THE COMPACT RANGE PROPERTY AND C_0

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The purpose of this short note is to make an observation about Dunford–Pettis operators from $L^1[0,1]$ to C_0. Recall that an operator $T: E \to F$ (where E and F are Banach spaces) is called *Dunford–Pettis* if T takes weakly convergent sequences of E into norm convergent sequences of F. A Banach space F has the *Compact Range Property* (CRP) if every operator $T: L^1[0,1] \to F$ is Dunford–Pettis. Talagrand shows in his book [**2**] that C_0 does not have the CRP. It is of interest (especially in mathematical economics [**3**]) to note that every positive operator from $L^1[0,1]$ to C_0 is Dunford–Pettis.

For a Banach lattice Y, call a Schauder basis *order-compatible* if the natural projections are positive operators. Certainly in this case, the basis elements themselves are positive.

THEOREM 1. *Let Y be a Banach lattice with order compatible Schauder basis. Then every positive linear operator $T: L^1[0,1] \to Y$ is a Dunford–Pettis operator.*

Proof. By a theorem of Bourgain given in [**1**], it suffices to show that $T \circ i$: $L^\infty[0,1] \to Y$ is a compact operator, where $i: L^\infty[0,1] \to L^1[0,1]$ is the natural injection. To fix notation, let $y_1, y_2, y_3, \ldots$ be the basis of Y and let P_n and Q_n be the associated projections defined by

$$P_n\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i=1}^{n} a_i x_i$$

and

$$Q_n = I - P_n \quad (n = 1, 2, 3, \ldots).$$

By hypothesis, each of these projection operators is positive. Now let f be in the unit ball of $L^\infty[0,1]$. We compute

$$\begin{aligned}(T \circ i)f &= (P_n \circ T \circ i)f + (Q_n \circ T \circ i)f \\ &= T_n f + R_n f\end{aligned}$$

so that

$$\begin{aligned}\|(T \circ i)f - T_n f\| &= \|R_n f\| \\ &\leq \|R_n \chi_{[0,1]}\|,\end{aligned}$$

since $|f| \leq \chi_{[0,1]}$ and Y is a Banach lattice. Given $\varepsilon > 0$, there is N such that $\|R_n \chi_{[0,1]}\| < \varepsilon$ for $n \geq N$. So for any f in the unit ball of L^∞, and $n \geq N$, we have

$$\|(T \circ i)f - T_n f\| < \varepsilon.$$

Consequently, $\|(T \circ i) - T_n\| < \varepsilon$. Since each T_n has finite dimensional range, $T \circ i$ is a compact operator. Thus T is Dunford–Pettis. ■

A consequence of this theorem is that every regular operator from $L^1[0,1]$ to Y is Dunford–Pettis. (Recall that an operator between two Banach lattices E and F is called

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regular if it is the difference of two positive operators.) In general, the regular operators are a proper subset of the bounded linear operators; in two special cases, however, it is well known that equality holds: one case is when F is an order complete $C(K)$ with K compact; the other is when E is $L^1(\mu)$ and F is the range of a positive, continuous projection in its bidual.

Examples of Banach lattices which fit into the framework of the theorem are C_0 and l_1. In the case of l_1, every operator from $L^1[0, 1]$ to l_1 is regular (second case above) and thus is Dunford–Pettis. (Of course, it is already obvious that this must be so because l_1 has the RNP and hence the CRP.) The space C_0 does not fit either of the two cases and, in fact as mentioned above, C_0 does not have the CRP. One final note: $L^1[0, 1]$ itself fits the second case and every operator from $L^1[0, 1] \to L^1[0, 1]$ is regular but not necessarily Dunford–Pettis, as the identity operator shows.

COROLLARY 2. *$L^1[0, 1]$ cannot have an order compatible Schauder basis.*

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