CAMBRIDGE
UNIVERSITY PRESS

RESEARCH ARTICLE

Calabi–Yau structures on (quasi-)bisymplectic algebras

Tristan Bozec [1], Damien Calaque [2] and Sarah Scherotzke [3]

[1]IMAG, Univ. Montpellier, CNRS, Place Eugène Bataillon, 34090 Montpellier, France; E-mail: tristan.bozec@umontpellier.fr.
[2]IMAG, Univ. Montpellier, CNRS, Place Eugène Bataillon, 34090 Montpellier, France; E-mail: damien.calaque@umontpellier.fr.
[3]Mathematical Institute, University of Luxembourg, 6, avenue de la Fonte, L-4364 Esch-sur-Alzette, Luxembourg;
E-mail: sarah.scherotzke@uni.lu.

Abstract

We show that relative Calabi–Yau structures on noncommutative moment maps give rise to (quasi-)bisymplectic structures, as introduced by Crawley-Boevey–Etingof–Ginzburg (in the additive case) and Van den Bergh (in the multiplicative case). We prove along the way that the fusion process (a) corresponds to the composition of Calabi–Yau cospans with 'pair-of-pants' ones and (b) preserves the duality between non-degenerate double quasi-Poisson structures and quasi-bisymplectic structures.

As an application, we obtain that Van den Bergh's Poisson structures on the moduli spaces of representations of deformed multiplicative preprojective algebras coincide with the ones induced by the 2-Calabi–Yau structures on (dg-versions of) these algebras.

Contents

1. Introduction

Throughout this paper, k is a field of characteristic zero.

Noncommutative algebraic geometry

The Kontsevich–Rosenberg principle of noncommutative algebraic geometry says that a structure on an associative algebra A has a (noncommutative) geometric meaning whenever it induces a genuine corresponding geometric structure on representation spaces. This principle led to the discovery of bisymplectic structures [9], double Poisson and double quasi-Poisson structures [30], and quasi-bisymplectic structures [31] on smooth algebras such that the associated representation spaces are respectively hamiltonian GL_n-varieties, Poisson and quasi-Poisson GL_n-varieties, and quasi-hamiltonian GL_n-varieties.

It turns out that the fusion procedure for (quasi-)hamiltonian spaces from [1, 2] has a noncommutative counterpart [30, 31] (also called fusion). This, in particular, allows for the construction of quasi-bisymplectic structures on (localisations of) path algebras of quivers by starting from several copies of A_2 and repeatedly applying the fusion procedure. Ultimately, this provides a construction of symplectic structures [32] on multiplicative quiver varieties [10].

Noncommutative algebra	Algebraic geometry
Smooth algebra A	Representation variety **Rep**(A)
Bisymplectic algebras	Hamiltonian GL-spaces
Quasi-bisymplectic algebras	Quasi-hamiltonian GL-spaces
Fusion	Fusion

Derived symplectic geometry

Hamiltonian and quasi-hamiltonian spaces actually find a nice interpretation (see [7, 23]) in the realm of shifted symplectic and lagrangian structures from [21] moment maps as well, as their multiplicative analogs naturally lead to lagrangian morphisms, and both the reduction and the fusion procedures can be understood in terms of derived intersections of these.

Algebraic geometry	Derived geometry
$G \subset X$	Quotientstack $[X/G]$
Hamiltonian G-space X	Lagrangian morphism $[X/G] \to [\mathfrak{g}^*/G]$
Quasi-hamiltonian G-space X	Lagrangian morphism $[X/G] \to [G/G]$
Reduction	Lagrangian intersection
Fusion	Composing Lagrangian correspondences

Calabi–Yau structures

More recently, absolute and relative Calabi–Yau structures [5] have turned out to be accurate noncommutative analogs of shifted symplectic and lagrangian structures [6, 26], *via* the moduli of object functor **Perf** from [27].

Higher algebra	Derived geometry
Finite type dg-category $\mathcal{C}$	Derived Artin stack **Perf**$(\mathcal{C})$
Shifted Calabi−Yau structure	Shifted symplecticstructure
Relative Calabi−Yau structure	Lagrangian structure
Calabi−Yau pushout	Lagrangian intersection
Composing Calabi−Yau cospans	Composing Lagrangian correspondences

It is therefore natural to wonder whether Calabi–Yau structures are hidden behind the aforementioned (quasi-)bisymplectic ones. More specifically, in our previous work [3, 4], we constructed relative Calabi–Yau structures on (multiplicative) noncommutative moment maps $k[x^{(\pm 1)}] \to A$ for (multiplicative) preprojective algebras associated with quivers, leading, in particular, to an alternative construction of symplectic structures on multiplicative quiver varities. Exhibiting a direct connection between Calabi–Yau and (quasi-)bisymplectic structures will then help identify the induced symplectic structures on multiplicative quiver varieties from both approaches.

