathcal{X}(N, \xi)$ the corresponding set of contact vector fields on N. The $\mathbb{T}^{n+1}$-action associates to every vector $\nu \in \mathfrak{t} \cong \mathbb{R}^{n+1}$ a vector field $R_\nu \in \mathcal{X}_X(W, \omega) \cong \mathcal{X}(N, \xi)$.

Definition 2.18. A contact form $\alpha \in \Omega^1(N, \xi)$ is called *Sasaki* if its Reeb vector field R_α satisfies

$$R_\alpha = R_\nu \quad \text{for some } \nu \in \mathbb{R}^{n+1}.$$

In this case we will say that $\nu \in \mathbb{R}^{n+1}$ is a *Reeb vector*.

In the context of their work on toric Sasaki geometry, Martelli *et al.* characterize in [MSY06] which $\nu \in \mathbb{R}^{n+1}$ are Reeb vectors of a Sasaki contact form on (N, ξ).

Proposition 2.19 (See [MSY06]). *Let $\nu_1, \dots, \nu_d \in \mathbb{R}^{n+1}$ be the defining integral normals of the moment cone $C \in \mathbb{R}^{n+1}$ associated with (W, ω, X) and (N, ξ). The vector field $R_\nu \in \mathcal{X}_X(W, \omega) \cong \mathcal{X}(N, \xi)$ is the Reeb vector field of a Sasaki contact form $\alpha_\nu \in \Omega^1(N, \xi)$ if and only if*

$$\nu = \sum_{j=1}^{d} a_j \nu_j \quad \textit{with } a_j \in \mathbb{R}^+ \textit{ for all } j = 1, \dots, d.$$

Proof (Outline). This result is well-known for $(\mathbb{C}^d \backslash \{0\}, \omega_{\mathrm{st}}, X_{\mathrm{st}})$. In fact, any such Reeb vector field R_ν corresponds to

$$\nu = \sum_{j=1}^{d} a_j e_j = (a_1, \dots, a_d) \in (\mathbb{R}^+)^d$$

and can be written in complex coordinates as

$$R_\nu = i \sum_{j=1}^{d} a_j \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right).$$

The corresponding Reeb flow is given by

$$(R_\nu)_s \cdot (z_1, \dots, z_d) = (e^{i a_1 s} z_1, \dots, e^{i a_d s} z_d),$$

and the contact form α_ν is the restriction of

$$\alpha_{\mathrm{st}} := \iota(X_{\mathrm{st}}) \omega_{\mathrm{st}} = \frac{i}{2} \sum_{j=1}^{d} (z_j \, d\bar{z}_j - \bar{z}_j \, dz_j)$$

to

$$\begin{aligned} S^{2d-1} &\cong \{ z \in \mathbb{C}^d : (\alpha_{\mathrm{st}})_z (R_\nu) = 1 \} \\ &= \left\{ z \in \mathbb{C}^d : \sum_{j=1}^{d} a_j |z_j|^2 = 1 \right\}. \end{aligned}$$

(Compare with the example in the Introduction and Example 2.5.)

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The result follows for any good toric symplectic cone (W, ω, X), with moment cone C, from the explicit reduction construction of the model (W_C, ω_C, X_C). Note in particular the definition of the linear map $\beta : \mathbb{R}^d \to \mathbb{R}^{n+1}$ given by (1). $\square$

3. The Conley–Zehnder index

3.1 The Maslov index for loops of symplectic matrices

Let $\mathrm{Sp}(2n)$ denote the symplectic linear group, i.e. the group of linear transformations of $\mathbb{R}^{2n}$ that preserve its standard linear symplectic form. The Maslov index provides an explicit isomorphism $\pi_1(\mathrm{Sp}(2n)) \cong \mathbb{Z}$. It assigns an integer $\mu_M(\varphi)$ to every loop $\varphi : S^1 = \mathbb{R}/2\pi\mathbb{Z} \to \mathrm{Sp}(2n)$, uniquely characterized by the following properties.

– Homotopy: two loops in $\mathrm{Sp}(2n)$ are homotopic if and only if they have the same Maslov index.
– Product: for any two loops $\varphi_1, \varphi_2 : S^1 \to \mathrm{Sp}(2n)$ we have
$$\mu_M(\varphi_1 \cdot \varphi_2) = \mu_M(\varphi_1) + \mu_M(\varphi_2).$$
In particular, the constant identity loop has Maslov index zero.
– Direct sum: if $n = n_1 + n_2$, we may regard $\mathrm{Sp}(2n_1) \oplus \mathrm{Sp}(2n_2)$ as a subgroup of $\mathrm{Sp}(2n)$ and
$$\mu_M(\varphi_1 \oplus \varphi_2) = \mu_M(\varphi_1) + \mu_M(\varphi_2).$$
– Normalization: the loop $\varphi : S^1 \to U(1) \subset \mathrm{Sp}(2)$ defined by $\varphi(\theta) = e^{i\theta}$ has Maslov index one.

3.2 The Conley–Zehnder index for paths of symplectic matrices

Robin and Salamon [RS93] defined a Conley–Zehnder index which assigns a half-integer $\mu_{CZ}(\Gamma)$ to any path of symplectic matrices $\Gamma : [a, b] \to \mathrm{Sp}(2n)$. This Conley–Zehnder index satisfies the following properties.

(1) Naturality: $\mu_{CZ}(\Gamma) = \mu_{CZ}(\psi\Gamma\psi^{-1})$ for all $\psi \in \mathrm{Sp}(2n)$.

(2) Homotopy: $\mu_{CZ}(\Gamma)$ is invariant under homotopies of Γ with fixed endpoints.

(3) Zero: if $\Gamma(a)$ is the identity matrix and $\Gamma(t)$ has no eigenvalue on the unit circle for $t \in (a, b]$, then $\mu_{CZ}(\Gamma) = 0$.

(4) Direct sum: if $n = n_1 + n_2$, we may regard $\mathrm{Sp}(2n_1) \oplus \mathrm{Sp}(2n_2)$ as a subgroup of $\mathrm{Sp}(2n)$ and
$$\mu_{CZ}(\Gamma_1 \oplus \Gamma_2) = \mu_{CZ}(\Gamma_1) + \mu_{CZ}(\Gamma_2).$$

(5) Loop: if $\varphi : [a, b] \to \mathrm{Sp}(2n)$ is a loop with $\varphi(a) = \varphi(b) =$ identity matrix, then
$$\mu_{CZ}(\varphi \cdot \Gamma) = 2\mu_M(\varphi) + \mu_{CZ}(\Gamma).$$

(6) Concatenation: for any $a < c < b$ we have
$$\mu_{CZ}(\Gamma) = \mu_{CZ}(\Gamma|_{[a,c]}) + \mu_{CZ}(\Gamma|_{[c,b]}).$$

(7) Signature: given a symmetric $(2n \times 2n)$-matrix S with $\|S\| < 1$, the Conley–Zehnder index of the path $\Gamma : [0, 1] \to \mathrm{Sp}(2n)$ defined by $\Gamma(t) = \exp(2\pi J_0 S t)$ is given by
$$\mu_{CZ}(\Gamma) = \tfrac{1}{2}\,\mathrm{sign}\, S.$$

Here $\|S\| := \max_{|v|=1} |Sv|$, using the standard Euclidean norm on $\mathbb{R}^{2n}$, $\mathrm{sign}(S) :=$ signature of the matrix S, i.e. the number of positive minus the number of negative eigenvalues, and J_0 is

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the matrix representing the standard complex structure on $\mathbb{R}^{2n}$, i.e.

$$J_0 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

(8) Shear axiom: the index of a symplectic shear

$$\Gamma(t) = \begin{pmatrix} I & B(t) \\ 0 & I \end{pmatrix}$$

is given by $\frac{1}{2}\operatorname{sign} B(a) - \frac{1}{2}\operatorname{sign} B(b)$.

Example 3.1. If $\Gamma : [a,b] \to \mathrm{Sp}(2n)$ is a loop then

$$\mu_{CZ}(\Gamma) = 2\mu_M(\Gamma).$$

Example 3.2. Let $T > 0$ and $\Gamma : [0,T] \to U(1) \subset \mathrm{Sp}(2)$ be defined by

$$\Gamma(t) = e^{2\pi i t} = \begin{bmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{bmatrix}.$$

Then

$$\mu_{CZ}(\Gamma) = \begin{cases} 2T & \text{if } T \in \mathbb{N}, \\ 2\lfloor T \rfloor + 1 & \text{otherwise,} \end{cases}$$

where $\lfloor T \rfloor := \max\{n \in \mathbb{Z}; n \leqslant T\}$.

