Canad. Math. Bull. Vol. **39** (2), 1996 pp. 199–202

ON POSITIVE INTEGER SOLUTIONS OF THE EQUATION $xy + yz + xz = n$

AL-ZAID HASSAN, B. BRINDZA AND Á. PINTÉR

ABSTRACT. As it had been recognized by Liouville, Hermite, Mordell and others, the number of non-negative integer solutions of the equation in the title is strongly related to the class number of quadratic forms with discriminant $-n$. The purpose of this note is to point out a deeper relation which makes it possible to derive a reasonable upper bound for the number of solutions.

For a positive integer n let $G(n)$ denote the class number of binary quadratic forms $aX^2 + 2bXY + cY^2$ with *determinant* $b^2 - ac = -n$. Generalizing some earlier results, Mordell ([M1], [M2]) proved that the number of non-negative integer solutions of the equation

$$xy + yz + xz = n \tag{1}$$

is $3G(n)$ if a weight is attached to a solution with $xyz = 0$. His argument is based upon a one-to-one correspondence between the reduced quadratic forms

$$AX^2 + 2BXY + CY^2,$$

and the non-negative solutions x, y, z of (1) given by $A = x + y$, $|B| = x$, $C = x + z$. However, the counting of strictly positive integer solutions seems to be a different and harder problem. It was verified ([K]) that the equation (1) (in positive integers) has solution for all $n \leq 10^7$ except the numbers $n = 1, 2, 4, 6, 10, 18, 22, 30, 42, 58, 70, 78, 102, 130, 190, 210, 330$ and 462 which is the biggest one. Let $h(D)$ and $\tilde{h}(D)$ denote the ideal class number of the field $\mathbb{Q}(\sqrt{-D})$ and the class number of the forms $aX^2 + bXY + cY^2$ with *discriminant* $b^2 - 4ac = -D$, respectively. In our equation n is not necessarily square-free and it does not satisfy certain prescribed congruences modulo 4, thus the relation between the class numbers $h(D)$, $\tilde{h}(D)$, and the number of solutions of (1) is not that straightforward, apart from the simple inequality $\max\{h(D), \tilde{h}(D)\} \leq G(D)$.

Let $S(n)$ denote the number of integer solutions of (1) with $0 < z \leq y \leq x$ and ϵ be a positive number. Then we have

---

The second and third authors were partially supported by Grant 4055 from the Hungarian National Foundation for Scientific Research.

Received by the editors October 14, 1994; revised March 7, 1995.

AMS subject classification: 11D09, 11R29.

Key words and phrases: class numbers, diophantine equations, quadratic forms.

https://doi.org/10.4153/CMB-1996-024-5 Published online by Cambridge University Press

THEOREM 1. *There exists an effectively computable constant c such that*

$$S(n) < c \cdot n^{\frac{1}{2}} \cdot \log n \cdot \prod_{\substack{p|n \\ p \text{ prime}}} \left(1 + \frac{1}{\sqrt{p} - 1}\right).$$

Furthermore, for every sufficiently large square-free n

$$n^{\frac{1}{2}-\epsilon} < S(n).$$

REMARKS. The proof is a combination of some known results, the crucial point is that a positive integer solution of (1) and the coefficients of the minimal polynomial of an element in the *modular domain* of $\mathbb{Q}(\sqrt{-n})$ satisfy quite similar relations. The second part of Theorem 1 is also effective, apart from at most one exceptional n.

Most likely $n = 462$ is the biggest number for which $S(n) = 0$, however it does not seem to be easy to prove. By taking a solution (x, y, z) to (1) with $x = y$, say, we have $x(x + 2z) = n$. The known effective lower bounds for the ideal class number $h(D)$ are not big enough comparing with the number of divisors of D. For instance, a deep result obtained by Oesterlè [O2] gives the lower bound

$$h(D) > \frac{1}{7000} \log D \prod_{\substack{p|D, p \neq D \\ p \text{ prime}}} \left(1 - \frac{[2\sqrt{p}]}{p+1}\right),$$

and the number of divisors of D can be around $\exp\{c\frac{\log D}{\log\log D}\}$.

However, an inequality of Tatuzawa [T] (see Lemma 2) leads to the following

THEOREM 2. *If $S(n) = 0$ then the square-free part of n belongs to a finite set which can be effectively determined up to at most one element.*

The proofs are based on some auxiliary results.

LEMMA 1. (Oesterlè [O1]) *If d is congruent* 0 *or* 3 *modulo* 4, *then*

$$\tilde{h}(d) = \sum_{\substack{0 \le b \le \sqrt{d/3} \\ b \equiv d \bmod 2}} \sum_{\substack{a|((b^2+d)/4) \\ b \le a \le \sqrt{(b^2+d)/4}}} n(a, b),$$

where $n(a, b) = 1$ if $ab(b - a) = 0$ or $a = \sqrt{(b^2 + d)/4}$ and $n(a, b) = 2$ for otherwise. Moreover,

$$\tilde{h}(d) = \sum_{f|F_d} h(df^{-2})$$

where F_d is the fundamental discriminant, that is F_d^2 is the biggest divisor of d such that dF_d^{-2} congruent 0 *or* 3 *modulo* 4.