Results

In a very satisfactory manner, relative Calabi–Yau structures on noncommutative moment maps do induce (quasi-)bisymplectic ones: the additive version is proved by our first main result (theorem 4.8), and the multiplicative one is given by theorem 5.5. The rough idea in each case is that the Calabi–Yau structure on $k[x^{(\pm 1)}] \to A$ is given by a family of noncommutative forms $\omega_n \in \Omega^{2n}A$, $n \geq 1$, satisfying conditions implying the required ones for the 2-form ω_1 to define a (quasi-)bisymplectic structure on A. In particular, non-degeneracy on the Calabi–Yau side implies non-degeneracy on the (quasi-)bisymplectic side.

Moreover, we prove that we retrieve for quivers the very same structures exhibited in [9, 30]: in the additive case in example 4.9, and in a much more involved way in the multiplicative case in section 5.4. This requires work on the elementary A_2 quiver as well as on the correct realization of fusion in the framework of Calabi–Yau cospans. For the latter, we need to prove in section 3 (along with theorem 4.10 and theorem 5.6) that fusion actually corresponds to composition of relative Calabi–Yau structures with a particular Calabi–Yau cospan studied in [4], the 'pair-of-pants' one; that is,

$$k[x^{(\pm 1)}] \amalg k[y^{(\pm 1)}] \longrightarrow k\langle x^{(\pm 1)}, y^{(\pm 1)}\rangle \longleftarrow k[z^{(\pm 1)}],$$

where z is mapped to $x + y$ in the additive version, and xy in the multiplicative one.

Higher algebra	Noncommutative algebra
Finite linear category $\mathcal{C}$	Path algebra $A_{\mathcal{C}}$
Object i	Primitive idempotent e_i
Calabi−Yau functor $\coprod_i k[x_i] \to \mathcal{C}$	Bisymplectic structure, with moment map $k[x] \to A_{\mathcal{C}}$
Calabi−Yau functor $\coprod_i k[x_i^{\pm 1}] \to \mathcal{C}$	Quasi-bisymplectic structure, with moment map $k[x^{\pm 1}] \to A_{\mathcal{C}}$
Pushing-out along the "pair-of-pants"	Fusion

We want to emphasize that section 5 contains what can be understood as the quasi-bisymplectic side of the fusion calculus for double quasi-Poisson algebra [30, §5.3]. Indeed, we know thanks to [31] that quasi-bisymplectic structures correspond to non-degenerate double quasi-Poisson ones, and we produce in proposition 5.4 the formula for fusion of quasi-bisymplectic structures, a noncommutative analog of [1, Proposition 10.7]. Because of this compatibility, we do not use double quasi-Poisson structures in this paper, but we prove that in the quiver case, the structures we get give back Van den Bergh's double quasi-Poisson structures from [30].

The last essential step for completeness is to check that when considering representation spaces, all these constructions yield the same symplectic structures, which is proved by our last main result, theorem 6.1. We prove specifically that the lagrangian structures induced by quasi-Hamiltonian ones thanks to [30], on the one hand, and by relative Calabi–Yau ones [6], on the other hand, are indeed the same. This achieves the proof of the conjectural program established in the open questions concluding [4], except the last part, which is rather independent.

Outline of the paper

In section 2, we recall the mixed structure on the graded vector space of noncommutative differential forms on an associative k-algebra, which yields a convenient construction of Hochschild and negative cyclic homology as shown by Ginzburg–Schedler [15]. We consider the example of $A = k[x^{\pm}]$ and identify the noncommutative differential form that yields the 1-Calabi–Yau structure from [4].

In section 3, we compare the fusion process introduced by Van den Bergh [30] with certain pushouts of categories involving the pair-of-pants cospan studied in [4]. Fusion has been introduced in order to glue idempotents in double (quasi-)Poisson algebras, but in this section, we only focus on the algebra structure and not on double brackets. Along the way, we show that the fusion of a 1-smooth (or formally smooth – see definition 3.10) algebra is 1-smooth.

The fourth section can be considered as an additive warm-up for the next one. We show that relative Calabi–Yau structures on additive noncommutative moment maps induce bisymplectic structures. Bisymplectic structures were first defined in [9] and are dual to non-degenerate double Poisson structures from [30]. We introduce, in analogy with Van den Bergh's fusion of double Poisson structures, the fusion of bisymplectic structures and show that it corresponds to composition with the additive pair-of-pants cospan from [4]. Furthermore, we show that the fusion process respects the duality between bisymplectic and double Poisson structures in the sense that a compatible pair of bisymplectic and double Poisson structures is sent by fusion to another compatible pair.

In section 5, we prove that relative Calabi–Yau structures on multiplicative noncommutative moment maps induce quasi-bisymplectic structures in the sense of [31]. Then we prove that the fusion of quasi-bisymplectic structures is induced by the composition of Calabi–Yau cospans with the multiplicative pair-of-pants, and that it is compatible with the duality between quasi-bisymplectic and double quasi-Poisson structures. We also show that in the case of multiplicative quiver varieties, the Calabi–Yau structure exhibited in [4] is compatible with the non-degenerate double quasi-Poisson structure defined in [31].