3.3 The Conley–Zehnder index for contractible periodic Reeb orbits

We will now define the Conley–Zehnder index of a periodic Reeb orbit which, for the sake of simplicity, we will assume to be contractible.

Let (N^{2n+1}, ξ) be a co-oriented contact manifold, with contact form α and Reeb vector field R_α. Given a contractible periodic Reeb orbit γ, consider a *capping disk* of γ, that is a map $\sigma_\gamma : D \to N$ that satisfies

$$\sigma_\gamma|_{\partial D} = \gamma.$$

Choose a (unique up to homotopy) symplectic trivialization

$$\Phi : \sigma_\gamma^* \xi \to D \times \mathbb{R}^{2n}.$$

We can define the symplectic path

$$\Gamma(t) = \Phi(\gamma(t)) \circ d(R_\alpha)_t(\gamma(0))|_\xi \circ \Phi^{-1}(\gamma(0)). \tag{4}$$

The Conley–Zehnder index of γ with respect to the capping disk σ_γ is defined by

$$\mu_{CZ}(\gamma, \sigma_\gamma) = \mu_{CZ}(\Gamma).$$

This is in general a half integer number, and it is an integer number if the periodic orbit is non-degenerate. This means that the linearized Poincaré map of γ has no eigenvalue equal to one.

This index in general does depend on the choice of the capping disk. More precisely, given another capping disk $\bar{\sigma}_\gamma$, we have

$$\mu_{CZ}(\gamma, \bar{\sigma}_\gamma) - \mu_{CZ}(\gamma, \sigma_\gamma) = 2\langle c_1(\xi), \bar{\sigma}_\gamma \#(-\sigma_\gamma)\rangle,$$

where $\bar{\sigma}_\gamma \#(-\sigma_\gamma)$ denotes the homology class of the gluing of the capping disks $\bar{\sigma}_\gamma$ and σ_γ with the reversed orientation. Notice however that the parity of the index of a non-degenerate closed orbit does not depend on the chosen capping disk. In particular, the index of a contractible non-degenerate periodic orbit is a well-defined element in $\mathbb{Z}/2c(\xi)\mathbb{Z}$, where

$$c(\xi) := \inf\{k > 0; \exists A \in \pi_2(N), \langle c_1(\xi), A\rangle = k\}$$

is the *minimal Chern number* of ξ (here we adopt the convention that the infimum over the empty set equals ∞).

Remark 3.3. We can define the Conley–Zehnder index of a contractible periodic orbit γ of a Hamiltonian flow on a symplectic manifold V in the same way, taking a capping disk σ_γ and a trivialization of TV over σ_γ. Analogously to periodic orbits of Reeb flows, the difference of the indexes with respect to two capping disks $\bar{\sigma}_\gamma$ and σ_γ is given by

$$\mu_{CZ}(\gamma, \bar{\sigma}_\gamma) - \mu_{CZ}(\gamma, \sigma_\gamma) = 2\langle c_1(TV), \bar{\sigma}_\gamma \#(-\sigma_\gamma)\rangle, \tag{5}$$

where $c_1(TV)$ is the first Chern class of TV.

3.4 Behavior of the Conley–Zehnder index under symplectic reduction

In this section we address the question of the relation between the Conley–Zehnder index of a periodic orbit and the Conley–Zehnder index of its symplectic reduction. This will be important later. Again, for the sake of simplicity, we will only consider contractible periodic orbits.

Let V be a symplectic manifold and $h : V \times \mathbb{R} \to \mathbb{R}$ a time-dependent Hamiltonian on V with a first integral $f : V \to \mathbb{R}$, that is, a function f constant along the orbits of h. Denote by X_h^t and X_f the Hamiltonian vector fields of h and f respectively. Consider a Riemannian metric on V induced by a compatible almost complex structure.

Let Z be a regular level of f, and suppose that X_f generates a free circle action on Z. Denote by W the Marsden–Weinstein reduced symplectic manifold Z/S^1. The Hamiltonian h induces a Hamiltonian g on W whose Hamiltonian flow ψ_t satisfies the relation $\pi \circ \varphi_t = \psi_t \circ \pi$, where φ_t is the Hamiltonian flow of h and $\pi : Z \to W$ is the quotient projection. In particular, every periodic orbit $\tilde{\gamma}$ of X_h^t gives rise to a periodic orbit $\gamma = \pi \circ \tilde{\gamma}$ of X_g^t with the same period.

Lemma 3.4. *Suppose that the linearized Hamiltonian flow of h on Z leaves the distribution* $\operatorname{span}\{\nabla f\}$ *invariant. Let $\tilde{\gamma}$ be a closed orbit of X_h contractible in Z and $\sigma_{\tilde{\gamma}} : D \to Z$ a capping disk for $\tilde{\gamma}$. Then the capping disk $\sigma_\gamma := \pi \circ \sigma_{\tilde{\gamma}}$ for the reduced periodic orbit γ satisfies*

$$\mu_{CZ}(\tilde{\gamma}, \sigma_{\tilde{\gamma}}) = \mu_{CZ}(\gamma, \sigma_\gamma).$$

Proof. Denote by $\mathcal{D}$ the symplectic distribution generated by X_f and ∇f. The hypothesis on ∇f and the fact that f is a first integral for h imply that $\mathcal{D}$ is invariant under $d\varphi_t$. Hence, the symplectic orthogonal complement $\mathcal{D}^\omega$ is also invariant under $d\varphi_t$.

Let $\Phi : \sigma_{\tilde{\gamma}}^* TV \to D^2 \times \mathbb{R}^{2d}$ be a (unique up to homotopy) trivialization of TV over $\sigma_{\tilde{\gamma}}$ ($\dim V = 2d$). Since X_f, ∇f and $\mathcal{D}^\omega$ are defined over the whole disk $\sigma_{\tilde{\gamma}}$, one can find a symplectic bundle isomorphism $\Psi : D^2 \times \mathbb{R}^{2d} \to D^2 \times \mathbb{R}^{2d}$ that covers the identity and satisfies:

(P1) $\pi_2(\Psi(\Phi(X_f))) = e_1$ and $\pi_2(\Psi(\Phi(\nabla f))) \in \operatorname{span}\{f_1\}$, where $\{e_1, \dots, e_d, f_1, \dots, f_d\}$ is a fixed symplectic basis in $\mathbb{R}^{2d}$ and $\pi_2 : D^2 \times \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is the projection onto the second factor;

(P2) $\pi_2(\Psi(\Phi(\sigma_{\tilde{\gamma}}^* \mathcal{D}^\omega))) = \operatorname{span}\{e_2, \dots, e_n, f_2, \dots, f_n\}$.

Note that $\sigma_\gamma = \pi \circ \sigma_{\tilde{\gamma}}$ is a capping disk for γ and the differential of π induces the identification $d\pi|_{\mathcal{D}^\omega} : \sigma_{\tilde{\gamma}}^* \mathcal{D}^\omega \to \sigma_\gamma^* TW$. Hence, in order that Φ induces a trivialization over σ_γ it is enough to

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choose it such that it sends $\mathcal{D}^\omega$ to a fixed symplectic subspace in $\mathbb{R}^{2d}$. Property (P2) ensures that the trivialization $\Lambda := \Psi \circ \Phi$ satisfies this property. In fact, consider the splitting $\mathbb{R}^{2d} = E_1 \oplus E_2$, where $E_1 = \operatorname{span}\{e_1, f_1\}$ and $E_2 = \operatorname{span}\{e_2, \dots, e_n, f_2, \dots, f_n\}$. We have $\Lambda(\sigma_{\tilde\gamma}^* \mathcal{D}^\omega) = D^2 \times E_2$, and the trivialization over σ_γ is then given by

$$\Lambda \circ (d\pi|_{\mathcal{D}^\omega})^{-1} : \sigma_\gamma^* TW \to D^2 \times E_2.$$

Now, define the symplectic path

$$\Gamma(t) = \Lambda(\tilde\gamma(t)) \circ d\varphi_t(\tilde\gamma(0)) \circ \Lambda^{-1}(\tilde\gamma(0)),$$

so that $\mu(\tilde\gamma, \sigma_{\tilde\gamma}) = \mu(\Gamma)$. Since f is a first integral, X_f is preserved by $d\varphi_t$ and, by hypothesis, $\operatorname{span}\{\nabla f\}$ is preserved as well. Thus, by property (P1), $\Gamma|_{E_1}$ is a symmetric symplectic path in $\mathbb{R}^2$ with an eigenvalue one. However, a symmetric symplectic isomorphism in $\mathbb{R}^2$ with an eigenvalue one is necessarily the identity.