LEMMA 2. (Tatuzawa [T]) *Let $0 < \epsilon < \frac{1}{2}$ and d be a square-free integer satisfying $d > \max(e^{1/\epsilon}, e^{11.2})$. Then*

$$h(d) \ge \frac{0.655}{\pi} \epsilon d^{\frac{1}{2}-\epsilon}$$

https://doi.org/10.4153/CMB-1996-024-5 Published online by Cambridge University Press

except for at most one exceptional d.

PROOFS. For a positive integer k we put

$$\mathcal{H}(k) = \{d \in \mathbb{Z} : 2k \le d \le \sqrt{n+k^2},\, d \mid n+k^2\}.$$

Let $0 < z \le y \le x$ be a solution to the equation (1). Then $z \le \sqrt{yz} \le \frac{n}{3}$,

$$n - yz \equiv n + z^2 \bmod (y+z)$$

and $2z \le y+z \le \sqrt{n+z^2}$, therefore

$$S(n) \le \sum_{z=1}^{\sqrt{\frac{n}{3}}} \sum_{\substack{d \mid n+z^2 \\ 2z \le d \le \sqrt{n+z^2}}} 1. \tag{2}$$

Applying Lemma 1 we get

$$\tilde{h}(4n) = \sum_{\substack{0 \le z \le \sqrt{\frac{4n}{3}} \\ z \text{ even}}} \sum_{\substack{d \mid \frac{4n+z^2}{4} \\ z \le d \le \sqrt{\frac{4n+z^2}{4}}}} n(d,z) = \sum_{0 \le z_1 \le \sqrt{\frac{n}{3}}} \sum_{\substack{d \mid n+z_1^2 \\ 2z_1 \le d \le \sqrt{n+z_1^2}}} n(d, 2z_1)$$

hence

$$S(n) \le \tilde{h}(4n) = \sum_{f \mid F_{4n}} h(4nf^{-2}) \le \sum_{d \mid 4n} h(d).$$

The well-known inequality

$$h(d) < c_1 d^{\frac{1}{2}} \log d,$$

where c_1 is an effective absolute constant (*cf.* [O1], [S]), yields

$$S(n) < c_1 \sum_{d \mid 4n} d^{\frac{1}{2}} \log d \le c_1 \log 4n \sum_{d \mid 4n} d^{\frac{1}{2}} \le c_1 (4n)^{\frac{1}{2}} \log 4n \prod_{\substack{p \mid 4n \\ p \text{ prime}}} \left(1 + \frac{1}{\sqrt{p}-1}\right).$$

As usual we denote by $d(n)$ the number of positive divisors of n. For every sufficiently large square-free n, $h(n) > n^{\frac{1}{2}-\epsilon}$ and $d(n) < n^{\epsilon}$. Therefore, the inequalities

$$3!S(n) \ge 3G(n) - 3d(n),$$
$$G(n) \ge h(n)$$

complete the proof of Theorem 1.

Theorem 2 is a simple consequence of these inequalities and Lemma 2.

ACKNOWLEDGEMENT. After the inequality (2) one may try to find a reasonable upper bound for the cardinalities of the sets $\mathcal{H}(z)$, however, thanks to some up-to-date information about the distribution of divisors provided by Prof. G. Tenenbaum, our feeling is that this approach would not lead to a better bound.

REFERENCES

[K] K. Kovács, *About some positive solutions of the diophantine equation* $\sum_{1\leq i<j\leq n} a_i a_j = m$, Publ. Math. Debrecen, **40**(1992), 207–210.

[M1] L. J. Mordell, *On the number of solutions in positive integers of the equation* $yz + zx + xy$, Amer. J. Math. **45**(1923), 1–4.

[M2] ———, *Diophantine Equations*, Academic Press, London 1969.

[O1] J. Oesterlè, *Le problème de Gauss sur le nombre de classes*, Enseign. Math. (2) **34**(1988), 43–67.

[O2] ———, *Nombres de classes des corps quadratiques imaginaires*, Sèminaire Nicolas Bourbaki, 1983–1984, Exp. 631.

[S] C. L. Siegel, *Abschzätzung von Einheiten*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. **2**(1969), 71–86.

[T] T. Tatuzawa, *On a theorem of Siegel*, Japan. J. Math. **21**(1951), 163–178.

Kuwait University
P.O.Box 5969
Safat 13060
Kuwait

Kuwait University
P.O.Box 5969
Safat 13060
Kuwait
e-mail: brindza@math-1.sci.kuniw.edu.kw

Department of Mathematics
Kossuth University
P.O. Box 12
4010 Debrecen
Hungary
E-mail: pinter@huklte51.bitnet