Finally, in the last section, we study the geometries induced by the aforementioned structures on representation spaces $X_V = \mathrm{Rep}(A, V)$ of algebras A in vector spaces V. Namely, assuming that we have a Calabi–Yau structure on $\coprod_{i \in I} k[x^{\pm 1}] \to \mathcal{C}$, with $A_{\mathcal{C}} = A$, we know thanks to [6] that it induces a lagrangian structure on $[X_V / \mathrm{GL}_V] \to [\mathrm{GL}_V / \mathrm{GL}_V]$. We also know that the double quasi-Poisson structure induced by our previous section yields a quasi-Hamiltonian structure on X_V (in the sense of [2]), and therefore a lagrangian structure on the very same morphism. We prove that these two lagrangian structures match.

Related works

A systematic comparison of noncommutative differential forms with Hochschild and cyclic complexes has been achieved by Yeung in [33]. There, the author uses [14], whereas we rely on [15]. We should also mention Pridham's [22], which presents a systematic way of producing shifted bisymplectic (resp. bilagrangian) structures out of absolute (resp. relative) Calabi–Yau structures (see Proposition 1.24 and Theorem 1.56 in [22]). One may be able to recover some of the results of the present paper using Pridham's general theory (but it would probably require as much work as here to derive these results from [22]).

2. Cyclic and noncommutative de Rham mixed complex

In this section, we first briefly recall some facts about Hochschild and negative cyclic homology, and then some constructions and results from [15]. In particular, in [15], Ginzburg and Schedler directly relate the negative cyclic homology of a unital algebra with the cohomology of a complex that is obtained from the mixed complex of noncommutative differential forms [17] on this algebra. We finally exhibit a closed noncommutative form representing the class in negative cyclic homology which defines the 1-Calabi-Yau structure on $k[x^{\pm 1}]$ in [4].

2.1. Hochschild and negative cyclic homology

We denote by Mod_k the category of *chain* complexes over k. We warn the reader that we use the homological grading instead of the cohomological grading used in our previous papers [3, 4]. In particular, differentials have degree -1, whereas mixed differentials have degree $+1$. Apart from this change, throughout this paper we borrow the convention and notation from *op. cit.*, to which we refer for more details. For instance, whenever $\mathscr{M}$ is a model category, we write **M** for the corresponding ∞-category obtained by localizing along weak equivalences.

A *dg-category* is a Mod_k-enriched category, and the category of dg-categories with dg-functors is denoted by Cat_k. We refer to [18, 24] for a detailed introduction to dg-categories and their homotopy theory. The *Hochschild chains* ∞-functor is then defined as

$$\mathrm{HH} \,:\, \mathbf{Cat}_k \longrightarrow \mathbf{Mod}_k \,;\, \mathcal{C} \longmapsto \mathcal{C} \overset{\mathbb{L}}{\underset{\mathcal{C}^e}{\otimes}} \mathcal{C}^{\mathrm{op}}\,,$$

where $\mathcal{C}^e := \mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}$. We write $\mathrm{HH}_i(\mathcal{C})$ for the i-th homology of $\mathrm{HH}(\mathcal{C})$.

There is an explicit description of the derived tensor product $\mathcal{C} \overset{\mathbb{L}}{\underset{\mathcal{C}e}{\otimes}} \mathcal{C}^{\mathrm{op}}$, which uses the normalized bar resolution of $\mathcal{C}$ as a $\mathcal{C}$-bimodule, and that leads to standard normalized Hochschild chains that we denote $(C_*(\mathcal{C}), b)$:

$$C_*(\mathcal{C}) = \bigoplus_{\substack{n \geq 0 \\ a_0, \ldots, a_n \in \mathrm{Ob}(\mathcal{C})}} \mathcal{C}(a_n, a_0) \otimes \bar{\mathcal{C}}(a_{n-1}, a_n) \otimes \cdots \otimes \bar{\mathcal{C}}(a_1, a_2) \otimes \bar{\mathcal{C}}(a_0, a_1)[-n],$$

with $\bar{\mathcal{C}}(a, a') = \mathcal{C}(a, a')$ if $a \neq a'$ and $\bar{\mathcal{C}}(a, a) = \mathcal{C}(a, a)/k \cdot \mathrm{id}_a$.

Hochschild chains carry a mixed structure (i.e., given on the standard normalized model by Connes's B-operator). We refer to [3, 4] and references therein for the homotopy theory of mixed complexes and explicit formulas.[1] The negative cyclic complex of $\mathscr{C}$, denoted by $\mathrm{HC}^-(\mathscr{C})$, is defined as the homotopy fixed points of $\mathrm{HH}(\mathcal{C})$ with respect to the mixed structure; it comes with a natural transformation $(-)^{\natural} : \mathrm{HC}^- \Rightarrow \mathrm{HH}$. In concrete terms, $\mathrm{HC}^-(\mathscr{C})$ is given by $(C_*(\mathcal{C})[[u]], b - uB)$, where u is a degree -2 variable.