Consequently, the direct sum property of the index yields

$$\mu(\Gamma) = \mu(\Gamma|_{E_1}) + \mu(\Gamma|_{E_2}) = \mu(\Gamma|_{E_2}) = \mu(\gamma, \sigma_\gamma),$$

finishing the proof of the lemma.

Remark 3.5. The assumption on $\operatorname{span}\{\nabla f\}$ is necessary. In order to show this, consider the Hamiltonian $h : \mathbb{C}^2 \to \mathbb{R}$ given by $h(z_1, z_2) = g(|z_1|^2 + |z_2|^2)$, where g is a smooth real function. It is obviously invariant under the Hamiltonian circle action generated by $f(z_1, z_2) = |z_1|^2 + |z_2|^2$ whose reduced symplectic manifold is S^2. Every reduced orbit is a constant solution whose constant capping disk has index zero. Consequently, by (5) and the fact that $c_1(TS^2) = 2$, the index of a reduced orbit is given by an integer multiple of four, whatever is the choice of the capping disk. However, one can show that a non-constant orbit $\tilde\gamma$ of h has index

$$\mu(\tilde\gamma) = \begin{cases} 7/2 & \text{if } g''(f(\tilde\gamma)) < 0, \\ 4 & \text{if } g''(f(\tilde\gamma)) = 0, \\ 9/2 & \text{if } g''(f(\tilde\gamma)) > 0. \end{cases}$$

The hint for showing this fact is the existence of a trivialization over a capping disk $\sigma_{\tilde\gamma}$ such that the linearized Hamiltonian flow restricted to the subspace spanned by X_f and ∇f is given by the symplectic shear

$$\begin{pmatrix} 1 & -g''(f(\tilde\gamma))t \\ 0 & 1 \end{pmatrix}.$$

Note that the linearized Hamiltonian flow of h preserves ∇f precisely when $g''(f(\tilde\gamma)) = 0$.

Remark 3.6. The hypothesis that $\tilde\gamma$ is contractible *in* Z is also necessary (notice that to define the Conley–Zehnder index of $\tilde\gamma$ we need only to suppose that $\tilde\gamma$ is contractible *in* V). As a matter of fact, let W be a symplectic manifold with first Chern class different from one, and consider on $V := W \times \mathbb{C}$ the circle action generated by $f(p, z) = |z|^2$. It is easy to see that every orbit $\tilde\gamma$ of f has a capping disk with index two. On the other hand, the reduced orbit is a constant solution γ whose constant capping disk has index zero. Consequently, the hypothesis on $c_1(TW)$ and (5) imply that there is no capping disk for γ with index two. A less trivial argument can give examples where the orbits are not contractible but homologically trivial.

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4. Cylindrical contact homology

There are several versions of contact homology (see [Bou09] for a survey). A suitable one for our purposes is cylindrical contact homology, whose definition is closer to the usual construction of Floer homology [Flo88a, Flo88b, Flo89] but with some rather technical differences. The aim of this section is to sketch this construction. Details can be found in [Bou03b, EGH00, van05, Ust99] and references therein.

Let α be a contact form on N^{2n+1} with contact structure $\xi = \ker\alpha$, and let R_α be its Reeb vector field. For the sake of simplicity, we will assume that $c_1(\xi) = 0$. Denote by $\mathcal{P}$ the set of periodic orbits of R_α, and suppose that R_α is non-degenerate, i.e. every closed orbit $\gamma \in \mathcal{P}$ is non-degenerate. A periodic orbit of R_α is called *bad* if it is an even multiple of a periodic orbit whose parities of the Conley–Zehnder index of odd and even iterates disagree. An orbit that is not bad is called *good*. Denote the set of good periodic orbits by $\mathcal{P}^0(\alpha)$.

Consider the chain complex $CC_*(\alpha)$ given by the graded group with coefficients in $\mathbb{Q}$ generated by good periodic orbits of R_α graded by their Conley–Zehnder index plus $n-2$. This extra term $n-2$ is not important in the definition of cylindrical contact homology but the reason for its use will be apparent later. Let us denote the degree of a periodic orbit by $|\gamma|$.

The boundary operator ∂ is given by counting rigid holomorphic cylinders in the symplectization $(W, \omega) := (\mathbb{R} \times N, d(e^t\alpha))$. More precisely, fix an almost complex structure J on W compatible with ω such that J is invariant by t-translations, $J(\xi) = \xi$ and $J(\partial/\partial t) = R_\alpha$. The space of these almost complex structures is contractible. Let $\Sigma = S^2 \backslash \Gamma$ be a punctured rational curve, where S^2 is endowed with a complex structure j and $\Gamma = \{x, y_1, \dots, y_s\}$ is the set of (ordered) punctures of Σ. We will consider holomorphic curves from Σ to the symplectization W, that is, smooth maps $F = (a, f) : \Sigma \to W$ satisfying $dF \circ j = J \circ dF$. We restrict ourselves to holomorphic curves such that, for polar coordinates (ρ, θ) centered at a puncture $p \in \Gamma$, the following conditions hold:

$$\lim_{\rho \to 0} a(\rho, \theta) = \begin{cases} +\infty & \text{if } p = x, \\ -\infty & \text{if } p = y_i \text{ for some } i = 1, \dots, s, \end{cases}$$

$$\lim_{\rho \to 0} f(\rho, \theta) = \begin{cases} \gamma(-T\theta/2\pi) & \text{if } p = x, \\ \gamma_i(T_i\theta/2\pi) & \text{if } p = y_i \text{ for some } i = 1, \dots, s, \end{cases}$$

where γ and γ_i are good periodic orbits of R_α of periods T and T_i respectively. Denote the set of such holomorphic curves by $\mathcal{M}(\gamma, \gamma_1, \dots, \gamma_s; J)$, and notice that j is not fixed. Define an equivalence relation $\simeq$ on $\mathcal{M}(\gamma, \gamma_1, \dots, \gamma_s; J)$ by saying that $(F = (a, f), j)$ and $(\tilde{F} = (\tilde{a}, \tilde{f}), \tilde{j})$ are equivalent if there is a shift $\tau \in \mathbb{R}$ and a biholomorphism $\varphi : (S^2, j) \to (S^2, \tilde{j})$ such that $\varphi(p) = p$ for every $p \in \Gamma$ and

$$(a, f) = (\tilde{a} \circ \varphi + \tau, \tilde{f} \circ \varphi).$$

Define the moduli space $\widehat{\mathcal{M}}(\gamma, \gamma_1, \dots, \gamma_s; J)$ as $\mathcal{M}(\gamma, \gamma_1, \dots, \gamma_s; J)/\simeq$. A crucial ingredient in order to understand the set $\widehat{\mathcal{M}}(\gamma, \gamma_1, \dots, \gamma_s; J)$ is the operator $D_{(F,j)} : T_F\mathcal{B}^{1,p,\delta}(\Sigma, V) \times T_j\mathcal{T} \to L^{p,\delta}(\Sigma, F^*TV)$ called the *vertical differential* and given by

$$D_{(F,j)}(\psi, y) = \nabla\psi + J \circ \nabla\psi \circ j + (\nabla_\psi J) \circ DF \circ j + J \circ DF \circ y,$$

where $p > 2$, $\delta > 0$ is sufficiently small, $\mathcal{B}^{1,p,\delta}(\Sigma, V)$ is the Banach manifold consisting of $W^{1,p}_{\mathrm{loc}}$ maps from Σ to W with a suitable behavior near the punctures, $\mathcal{T}$ stands for a Teichmüller slice through j as defined in [Wen10], and $L^{p,\delta}(\Sigma, F^*TV)$ is a weighted Sobolev space given by

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the completion of the space of smooth anti-holomorphic 1-forms $\Omega^{0,1}(\Sigma, F^*TV)$ with respect to suitable norms, see [Wen10] for details. Notice that we are tacitly taking the Levi-Civita connection given by the metric induced by the symplectic form and the almost complex structure.