We can view every dg-algebra with a finite set $(e_i)_{i \in I}$ of orthogonal nonzero idempotents such that $1 = \sum_{i \in I} e_i$ is a dg-category with object set I. Conversely, we can associate to every dg-category $\mathcal{C}$ with finitely many objects its *path algebra* given by the complex

$$A_{\mathcal{C}} := \bigoplus_{(a,b) \in Ob(\mathcal{C}) \times Ob(\mathcal{C})} \mathcal{C}(a, b)$$

with product given by composition of morphisms. The dg-algebra $A_{\mathcal{C}}$ is an R-algebra, where $R = \oplus_{c \in Obj(c)} k e_c$. Note that the construction is in general not functorial, meaning that a functor does not necessarily give a morphism between the corresponding dg-algebras (unless the functor is injective on

---

[1] Beware of the change of (co)homological grading convention though.

objects). This can be seen very easily in the following example, which will play an important role in the next section.

Example 2.1. The dg-category coproduct $k \coprod k$ is the dg-category given by two objects 1 and 2 and endomorphism ring $k = \text{End}(1)$ respectively $k = \text{End}(2)$ at each object, but zero Hom-spaces between the two objects. Hence, its path algebra $A_{k \coprod k}$ is isomorphic to $k \oplus k$. There is a dg-functor

$$k \coprod k \to k$$

sending 1 and 2 to pt, which denotes the only object of k, but there is no map of k-linear dg-algebras $k \oplus k \to k$.

Nevertheless, $\mathcal{C}$ and $A_{\mathcal{C}}$ are Morita equivalent, so that their Hochschild (resp. negative cyclic) homology is isomorphic. More precisely, we have an inclusion of mixed complexes $\big(C_*(\mathcal{C}), b, B\big) \hookrightarrow \big(C_*(A_{\mathcal{C}}), b, B\big)$, which is a weak equivalence (here, we view $A_{\mathcal{C}}$ as a dg-category with one object).

2.2. Noncommutative forms

Consider a unital associative k-algebra A, along with a subalgebra R. We fix a complementary subspace $\bar{A} \simeq A/R$ of R. Denote by $d : A \to \bar{A}$ the associated quotient map. We will systematically use the $\bar{}$ notation for the quotient by R. The graded algebra $\Omega_R^* A$ of noncommutative differential forms is defined as the quotient of $T_R(A \oplus \bar{A}[-1])$ by the relations

$$a \otimes b = ab \qquad \text{and} \qquad d(ab) = a \otimes d(b) + d(a) \otimes b$$

for every $a, b \in A$. It comes equipped with a mixed differential, that is the derivation induced by d and that we denote by the same symbol. The mixed differential d, descends to the *Karoubi–de Rham* graded vector space $\text{DR}_R^* A := \Omega_R^* A / [\Omega_R^* A, \Omega_R^* A]$, first introduced in [17].

In order to define a differential on $\Omega_R^* A$, turning it into a mixed complex, we consider the distinguished double derivation $E : a \mapsto a \otimes 1 - 1 \otimes a$, denoted by Δ in [9]. Recall that the A-bimodule of (R-linear) double derivations is defined as

$$D_{A/R} := \text{Der}_R(A, A \otimes A) \simeq \Omega_R^1 A^\vee,$$

where the derivations are taken with respect to the outer A-bimodule structure on $A \otimes A$, and the remaining A-bimodule structure on $D_{A/R}$ comes from the inner one on $A \otimes A$. Here, $\Omega_R^1 A$ is the kernel of the multiplication $A \otimes_R A \to A$ and inherits its A-bimodule structure from the outer one on $A \otimes A$; it is isomorphic to $A \otimes_R \bar{A}$ as a left A-module ($1 \otimes da \in A \otimes_R \bar{A}$ being identified with $E(a) \in \Omega_R^1 A$). As a matter of notation, we will often write $\Omega_{A/R} := \Omega_R^1 A$.

There is an obvious graded algebra isomorphism $\Omega_R^* A \simeq T_A(\Omega_R^1 A[-1])$, as well as a left A-module isomorphism $\Omega_R^n A \simeq A \otimes_R \bar{A}^{\otimes_R n}$ (see [11]). For later purposes, we also introduce the graded algebra of polyvector fields $D_R^* A = T_A(D_{A/R}[-1])$ from [30].

Following [9], we define, for any R-linear double derivation $\delta \in D_{A/R}$ of A, a graded double derivation

$$i_\delta : \Omega_R^* A \to \Omega_R^* A \otimes \Omega_R^* A$$

of $\Omega_R^* A$ by setting

$$i_\delta(a) := 0 \qquad \text{and} \qquad i_\delta(da) := \delta(a)$$

for any $a \in A$. On $\Omega_R^2 A$, we thus have, for instance,

$$i_\delta(pdqdr) = p\delta(q)' \otimes \delta(q)'' dr - pdq\delta(r)' \otimes \delta(r)'' \in A \otimes \Omega_R^1 A + \Omega_R^1 A \otimes A,$$

where we use Sweedler's sumless notation $\delta(a) = \delta(a)' \otimes \delta(a)''$. The graded double derivation i_δ induces a linear contraction operator

$$\iota_\delta := {}^\circ i_\delta : \Omega_R^* A \to \Omega_R^{*-1} A,$$

where $^\circ(\alpha \otimes \beta) = (-1)^{kl} \beta \otimes \alpha$ for $\alpha \otimes \beta \in \Omega_R^k A \otimes \Omega_R^l A$. Our differential will be given by the contraction operator $\iota_E : \Omega_R^* A \to \Omega_R^{*-1} A$, which has the following properties thanks to [9, Lemma 3.1.1]: it is explicitly given by the formula

$$\iota_E(a_0 da_1 \ldots da_n) = \sum_{l=1}^{n} (-1)^{(l-1)(n-1)+1} [a_l, da_{l+1} \ldots da_n a_0 da_1 \ldots da_{l-1}].$$

It vanishes on $[\Omega_R^* A, \Omega_R^* A]$ (and thus factors though $\mathrm{DR}_R^* A$), and it takes vales in $[\Omega_R^* A, \Omega_R^* A]^R$ (in particular, $\iota_E^2 = 0$), and $[\iota_E, d] = 0$. As a consequence, we obtain that $(\Omega_R^* A, \iota_E, d)$ is a mixed complex.

2.3. Hochschild chains versus noncommutative forms

Below, we rephrase some constructions and results of [15] in terms of mixed complexes. Beware that the notation used here is not exactly the same as in *op. cit.*. For the moment, we only assume that A is a k-algebra.

Through the identification $C_*(A) \simeq \Omega_k^* A$, the Hochschild differential b reads as

$$b(\alpha da) = (-1)^{|\alpha|} [\alpha, a].$$

The Karoubi operator on $\Omega_k^* A$, given by

$$\kappa(\alpha da) = (-1)^{|\alpha|} da \alpha,$$

allows one to define a harmonic decomposition $\bar{\Omega}_k^* A = P\bar{\Omega}_k^* A \oplus P^\perp \bar{\Omega}_k^* A$, where

$$P\bar{\Omega}_k^* A = \ker(1-\kappa)^2 \qquad \text{and} \qquad P^\perp \bar{\Omega}_k^* A = \mathrm{ima}(1-\kappa)^2.$$

The following identites hold:

$$\iota_E = bN|_P \qquad \text{and} \qquad B = Nd|_P,$$

where N is the grading operator and B is the Connes mixed differential.

Hence, we have the following chain of morphisms of mixed complexes

$$(\bar{\Omega}_k^* A, \iota_E, d) \xrightarrow{P} (P\bar{\Omega}_k^* A, \iota_E, d) \xrightarrow{N!} (P\bar{\Omega}_k^* A, b, B) \lhook\joinrel\longrightarrow (\bar{\Omega}_k^* A, b, B) \qquad (2.1)$$

such that, according to [15], $\overline{[d\Omega_k^* A, d\Omega_k^* A]} \hookrightarrow (\ker(P)[[u]], \iota_E - ud)$ is a quasi-isomorphism, where u is a degree -2 formal variable, $N!$ is an isomorphism and the rightmost inclusion is a quasi-isomorphism. We thus get a quasi-isomorphism

$$\left(\frac{\bar{\Omega}_k^* A [[u]]}{\overline{[d\Omega_k^* A, d\Omega_k^* A]}}, \iota_E - ud \right) \longrightarrow (\bar{\Omega}_R A [[u]], b - uB),$$

and the homology of both complexes yields the reduced negative cyclic homology $\overline{\mathrm{HC}}^-(A)$.

Hence, when $A = A_{\mathcal{C}}$, for $\mathcal{C}$ a genuine k-linear category with a finite set I of objects, and $R = \oplus_{i \in I} k e_i$, we have a zig-zag

$$
\begin{array}{ccc}
\left(\bar{C}_*(\mathcal{C})\,[[u]], b - uB \right) & \xrightarrow{\ \ \sim\ \ } & \left(\bar{C}_*(A)\,[[u]], b - uB \right) \\[1em]
& \nearrow^{\sim} & \\[1em]
\left(\dfrac{\bar{\Omega}_k^* A\,[[u]]}{[d\Omega_k^* A, d\Omega_k^* A]}, \iota_E - ud \right) & \xrightarrow{\hspace{3em}} & \left(\dfrac{\bar{\Omega}_R^* A\,[[u]]}{[d\Omega_R^* A, d\Omega_R^* A]}, \iota_E - ud \right),
\end{array}
$$

where only the last bottom arrow may not be a quasi-isomorphism.

2.4. Computations for $A = k[x^{\pm 1}]$

As a matter of convention, we always mean $(dx)y$ if no brackets appear in dxy. We want to find a harmonic cyclic lift for $\alpha_1 := x^{-1}dx \in \bar{\Omega}^1 A$ which is closed for the mixed structure $(P\bar{\Omega}, \iota_E, d)$. That means that A is 1-pre-Calabi–Yau according to the terminology of [3]. This was already proved in [4] using the standard normalized Hochschild complex, but we reprove it here on the 'de Rham side' and check consistency afterwards to illustrate (2.1).