This is a Fredholm operator with index given by

$$|\gamma| - \sum_{i=1}^{s} |\gamma_i| + \dim \operatorname{Aut}(\Sigma, j),$$

where $\operatorname{Aut}(\Sigma, j)$ is the group of automorphisms of (Σ, j), see [Wen10, p. 376]. We say that J is *regular* if $D_{(F,j)}$ is surjective for every holomorphic curve (F, j) (it does not depend on the choice of the Teichmüller slice, see [Wen10, Lemma 3.11]). It turns out that if J is regular then $\widehat{\mathcal{M}}(\gamma, \gamma_1, \dots, \gamma_s; J)$ is a smooth manifold with dimension given by

$$|\gamma| - \sum_{i=1}^{s} |\gamma_i| - 1.$$

Moreover, $\widehat{\mathcal{M}}(\gamma, \gamma_1, \dots, \gamma_s; J)$ admits a compactification $\overline{\mathcal{M}}(\gamma, \gamma_1, \dots, \gamma_s; J)$ with a coherent orientation [BM04] whose boundary is given by holomorphic buildings [BEHWZ03]. In particular, if J is regular, then $\overline{\mathcal{M}}(\gamma, \gamma_1, \dots, \gamma_s; J)$ is a finite set with signs whenever $|\gamma| - \sum_{i=1}^{s} |\gamma_i| = 1$.

However, unlike Floer homology in the monotone case, regularity is not achieved in general by a generic choice of J. Instead, one needs to use multi-valued perturbations equivariant with respect to the action of biholomorphisms, and this turns out to be a very delicate issue. Several ongoing approaches have been developed to give a rigorous treatment to this problem, see [CM07, HWZ07, HWZ09a, HWZ09b]. Consequently, following [BO09, Remark 9], we will assume the following technical condition throughout this work.

Transversality assumption. We suppose that the almost complex structure J is regular for holomorphic curves with index less or equal than two (the index of a holomorphic curve is defined as the degree of the positive periodic orbit minus the sum of the degrees of the negative ones). Moreover, we will also assume the existence of regular almost complex structures for holomorphic curves with index less or equal than one in cobordisms and with index less or equal than zero in 1-parameter families of cobordisms.

We have then the following result on the structure of moduli spaces of holomorphic cylinders in symplectizations.

Proposition 4.1 [EGH00]. *Under the previous transversality assumption, the moduli spaces $\overline{\mathcal{M}}(\gamma, \gamma_1)$ of dimension zero consist of finitely many points with rational weights. The moduli spaces $\overline{\mathcal{M}}(\gamma, \gamma_1)$ of dimension one have boundary given by finitely many points corresponding to holomorphic buildings with rational weights whose sum counted with orientations vanishes. Moreover, if a holomorphic building in the boundary consists of a broken cylinder then its weight is given by the product of the weights of each cylinder.*

We expect the transversality assumption to be completely removed using the polyfold theory developed by Hofer, Wysocki and Zehnder, see [HWZ07, HWZ09a, HWZ09b].

Thus, fix $s = 1$; that is, let Σ be a cylinder. By the discussion above, if two periodic orbits γ and $\bar{\gamma}$ satisfy $|\gamma| = |\bar{\gamma}| + 1$ then $\overline{\mathcal{M}}(\gamma, \bar{\gamma})$ is a finite set. This enables us to define the boundary operator in the following way. Let γ be a periodic orbit of multiplicity $m(\gamma)$, i.e. γ is a covering of degree $m(\gamma)$ of a simple closed orbit. Define

$$\partial\gamma = m(\gamma) \sum_{\bar{\gamma}\in\mathcal{P}^0(\alpha),|\bar{\gamma}|=|\gamma|-1} \sum_{F\in\overline{\mathcal{M}}(\gamma,\bar{\gamma})} \mathrm{sign}(F)\ \mathrm{weight}(F)\bar{\gamma},$$

where $\mathrm{sign}(F)$ is the sign of F determined by the coherent orientation of $\overline{\mathcal{M}}(\gamma,\bar{\gamma})$ and $\mathrm{weight}(F)$ is the weight established in the previous proposition. A somewhat different definition of the boundary operator is given in [EGH00] using asymptotic markers, but one can check that this is equivalent to the definition above. Notice the similarity with Floer homology, but we have to consider weights in the boundary operator.

The next proposition is a generalization of [EGH00, § 1.9.2], where it is shown that cylindrical contact homology is well defined and an invariant of the contact structure for nice contact forms. The specific nature of the weights in the boundary operator does not play any role in the proof; the point is to avoid the presence of certain tree-like curves in the boundary of moduli spaces of dimension one, and it is here that the hypothesis on the contact forms comes in. As a matter of fact, as will be accounted in the proof, the assumption that the contact form is even implies that there is no holomorphic curve of index one in the symplectization, and the hypothesis of non-existence of periodic orbits of degree 1, 0 and -1 is to avoid rigid planes (rigid means that it belongs to a moduli space of dimension zero) in symplectizations, cobordisms and cobordisms in 1-parameter families of cobordisms respectively.

Following exactly as in the proof in [Bou03b, EGH00, Ust99], one can extend the argument to even contact forms, and prove Proposition 1.2, which we restate here for the convenience of the reader.

Proposition 4.2. *Let (N,ξ) be a contact manifold with an even or nice non-degenerate contact form α. Then the boundary operator $\partial : C_*(N,\alpha) \to C_{*-1}(N,\alpha)$ satisfies $\partial^2 = 0$, and the homology of $(C_*(N,\alpha),\partial)$ is independent of the choice of even or nice non-degenerate contact form α. Hence, the cylindrical contact homology $HC_*(N,\xi;\mathbb{Q})$ is a well-defined invariant of the contact manifold (N,ξ).*

Proof. The proof follows the proofs in [Bou03b, EGH00, Ust99]. We will just recall the main steps and explain how to proceed with even contact forms.

If α is even then obviously $\partial^2 = 0$, since $\partial = 0$. To deal with the case that α is nice, notice that ∂^2 counts broken rigid holomorphic cylinders in the symplectization of α that appear (by a gluing argument) as points in the boundary of the moduli space of cylinders connecting orbits with index difference equal to two. The condition that R_α has no periodic orbit of degree one implies that there is nothing else in the boundary of this moduli space. Indeed, we could have in the boundary a tree-like curve with one level of index 1 and the other level consisting of a rigid plane and a vertical cylinder. However, the non-existence of orbits of degree one excludes the existence of these planes.

Thus, ∂^2 counts points in the boundary of the moduli space of dimension one, and the sum of the weights of these points counted with orientations vanishes. These weights are the products of the weights of the rigid holomorphic cylinders in each level and the number of ways that such cylinders can be glued to each other is given precisely by the multiplicity of the closed orbit where we glue. This is the reason why the factor $m(\gamma)$ appears in the definition of ∂. Hence it follows that $\partial^2 = 0$.

Now, let us consider the invariance problem. To carry it out, we will construct an isomorphism $\Phi : HC_*(N,\alpha) \to HC_*(N,\tilde{\alpha})$. Since $\tilde{\alpha}$ defines the same contact structure as α we can write $\tilde{\alpha} = f\alpha$, where $f : N \to \mathbb{R}$ is a smooth positive function. Take a function $g : \mathbb{R} \times N \to \mathbb{R}$ such that $g(t, x) = e^t$ for $t > R$, $g(t, x) = e^t f(x)$ for $t < -R$ and $\partial_t g > 0$, where $R > 0$ is a constant big enough. It is easy to check that $d(g\alpha)$ is a symplectic form on $\mathbb{R} \times N$. We call $(W, \omega) := (\mathbb{R} \times N, d(g\alpha))$ a symplectic cobordism with ends $W_+ = (R, +\infty) \times N$ and $W_- = (-\infty, -R) \times N$, restricted to which ω coincides with the symplectic forms of the symplectizations of α and $\tilde{\alpha}$ respectively. Denote by J_+ and J_- the corresponding almost complex structures on W_+ and W_- as defined previously, and consider a compatible almost complex structure J_W on W that extends J_- and J_+.

In order to define Φ, we need to consider holomorphic curves on W in a similar fashion to what we did in symplectizations. More precisely, let Σ be as before, and fix a periodic orbit γ of R_α and periodic orbits $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_s$ of $R_{\tilde{\alpha}}$. We look at holomorphic curves $F : \Sigma \to W$ that are asymptotic to γ and $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_s$ at the positive and negative punctures respectively. Denote the set of such curves by $\mathcal{M}(\gamma, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_s; J_W)$.