Set $\alpha_n = (x^{-1}dx)^{2n-1}, \beta_n = \kappa(\alpha_n) = (dxx^{-1})^{2n-1} \in \bar{\Omega}^{2n-1} A$. Then

$$\kappa(\beta_n) = \kappa(-\beta_{n-1}dxdx^{-1}) = -dx^{-1}\beta_{n-1}dx = \alpha_n.$$

Hence, $\alpha_n + \beta_n \in P\bar{\Omega}A$ and $\alpha_n - \beta_n = \frac{1}{2}(1 - \kappa)^2(\alpha_n) \in P^\perp \bar{\Omega}A$. Then

$$
\begin{aligned}
\iota_E \alpha_n &= \frac{1}{2}(2n-1)b(\alpha_n + \beta_n) \\
&= \frac{1}{2}(2n-1)([\alpha_{n-1}x^{-1}dxx^{-1}, x] - [\beta_{n-1}dx, x^{-1}]) \\
&= \frac{1}{2}(2n-1)(x^{-1}\beta_{n-1}dx + \alpha_{n-1}x^{-1}dx - \beta_{n-1}dxx^{-1} - x\alpha_{n-1}x^{-1}dxx^{-1}) \\
&= (2n-1)((x^{-1}dx)^{2n-2} - (dxx^{-1})^{2n-2}).
\end{aligned}
$$

However, $d\alpha_1 = -(x^{-1}dx)^2$, and if we assume $d\alpha_{n-1} = -(x^{-1}dx)^{2n-2}$, we get

$$
\begin{aligned}
d\alpha_n &= d(x^{-1}dx(x^{-1}dx)^{2n-2}) \\
&= d(x^{-1}dx)(x^{-1}dx)^{2n-2} - x^{-1}dxd((x^{-1}dx)^{2n-2}) \\
&= -x^{-1}dxx^{-1}dx(x^{-1}dx)^{2n-2} - x^{-1}dxd^2\alpha_{n-1} \\
&= -(x^{-1}dx)^{2n}.
\end{aligned}
$$

Similarly, $d\beta_n = (dxx^{-1})^{2n}$ for all n. Thus, as $\iota_E \alpha_n = \iota_E \beta_n$,

$$\iota_E(\alpha_n + \beta_n) = 2\iota_E \alpha_n = -2(2n-1)d(\beta_{n-1} + \alpha_{n-1}).$$

As a consequence, $(\iota_E - ud)(\gamma) = 0$, where $\gamma_k = \frac{1}{2}(\alpha_k + \beta_k) \in P\bar{\Omega}^{2k-1}k[x^{\pm 1}]$ and

$$\gamma = \sum_{k \geq 0} \frac{k!}{(2k+1)!}(-u)^k \gamma_{k+1},$$

where u is a formal degree -2 variable.

Let us check now that this is coherent with [4]. Through (2.1) and the isomorphism $\Omega^n A \simeq A \otimes \bar{A}^{\otimes n}$, γ is mapped to

$$\sum_{k \geq 0} k! u^k \frac{(x^{-1} \otimes x)^{\otimes(k+1)} - (x \otimes x^{-1})^{\otimes(k+1)}}{2}$$

as

$$\alpha_{k+1} = (x^{-1}dx)^{2k+1} = (-1)^k x^{-1}(dxdx^{-1})^k dx,$$
$$\beta_{k+1} = (dxx^{-1})^{2k+1} = (-1)^{k+1} x(dx^{-1}dx)^k dx^{-1},$$
$$\text{and } \gamma_{k+1} \in P\bar{\Omega}^{2k+1},$$

all of which is consistent with [4, 3.1.1].

3. Fusion

In this section, we compare certain pushouts of k-linear dg-categories with the fusion formalism introduced by Van den Bergh [30] for algebras. Fusion is a process which glues two pairwise orthogonal idempotents into one. Given an algebra with a double (quasi-)Poisson structure, the new algebra obtained by fusion inherits a double (quasi-)Poisson structure from the original one as shown in [30, 12].

This will be relevant in the next sections, where we will compare fusion of bisymplectic and quasi-bisymplectic structures with compositions of Calabi–Yau cospans.

3.1. Fusion as a pushout

Recall that Van den Bergh defines in [30] the fusion algebra which identifies two pairwise orthogonal idempotents. We use the notation $(-)^+$ instead of $\overline{(-)}$ as in [30] since it is already used.

Definition 3.1. Let $R = ke_1 \oplus \cdots \oplus ke_n$ be a semi-simple algebra with pairwise orthogonal idempotents e_i, and A an R-algebra. Set $\mu = 1 - e_1 - e_2$ and $\epsilon = 1 - e_2$. Then the fusion algebra A^f is defined as $\epsilon A^+ \epsilon$, where $A^+ := A \coprod_{ke_1 \oplus ke_2 \oplus k\mu}(M_2(k) \oplus k\mu)$. Here, $M_2(k)$ denotes the $(ke_1 \oplus ke_2)$-algebra of 2×2 matrices, and the idempotent e_i is sent to e_{ii}, where e_{ij}'s are matrix units.

One can see that A^+ is isomorphic to $A \coprod_R R^+$ and that $R^+ = M_2(k) \oplus R_{\geq 3}$ and $R^f = ke_1 \oplus R_{\geq 3}$, where $R_{\geq 3} := ke_3 \oplus \cdots \oplus ke_n$.

Now, let $\mathcal{C}$ be a dg-category with a finite set of objects $I = \{1, \ldots, n\}$, $n \geq 2$. We define

$$\mathcal{C}^f := \mathcal{C} \coprod_{k \amalg k} k,$$

where the functor $k \amalg k \to \mathcal{C}$ is given by the units of the first two objects 1 and 2. Note that the strict pushout is (categorically equivalent to) a homotopy pushout.

Examples 3.2. (1) The category $(k[x] \amalg k[y])^f$ (when defined using the strict pushout) is isomorphic to $k\langle x, y \rangle$. Similarly, $(k[x^{\pm 1}] \amalg k[y^{\pm 1}])^f$ is isomorphic $k\langle x^{\pm}, y^{\pm} \rangle$. As a consequence, we get that

$$\mathcal{C}^f \simeq \mathcal{C} \coprod_{k[x_1^\square] \amalg k[x_2^\square]} k\langle x_1^\square, x_2^\square \rangle,$$

where $\square \in \{\emptyset, \pm 1\}$ and $k[x_i^\square] \to \text{End}_{\mathcal{C}}(i)$.

(2) If $\mathcal{R} = \coprod_{i \in I} k$, then $\mathcal{R}^f = k \amalg \mathcal{R}_{\geq 3}$, where $\mathcal{R}_{\geq 3} := \coprod_{i \geq 3} k$. As a consequence, we get that

$$\mathcal{C}^f := \mathcal{C} \coprod_{\mathcal{R}} (k \amalg \mathcal{R}_{\geq 3}),$$

where the functor $\mathcal{R} \to \mathcal{C}$ is uniquely determined by mapping the object of the i-th copy of k to i, and the functor $\mathcal{R} \to k \amalg \mathcal{R}_{\geq 3}$ maps the first two objects of $\mathcal{R}$ to the object of the first copy of k.

Proposition 3.3. *Let $\mathcal{C}$ be a k-linear dg-category with set of objects I. Then $A_{\mathcal{C}^f}$ is isomorphic to $(A_{\mathcal{C}})^f$.*

Proof. We can assume without loss of generality that $\mathcal{C}$ has only two objects 1 and 2. We denote e_1 and e_2 their respective identity map. The dg-category $\mathcal{C} \coprod_{k \amalg k} k$ has exactly one object which we denote pt. Let us show that the endomorphism ring $B := \mathrm{End}(pt)$ is isomorphic to the fusion algebra A^f of $A := A_{\mathcal{C}}$. By the pushout property, there are algebra homomorphisms

$$f : \mathrm{End}_{\mathcal{C}}(1) \simeq e_1 A e_1 \to B$$
$$g : \mathrm{End}_{\mathcal{C}}(2) \simeq e_2 A e_2 \to B,$$

and bimodule morphisms $e_1 A e_2 \simeq \mathcal{C}(2,1) \to B, e_1 a e_2 \mapsto e_1 a e_{21}$ and $e_2 A e_1 \simeq \mathcal{C}(1,2) \to B, e_2 a e_1 \mapsto e_{12} a e_1$ such that

$$
\begin{array}{ccc}
\mathcal{C}(2,1) \otimes \mathcal{C}(1,2) & \longrightarrow & B \otimes B \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\
\mathrm{End}_{\mathcal{C}}(1) & \xrightarrow{\ g\ } & B
\end{array}
$$

commutes. The algebra homomorphism $k \to B$ is then uniquely determined.

We have injective algebra morphisms $\mathrm{End}_{\mathcal{C}}(1) \simeq e_1 A e_1 \to A^f, a \mapsto a$, $\mathrm{End}_{\mathcal{C}}(2) \simeq e_2 A e_2 \to A^f, a \mapsto e_{12} a e_{21}$. Similarly, we have injective morphisms of bimodules $\mathcal{C}(2,1) \simeq e_1 A e_2 \to A^f, a \mapsto a e_{21}$ and $\mathcal{C}(1,2) \simeq e_2 A e_1 \to A^f, a \mapsto e_{12} a$ compatible with the composition of morphisms. Hence, we obtain a unique injective algebra homomorphism $B \to A^f$. As the image of the above maps generates A^f, this morphism is also surjective, and hence, $B = A_{\mathcal{C}^f} \simeq A^f$. $\qquad\square$

3.2. Trace maps

Acccording to Van den Bergh [30], we consider the following situation: an R-algebra A and an idempotent e in R such that $ReR = R$. One writes $1 = \sum_i p_i e q_i$ with $p_i, q_i \in R$ and defines a *trace map*

$$\mathrm{Tr} : A \to eAe \; ; \; a \mapsto \sum_i e q_i a p_i e.$$

We recall a series of standard results, for which we provide full proofs for the sake of completeness; the main point is to be able to describe the trace map on $\Omega_R A$ and $\mathrm{DR}_R A$.