Analogously to symplectizations, define an equivalence relation $\simeq$ on $\mathcal{M}(\gamma, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_s; J_W)$ by saying that $F = (a, f)$ and $\tilde{F} = (\tilde{a}, \tilde{f})$ are equivalent if there is a biholomorphism $\varphi : S^2 \to S^2$ that restricted to Γ is the identity and

$$(a, f) = (\tilde{a} \circ \varphi, \tilde{f} \circ \varphi).$$

The moduli space $\widehat{\mathcal{M}}(\gamma, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_s; J_W) := \mathcal{M}(\gamma, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_s; J_W)/\simeq$ admits a compactification

$$\overline{\mathcal{M}}(\gamma, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_s; J_W)$$

whose boundary is given by holomorphic buildings.

Under our transversality assumption, a result similar to Proposition 4.1 holds for symplectic cobordisms, and it establishes that one can choose J_W such that the moduli spaces $\overline{\mathcal{M}}(\gamma, \tilde{\gamma}_1; J_W)$ of dimension zero or one have the desired properties. The dimension is given by $|\gamma| - |\gamma_1|$.

We define a map $\Psi : CC_*(N, \alpha) \to CC_*(N, \widetilde{\alpha})$ by

$$\Psi(\gamma) = m(\gamma) \sum_{\tilde{\gamma} \in \mathcal{P}^0(\tilde{\alpha}), |\tilde{\gamma}| = |\gamma|} \sum_{F \in \overline{\mathcal{M}}(\gamma, \tilde{\gamma})} \operatorname{sign}(F) \operatorname{weight}(F) \tilde{\gamma}.$$

In order to show that Ψ is a chain map, the idea, as in the proof of $\partial^2 = 0$, is to identify $\partial_{\tilde{\alpha}} \Psi - \Psi \partial_\alpha$ with the boundary of a moduli space of dimension one. To achieve this identification, consider the moduli space

$$\overline{\mathcal{M}}(\gamma; J_W) := \bigcup_{\tilde{\gamma} \in \mathcal{P}^0(\tilde{\alpha}), |\gamma| = |\tilde{\gamma}| + 1} \overline{\mathcal{M}}(\gamma, \tilde{\gamma}; J_W).$$

By a gluing argument, the broken cylinders counted in $(\partial_{\tilde{\alpha}} \Psi - \Psi \partial_\alpha)(\gamma)$ are contained in $\partial \overline{\mathcal{M}}(\gamma; J_W)$. We need to show that there is nothing else than these broken cylinders in $\partial \overline{\mathcal{M}}(\gamma; J_W)$. However, the compactness results show that the boundary is given by holomorphic buildings with two levels. Hence, we may have two possibilities:

- a pair of pants of index 0 in the cobordism W, and a rigid plane and a vertical cylinder in the symplectization of $\tilde{\alpha}$;
- a punctured sphere of index 1 in the symplectization of α, and (possibly several) rigid planes and a rigid cylinder in the cobordism W.

The first possibility does not hold because if $\tilde{\alpha}$ is even or nice then there is no rigid holomorphic plane in the symplectization of $\tilde{\alpha}$. The second possibility, in turn, is forbidden because, if α is even, there is no rigid holomorphic curve in the symplectization and, if α is nice,

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there is no rigid plane in the cobordism from α to $\tilde{\alpha}$, since there is no orbit of degree zero. This shows that Ψ is a chain map, and consequently it induces a map Φ in the homology.

To prove that Φ is an isomorphism we construct its inverse. Consider the map $\tilde{\Psi}: CC_*(N, \tilde{\alpha}) \to CC_*(N, \alpha)$ obtained by the construction above switching α and $\tilde{\alpha}$. We claim that $\tilde{\Psi} \circ \Psi$ is chain homotopic to the identity. Indeed,

$$\tilde{\Psi} \circ \Psi - \mathrm{Id} = \partial_\alpha \circ A + A \circ \partial_\alpha,$$

where $A: CC_*(N, \alpha) \to CC_{*+1}(N, \alpha)$ is a map of degree one obtained in the following way.

Consider a 1-parameter family of symplectic cobordisms $W_\lambda := (W, \omega_\lambda)$, $\lambda \in [0, 1]$, such that W_0 is the symplectic cobordism given by the gluing of the cobordisms from α to $\tilde{\alpha}$ and from $\tilde{\alpha}$ to α, and W_1 is the symplectization of α. Let J_λ be a smooth family of almost complex structures compatible with ω_λ, and consider the set

$$\overline{\mathcal{M}}(\gamma, \gamma_1, \ldots, \gamma_s; \{J_\lambda\}) = \{(\lambda, F); 0 \leqslant \lambda \leqslant 1, F \in \overline{\mathcal{M}}(\gamma, \gamma_1, \ldots, \gamma_s; J_\lambda)\}.$$

Once again, a result similar to Proposition 4.1 holds for 1-parameter families of symplectic cobordisms establishing that one can choose J_λ such that the moduli spaces $\overline{\mathcal{M}}(\gamma, \gamma_1; \{J_\lambda\})$ of dimension zero or one have the desired properties. Now, the dimension is given by $|\gamma| - |\gamma_1| + 1$. Hence if γ and $\bar{\gamma}$ are good periodic orbits of R_α such that $|\gamma| - |\bar{\gamma}| = -1$ then $\mathcal{M}(\gamma, \bar{\gamma}; \{J_\lambda\})$ is a finite set. Define

$$A(\gamma) = m(\gamma) \sum_{\bar{\gamma} \in \mathcal{P}^0(\alpha), |\bar{\gamma}| = |\gamma| + 1} \; \sum_{F \in \overline{\mathcal{M}}(\gamma, \bar{\gamma}; \{J_\lambda\})} \mathrm{sign}(F) \; \mathrm{weight}(F) \bar{\gamma}.$$

By compactness results, if $|\gamma| = |\bar{\gamma}|$ then the boundary of $\overline{\mathcal{M}}(\gamma, \bar{\gamma}; \{J_\lambda\})$ is given by components coming from the boundary of $[0, 1]$ and holomorphic buildings of height two. This first component is the union of $\overline{\mathcal{M}}(\gamma, \bar{\gamma}; J_0)$ and $\overline{\mathcal{M}}(\gamma, \bar{\gamma}; J_1)$, and it counts as $\tilde{\Psi} \circ \Psi - \mathrm{Id}$, since every cylinder of index zero in the symplectization W_1 is trivial. The second one is given by broken cylinders of index zero counted by $\partial_\alpha \circ A + A \circ \partial_\alpha$, and, besides these broken cylinders, we might have three possibilities:

- a pair of pants of index -1 in a cobordism W_λ, and a rigid plane and a vertical cylinder in the symplectization of α;
- a pair of pants of index 1 in the symplectization of α, and a plane of index -1 and a cylinder of index 0 in a cobordism W_λ;
- a punctured sphere of index 1 in the symplectization of α, (possibly several) planes of index 0, and a cylinder of index -1 in a cobordism W_λ.

The first case is discarded because there is no rigid plane in the symplectization if α is even or nice. The second one does not hold if α is even since there is no rigid curve in the symplectization, and, if α is nice, there is no plane of index -1 in the cobordism. Finally, the third possibility cannot happen because, if α is even, there is no rigid curve in the symplectization and, if α is nice, there is no plane of index 0 in the cobordism. □

5. Proof of Theorem 1.3

Let us first describe the idea of the proof. We consider a Sasaki contact form on N whose Reeb flow has finitely many non-degenerate simple periodic orbits γ_ℓ, $\ell = 1, \ldots, m$, where m is the number of edges of the good moment cone. Let X be the Liouville vector field of the

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corresponding good symplectic cone, and consider the X-invariant Hamiltonian flow associated to the Reeb flow. For each $\ell = 1, \ldots, m$ we choose a suitable lift of this Hamiltonian flow to a linear flow on $\mathbb{R}^{2d}$ using the symplectic reduction process described after Remark 2.14. More precisely, we require that the lift $\tilde{\gamma}_\ell$ of γ_ℓ is a *closed* orbit in $\mathbb{R}^{2d}$. This enables us to apply Lemma 3.4, and consequently reduces the proof to the computation of the Conley–Zehnder index of $\tilde{\gamma}_\ell$. For this computation, we can use the global trivialization of $T\mathbb{R}^{2d}$, and, since the lifted flow is given by a 1-parameter subgroup of $\mathbb{T}^d$ via the usual $\mathbb{T}^d$-action in $\mathbb{R}^{2d} \simeq \mathbb{C}^d$, the index is easily computed by the corresponding vector in the Lie algebra. This vector, given by (6), is completely determined by the associated good moment cone, and it turns out that the degree of every orbit is an even number.

Let (W, ω, X) be a good toric symplectic cone determined by a good moment cone $C \subset (\mathbb{R}^{n+1})^*$ defined by

$$C = \bigcap_{j=1}^{d} \{x \in (\mathbb{R}^{n+1})^* : \ell_j(x) := \langle x, \nu_j \rangle \geqslant 0\}$$

where $d \geqslant n+1$ is the number of facets and each ν_j is a primitive element of the lattice $\mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1}$ (the inward-pointing normal to the jth facet of C).

Let $\nu \in \mathfrak{t} \cong \mathbb{R}^{n+1}$ be any vector in the Lie algebra of the torus $\mathbb{T}^{n+1}$ satisfying the following two conditions.

(i) It can be expressed as

$$\nu = \sum_{j=1}^{d} a_j \nu_j \quad \text{with } a_j \in \mathbb{R}^+ \text{ for all } j = 1, \ldots, d.$$

(ii) The 1-parameter subgroup generated by ν is dense in $\mathbb{T}^{n+1}$.

Let $R_\nu \in \mathcal{X}_X(W, \omega) \cong \mathcal{X}(N, \xi)$ be the Reeb vector field of the Sasaki contact form $\alpha_\nu \in \Omega^1(N, \xi)$.

Lemma 5.1. *The Reeb vector field R_ν has exactly m simple closed orbits, where*

$$m = \textit{number of edges of } C.$$

Proof. Under the moment map $\mu : W \to C \in (\mathbb{R}^{n+1})^*$, any R_ν-orbit γ is mapped to a single point $p \in C$. The pre-image $\mu^{-1}(p)$ is a $\mathbb{T}^{n+1}$-orbit, and the fact that ν generates a dense 1-parameter subgroup of $\mathbb{T}^{n+1}$ implies that the R_ν-orbit γ is dense in $\mu^{-1}(p)$. Hence, γ is closed if and only if $\dim(\mu^{-1}(p)) = 1$, and this happens if and only if p belongs to a one-dimensional face of P, i.e. an edge. □

Remark 5.2. If the toric symplectic cone W is not simply connected, the simple closed Reeb orbit γ associated to an edge E of the moment cone C might not be contractible. However, it follows from Proposition 2.15 that a finite multiple of γ is contractible, and 'simple closed Reeb orbit associated to E' will always mean 'smallest multiple of γ that is contractible'.

Let $E_1, \ldots, E_m$ denote the edges of C, and $\gamma_1, \ldots, \gamma_m$ the corresponding simple closed orbits of the Reeb vector field R_ν. Since C is a good cone, each edge E_ℓ is the intersection of exactly n facets $F_{\ell_1}, \ldots, F_{\ell_n}$, whose set of normals

$$\nu_{\ell_1}, \ldots, \nu_{\ell_n}$$

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can be completed to an integral base of $\mathbb{Z}^{n+1}$. Hence, for each $\ell = 1, \ldots, m$, we can choose an integral vector $\eta_\ell \in \mathbb{Z}^{n+1}$ such that

$$\{\nu_{\ell_1}, \ldots, \nu_{\ell_n}, \eta_\ell\} \text{ is an integral base of } \mathbb{Z}^{n+1}.$$

The map $\beta : \mathbb{R}^d \to \mathbb{R}^{n+1}$ defined by (1) is surjective and integral ($\beta(\mathbb{Z}^d) \subset \mathbb{Z}^{n+1}$). Hence, for each $\ell = 1, \ldots, m$, there is a smallest natural number $N_\ell \in \mathbb{N}$ and an integral vector $\tilde{\eta}_\ell \in \mathbb{Z}^d$ such that

$$\beta(\tilde{\eta}_\ell) = N_\ell \, \eta_\ell.$$

The Reeb vector field R_ν can be uniquely written as

$$R_\nu = \sum_{i=1}^{n} b_i^\ell \nu_{\ell_i} + b^\ell N_\ell \eta_\ell \quad \text{with } b_1^\ell, \ldots, b_n^\ell, b^\ell \in \mathbb{R},$$

and we can then lift it to a vector $\tilde{R}_\nu^\ell \in \mathbb{R}^d$ as

$$\tilde{R}_\nu^\ell = \sum_{i=1}^{n} b_i^\ell e_{\ell_i} + b^\ell \tilde{\eta}_\ell, \tag{6}$$

so that

$$\beta(\tilde{R}_\nu^\ell) = R_\nu.$$

Remark 5.3. $N_\ell =$ 'smallest multiple' considered in Remark 5.2. If the moment cone C determines a simply connected toric symplectic cone W, then $N_\ell = 1$, for all $\ell = 1, \ldots, m$.

Recall from § 2.3 that $W = Z/K$, where $K = \ker \beta \subset \mathbb{T}^d$ and

$$Z = \phi_K^{-1}(0) \backslash \{0\} \equiv \text{zero level set of moment map in } \mathbb{C}^d \backslash \{0\}.$$

The restriction to $Z \subset \mathbb{C}^d$ of the linear flow on $\mathbb{C}^d$ generated by $\tilde{R}_\nu^\ell$ is a lift of the Reeb flow on W generated by R_ν. Consider

$$\begin{aligned} Z &\longrightarrow W = Z/K \xrightarrow{\mu} C \subset (\mathbb{R}^{n+1})^* \\ z &\longmapsto [z]. \end{aligned}$$

We have

$$[z] \in \mu^{-1}(E_\ell) \Leftrightarrow z_{\ell_1} = \cdots = z_{\ell_n} = 0.$$

This implies that γ_ℓ can be lifted to Z as a closed orbit $\tilde{\gamma}_\ell$ of $\tilde{R}_\nu^\ell$. The periods of γ_ℓ and $\tilde{\gamma}_\ell$ are both given by

$$T_\ell = \frac{2\pi}{b^\ell},$$

and the linearization of the lifted Hamiltonian Reeb flow on $\mathbb{C}^d$ along $\tilde{\gamma}_\ell$ is the linear flow generated by $\tilde{R}_\nu^\ell$. Note that, by replacing η_ℓ with $-\eta_\ell$ if necessary, we can, and will, assume that $b^\ell > 0$ for all $\ell = 1, \ldots, m$. We can now use Lemma 3.4 to assert that

$$\mu_{CZ}(\gamma_\ell^N) = \mu_{CZ}(\tilde{\gamma}_\ell^N)$$

for all $\ell = 1, \ldots, m$ and all iterates $N \in \mathbb{N}$.

To compute $\mu_{CZ}(\tilde{\gamma}_\ell^N)$, note first that, since R_ν is assumed to generate a dense 1-parameter subgroup of the torus $\mathbb{T}^{n+1}$, we have that the closure of the 1-parameter subgroup of $\mathbb{T}^d$ generated by $\tilde{R}_\nu^\ell$ is a torus of dimension $n+1$. That immediately implies that the $n+1$ real numbers

$$\{b_1^\ell, \ldots, b_n^\ell, b^\ell\}$$

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are $\mathbb{Q}$-independent. We can then use Example 3.2 and the direct sum property of the Conley–Zehnder index to conclude that

$$\begin{aligned}\mu_{CZ}(\tilde{\gamma}_\ell^N) &= \sum_{i=1}^{n}\left(2\left\lfloor N\frac{b_i^\ell}{b^\ell}\right\rfloor + 1\right) + 2N\left(\sum_{j=1}^{d}(\tilde{\eta}_\ell)_j\right)\\ &= 2\left(\sum_{i=1}^{n}\left\lfloor N\frac{b_i^\ell}{b^\ell}\right\rfloor + N\left(\sum_{j=1}^{d}(\tilde{\eta}_\ell)_j\right)\right) + n\\ &= \text{even} + n.\end{aligned}$$

This implies that the contact homology degree is given by

$$\deg(\tilde{\gamma}_\ell^N) = \mu_{CZ}(\tilde{\gamma}_\ell^N) + n - 2 = \text{even} + n + n - 2 = \text{even},$$

which finishes the proof of Theorem 1.3.

6. Examples and proof of Theorem 1.4

6.1 A particular family of good moment cones

Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. For each $k \in \mathbb{N}_0$ consider the cone $C(k) \subset \mathbb{R}^3$ with four facets defined by the following four normals:

$$\begin{aligned}\nu_1 &= e_1 + e_3 = (1, 0, 1),\\ \nu_2 &= -e_2 + e_3 = (0, -1, 1),\\ \nu_3 &= ke_2 + e_3 = (0, k, 1),\\ \nu_4 &= -e_1 + (2k-1)e_2 + e_3 = (-1, 2k-1, 1).\end{aligned}$$

Each of these cones is good, and hence defines a smooth, connected, closed toric contact 5-manifold (N_k, ξ_k). Because all the normals have last coordinate equal to one, Remark 2.17 implies that the first Chern class of all these contact manifolds is zero. Moreover, one can use Proposition 2.15 to easily check that N_k is simply connected for all $k \in \mathbb{N}$. In fact, this family of good cones is $SL(3, \mathbb{Z})$ equivalent to the family of moment cones associated to the Sasaki–Einstein toric manifolds $Y^{p,q}$, with $q = 1$ and $p = k + 1$, constructed by Gauntlett *et al.* in [GMSW04a] (see also [MSY06]). Hence we have that

$$(N_k, \xi_k) \cong (S^2 \times S^3, \xi_k) \quad \text{with } c_1(\xi_k) = 0$$

and, as hyperplane distributions, the ξ_k are all homotopic to each other.

When $k = 0$ there is a direct way of identifying the toric contact manifold (N_0, ξ_0). In fact, the cone $C(0) \subset \mathbb{R}^3$ is $\mathrm{SL}(3, \mathbb{Z})$ equivalent to the cone $C' \subset \mathbb{R}^3$ defined by the following four normals:

$$\begin{aligned}\nu_1' &= e_1 = (1, 0, 0),\\ \nu_2' &= -e_2 + e_3 = (0, -1, 1),\\ \nu_3' &= e_2 = (0, 1, 0),\\ \nu_4' &= -e_1 + e_3 = (-1, 0, 1).\end{aligned}$$

One easily checks that C' is the standard cone over the square $[0, 1] \times [0, 1] \subset \mathbb{R}^2$. Hence, (N_0, ξ_0) can be described as the Boothby–Wang manifold over $(S^2 \times S^2, \omega = \sigma \times \sigma)$, where $\sigma(S^2) = 2\pi$. This is also the unit cosphere bundle of S^3, and its Calabi–Yau symplectic cone is known in the physics literature as the conifold.

Remark 6.1. Gauntlett *et al.* construct in [GMSW04b] a family of higher dimensional generalizations of the manifolds $Y^{p,q}$. They do not describe their exact diffeomorphism type, and they do not write down the associated moment cones. The latter are described in [Abr10] and can be used to show that, contrary to what happens in dimension five, different cones in this higher dimensional family give rise to non-diffeomorphic manifolds.

6.2 Contact homology computations

We will now apply the algorithm of § 5 to this family of good moment cones: $C(k) \subset \mathbb{R}^3$, $k \in \mathbb{N}_0$. We will do it for two different types of Reeb vector fields.

First, we consider the case when the Reeb vector field $R_\nu \in \mathcal{X}(S^2 \times S^3, \xi_k)$ is induced by a Lie algebra vector $\nu \in \mathfrak{t}^3 \cong \mathbb{R}^3$ of the form

$$\nu = (a_1, a_2, a_3) \approx (0, 0, 1),$$

with the a_i $\mathbb{Q}$-independent.

Remark 6.2. When $k > 0$, these vectors satisfy the requirement of Proposition 2.19 because the vector $(0, 0, 1)$ can be written as a positive linear combination of the normals to $C(k)$:

$$\frac{1}{3k+2}(\nu_1 + (3k-1)\nu_2 + \nu_3 + \nu_4) = (0, 0, 1).$$

When $k = 0$, the second coordinate of all the normals is either zero or negative and so we must have $a_2 < 0$.

Each cone $C(k)$ has four edges.

(1) The edge E_1, with γ_1 the corresponding simple closed R_ν-orbit, is the intersection of the facets F_1 and F_3 with normals

$$\nu_1 = (1, 0, 1) \quad \text{and} \quad \nu_3 = (0, k, 1).$$

The vector $\eta_1 \in \mathbb{Z}^3$ can be chosen to be

$$\eta_1 = \nu_4 = (-1, 2k-1, 1).$$

In fact, $\{\nu_1, \nu_3, \eta_1 = \nu_4\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^3$ and

$$R_\nu = b_1^1 \nu_1 + b_2^1 \nu_3 + b^1 \eta_1$$

with

$$\begin{aligned} b_1^1 &= (1-k)a_1 - a_2 + ka_3, \\ b_2^1 &= (2k-1)a_1 + 2a_2 - (2k-1)a_3, \\ b^1 &= -ka_1 - a_2 + ka_3. \end{aligned}$$

(2) The edge E_2, with γ_2 the corresponding simple closed R_ν-orbit, is the intersection of the facets F_1 and F_2 with normals

$$\nu_1 = (1, 0, 1) \quad \text{and} \quad \nu_2 = (0, -1, 1).$$

The vector $\eta_2 \in \mathbb{Z}^3$ can be chosen to be

$$\eta_2 = 2\nu_3 - \nu_4 = (1, 1, 1).$$

In fact, $\{\nu_1, \nu_2, \eta_1 = 2\nu_3 - \nu_4\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^3$ and

$$R_\nu = b_1^2 \nu_1 + b_2^2 \nu_2 + b^2 \eta_2$$

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with

$$\begin{aligned} b_1^2 &= 2a_1 - a_2 - a_3, \\ b_2^2 &= -a_1 + a_3, \\ b^2 &= -a_1 + a_2 + a_3. \end{aligned}$$

(3) The edge E_3, with γ_3 the corresponding simple closed R_ν-orbit, is the intersection of the facets F_3 and F_4 with normals

$$\nu_3 = (0, k, 1) \quad \text{and} \quad \nu_4 = (-1, 2k-1, 1).$$

The vector $\eta_3 \in \mathbb{Z}^3$ can be chosen to be

$$\eta_3 = \nu_1 = (1, 0, 1).$$

In fact, $\{\nu_3, \nu_4, \eta_3 = \nu_1\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^3$ and

$$R_\nu = b_1^3 \nu_3 + b_2^3 \nu_4 + b^3 \eta_3$$

with

$$\begin{aligned} b_1^3 &= (2k-1)a_1 + 2a_2 - (2k-1)a_3, \\ b_2^3 &= -ka_1 - a_2 + ka_3, \\ b^3 &= (1-k)a_1 - a_2 + ka_3. \end{aligned}$$

(4) The edge E_4, with γ_4 the corresponding simple closed R_ν-orbit, is the intersection of the facets F_2 and F_4 with normals

$$\nu_2 = (0, -1, 1) \quad \text{and} \quad \nu_4 = (-1, 2k-1, 1).$$

The vector $\eta_4 \in \mathbb{Z}^3$ can be chosen to be

$$\eta_4 = 2\nu_3 - \nu_1 = (-1, 2k, 1).$$

In fact, $\{\nu_2, \nu_4, \eta_4 = 2\nu_3 - \nu_1\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^3$ and

$$R_\nu = b_1^4 \nu_2 + b_2^4 \nu_4 + b^4 \eta_4$$

with

$$\begin{aligned} b_1^4 &= a_1 + a_3, \\ b_2^4 &= -(2k+1)a_1 - a_2 - a_3, \\ b^4 &= 2ka_1 + a_2 + a_3. \end{aligned}$$

We can now compute the Conley–Zehnder index of all closed R_ν orbits, which coincides in the $n = 2$ case with the contact homology degree:

$$\begin{aligned} \mu_{CZ}(\gamma_1^N) &= 2\left\lfloor \frac{N}{k} \right\rfloor + \operatorname{sign}(a_1) + \begin{cases} 1 & \text{if } N \neq \text{multiple of } k, \\ \operatorname{sign}(a_2) & \text{if } N = \text{multiple of } k, \end{cases} \\ \mu_{CZ}(\gamma_2^N) &= 2N + \operatorname{sign}(a_1) - \operatorname{sign}(a_2), \\ \mu_{CZ}(\gamma_3^N) &= 2\left\lfloor \frac{N}{k} \right\rfloor - \operatorname{sign}(a_1) + \begin{cases} 1 & \text{if } N \neq \text{multiple of } k, \\ \operatorname{sign}((2k-1)a_1 + a_2) & \text{if } N = \text{multiple of } k, \end{cases} \\ \mu_{CZ}(\gamma_4^N) &= 2N - \operatorname{sign}((2k-1)a_1 + a_2) - \operatorname{sign}(a_1). \end{aligned}$$

To determine the rank of the contact homology groups, we can assume, for example, that $a_1, a_2 < 0$, and get the following table.

deg	0	2	4	6	8	$\cdots$
γ_1	k	k	k	k	k	$\cdots$
γ_2	$--$	1	1	1	1	$\cdots$
γ_3	$--$	k	k	k	k	$\cdots$
γ_4	$--$	$--$	1	1	1	$\cdots$
rank	k	$2k+1$	$2k+2$	$2k+2$	$2k+2$	$\cdots$

Another possibility would be to assume that $a_1 > 0$, $a_2 < 0$ and $(2k-1)a_1 + a_2 < 0$. We would then get the following table.

deg	0	2	4	6	8	$\cdots$
γ_1	$--$	k	k	k	k	$\cdots$
γ_2	$--$	$--$	1	1	1	$\cdots$
γ_3	k	k	k	k	k	$\cdots$
γ_4	$--$	1	1	1	1	$\cdots$
rank	k	$2k+1$	$2k+2$	$2k+2$	$2k+2$	$\cdots$

Following a suggestion of Viktor Ginzburg, we will now consider a second type of Reeb vector fields, namely those that are arbitrarily close to one of the normals of the cone $C(k) \in \mathbb{R}^3$.

More precisely, consider

$$R_\nu = \sum_{i=1}^{4} \varepsilon_i \nu_i = (a_1, a_2, a_3),$$

which means that

$$a_1 = \varepsilon_1 - \varepsilon_4, \quad a_2 = -\varepsilon_2 + k\varepsilon_3 + (2k-1)\varepsilon_4 \quad \text{and} \quad a_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4, \quad \varepsilon_i > 0, i = 1, \ldots, 4.$$

Using the already determined formulas for R_ν, we have the following.

(1) On the edge E_1, where $\{\nu_1, \nu_3, \eta_1 = \nu_4\}$ is the relevant $\mathbb{Z}$-basis, we can write

$$R_\nu = (\varepsilon_1 + (k+1)\varepsilon_2)\nu_1 + (-(2k+1)\varepsilon_2 + \varepsilon_3)\nu_3 + ((k+1)\varepsilon_2 + \varepsilon_4)\eta_1.$$

(2) On the edge E_2, where $\{\nu_1, \nu_2, \eta_2 = 2\nu_3 - \nu_4\}$ is the relevant $\mathbb{Z}$-basis, we can write

$$R_\nu = (\varepsilon_1 - (k+1)\varepsilon_3 - 2(k+1)\varepsilon_4)\nu_1 + (\varepsilon_2 + \varepsilon_3 + 2\varepsilon_4)\nu_2 + ((k+1)\varepsilon_3 + (2k+1)\varepsilon_4)\eta_2.$$

(3) On the edge E_3, where $\{\nu_3, \nu_4, \eta_3 = \nu_1\}$ is the relevant $\mathbb{Z}$-basis, we can write

$$R_\nu = (-2(k+1)\varepsilon_2 + \varepsilon_3)\nu_3 + ((k+1)\varepsilon_2 + \varepsilon_4)\nu_4 + (\varepsilon_1 + (k+1)\varepsilon_2)\eta_3.$$

(4) On the edge E_4, where $\{\nu_2, \nu_4, \eta_4 = 2\nu_3 - \nu_1\}$ is the relevant $\mathbb{Z}$-basis, we can write

$$R_\nu = (2\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\nu_2 + (-2(k+1)\varepsilon_1 - (k+1)\varepsilon_3 + \varepsilon_4)\nu_4 + ((2k+1)\varepsilon_1 + (k+1)\varepsilon_3)\eta_4.$$

We can now make R_ν arbitrarily close to a normal ν_j by considering the ε_i, with $i \neq j$, to be arbitrarily small positive numbers and $\varepsilon_j \approx 1$.

Let us start with the case

$$R_\nu \approx \nu_1.$$

(1) On the edge E_1,

$$R_\nu \approx \nu_1 + \varepsilon\nu_3 + \varepsilon\eta_1,$$

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with $\varepsilon > 0$ an arbitrarily small number. This implies that

$$\mu_{CZ}(\gamma_1^N) \approx \frac{2N}{\varepsilon}$$

can be made arbitrarily large for any $N \in \mathbb{N}$, and so γ_1^N gives no contribution to contact homology up to an arbitrarily large degree.

(2) The same happens for γ_2^N since, on the edge E_2,

$$R_\nu \approx \nu_1 + \varepsilon\nu_2 + \varepsilon\eta_2.$$

(3) On the edge E_3,

$$R_\nu \approx \varepsilon\nu_3 + \varepsilon\nu_4 + \eta_3,$$

with $\varepsilon > 0$ arbitrarily small. This implies that

$$\mu_{CZ}(\gamma_3^N) = 2N \quad \text{for } N \approx 1, \ldots, \frac{1}{\varepsilon},$$

and so γ_3^N gives a rank-one contribution to contact homology in all positive even degrees up to the arbitrarily large $1/\varepsilon$.

(4) On the edge E_4,

$$R_\nu \approx (2+\varepsilon)\nu_2 + (-2(k+1)+\varepsilon)\nu_4 + (2k+1)\eta_4,$$

with ε arbitrarily small. This implies a particularly interesting behavior for the Conley–Zehnder index of γ_4^N. In fact, when $m(2k+1) - 2k \leqslant N \leqslant m(2k+1)$ for some $m \in \mathbb{N}$ and up to an arbitrarily large $N \in \mathbb{N}$,

$$\mu_{CZ}(\gamma_4^N) = \begin{cases} 2m-2 & \text{if } m(2k+1) - 2k \leqslant N \leqslant m(2k+1) - k - 1, \\ 2m & \text{if } m(2k+1) - k \leqslant N \leqslant m(2k+1) - 1, \\ 2m+2 & \text{if } N = m(2k+1). \end{cases}$$

For $N \geqslant k+1$, this can also be written as

$$\mu_{CZ}(\gamma_4^N) = \begin{cases} 2m & \text{if } m(2k+1) - k \leqslant N \leqslant m(2k+1) - 1, \\ 2m+2 & \text{if } N = m(2k+1), \\ 2m & \text{if } m(2k+1) + 1 \leqslant N \leqslant m(2k+1) + k, \end{cases}$$

and we see that in this case the Conley–Zehnder index is not monotone with respect to N.

Hence, when $R_\nu \approx \nu_1$, the rank of the contact homology groups is determined from the following table.

deg	0	2	4	6	8	$\cdots$
γ_1	$--$	$--$	$--$	$--$	$--$	$\cdots$
γ_2	$--$	$--$	$--$	$--$	$--$	$\cdots$
γ_3	$--$	1	1	1	1	$\cdots$
γ_4	k	$2k$	$2k+1$	$2k+1$	$2k+1$	$\cdots$
rank	k	$2k+1$	$2k+2$	$2k+2$	$2k+2$	$\cdots$

In this case we have that all the interesting contact homology information is concentrated on just one closed Reeb orbit (and its multiples): namely, γ_4.

When $R_\nu \approx \nu_4$, we obtain a similar picture, with all interesting contact homology information concentrated on γ_2 and its multiples. In this case, and up to an arbitrarily large contact homology degree, γ_3^N and γ_4^N contribute nothing, while γ_1^N gives a rank-one contribution to degree $2N$.

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When $R_\nu \approx \nu_2$, we have that γ_2^N and γ_4^N contribute nothing, while γ_1^N and γ_3^N contribute about half the rank of contact homology each. When $R_\nu \approx \nu_3$, we have that γ_1^N and γ_3^N contribute nothing, while γ_2^N and γ_4^N contribute about half the rank of contact homology each.

In any case, and for any $k \in \mathbb{N}_0$, the final result is

$$\operatorname{rank} HC_*(S^2 \times S^3, \xi_k; \mathbb{Q}) = \begin{cases} k & \text{if } * = 0, \\ 2k+1 & \text{if } * = 2, \\ 2k+2 & \text{if } * > 2 \text{ and even}, \\ 0 & \text{otherwise.} \end{cases}$$

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