Lemma 3.4. *The trace map Tr descends to an isomorphism $A/[A, A] \to eAe/[eAe, eAe]$ that does not depend on the choice of decomposition $1 = \sum_i p_i e q_i$.*

Proof. First of all, the trace map Tr sends commutators to commutators. Indeed,

$$
\text{Tr}(ab - ba) = \sum_i (eq_i ab p_i e - eq_i ba p_i e)
$$

$$
= \sum_{i,j} eq_i ap_j eq_j bp_i e - eq_i bp_j eq_j ap_i e
$$

$$
= \sum_{i,j} eq_i ap_j eq_j bp_i e - eq_j bp_i eq_i ap_j e \in [eAe, eAe].
$$

Then, one can check that it is a k-linear inverse modulo commutator, to the algebra morphism $eAe \to A$. Indeed, on the one hand, $a = \sum_i p_i eq_i a = \text{Tr}(a)$ mod $[A, A]$, and on the other hand, $eae = \sum_i ep_i eq_i eae = \text{Tr}(eae)$ mod $[eAe, eAe]$. Since the morphism $eAe \to A$ does not depend on the decomposition of 1, its inverse (modulo commutator) does not either. $\qquad\square$

Lemma 3.5. *For any two A-bimodules M and N, the canonical morphism $Me \otimes_{eRe} eN \to M \otimes_R N$ of A-bimodules is invertible with the inverse given by*

$$
\Psi_{M,N} : M \otimes_R N \to Me \otimes_{eRe} eN ; \; m \otimes n \mapsto \sum_i mp_i e \otimes eq_i n.
$$

Proof. Let us check that it is well-defined. Consider $r \in R$ and write $r = \sum_j h_j el_j$ for some $h_j, l_j \in R$. Then

$$
\Psi_{M,N}(mr \otimes n) = \sum_i mr p_i e \otimes eq_i n = \sum_{i,j} mh_j el_j p_i e \otimes eq_i n
$$

$$
= \sum_{i,j} mh_j e \otimes el_j p_i eq_i n = \sum_j mh_j e \otimes el_j n
$$

$$
= \sum_{i,j} mp_i eq_i h_j e \otimes el_j n = \sum_{i,j} mp_i e \otimes eq_i h_j el_j n
$$

$$
= \sum_i mp_i e \otimes eq_i rn = \Psi_{M,N}(m \otimes rn).
$$

We finally observe that $\Psi_{M,N}$ is an inverse to the canonical morphism $Me \otimes_{eRe} eN \to M \otimes_R N$. Indeed, in $M \otimes_R N$, $\sum_i mp_i e \otimes eq_i n = \sum_i m \otimes p_i eq_i n = m \otimes n$, and in $Me \otimes_{eRe} eN$, $\sum_i mep_i e \otimes eq_i en = \sum_i me \otimes p_i eq_i en = me \otimes en$. $\qquad\square$

As a matter of notation, we introduce $\Psi_M := \Psi_{M,M}$.

Lemma 3.6. *The isomorphism $\Psi_{\Omega_{A/R}}$ induces an isomorphism $e(\Omega_R A)e \simeq \Omega_{eRe}(eAe)$, through which the trace map of $\Omega_R A$ reads as follows:*

$$
\text{Tr} : \Omega_R A \to e(\Omega_R A)e \simeq \Omega_{eRe}(eAe)
$$

$$
a_0 da_1 \dots da_m \mapsto \sum_{i_0,\dots,i_m} eq_{i_0} a_0 p_{i_1} ed(eq_{i_1} a_1 p_{i_2} e) \dots d(eq_{i_m} a_m p_{i_0} e).
$$

Moreover, it induces a k-linear isomorphism

$$
\text{Tr} : \text{DR}_R(A) \to \text{DR}_{eRe}(eAe)
$$

that does not depend on the decomposition $1 = \sum_i p_i eq_i$.

Proof. Thanks to the previous lemma, the isomorphism $\Psi_{\Omega_{A/R}}$ induces an isomorphism of tensor algebras $e(T_A\Omega_{A/R})e \simeq T_{eAe}(e\Omega_{A/R}e)$. Using Ψ_A, we also have

$$
\begin{aligned}
\Omega_{eAe/eRe} &= \ker(eAe \otimes_{eRe} eAe \to eAe) \\
&\simeq \ker(eA \otimes_R Ae \to eAe) \\
&= e\ker(A \otimes_R A \to A)e \\
&= e\Omega_{A/R}e.
\end{aligned}
$$

Combining these, we get

$$
e(\Omega_R A)e := e(T_A\Omega_{A/R})e \simeq T_{eAe}(e\Omega_{A/R}e) \simeq T_{eAe}\Omega_{eAe/eRe} =: \Omega_{eRe}(eAe).
$$

Through this identification, an element $edae = ea \otimes e - e \otimes ae \in e\Omega_{A/R}e$ becomes, in $\Omega_{eAe/eRe}$,

$$
\sum_i eap_ie \otimes eq_ie - ep_ie \otimes eq_iae = eae \otimes e - e \otimes eae =: d(eae) \in \Omega_{eAe/eRe}.
$$

Thus, the trace map reads

$$
\begin{aligned}
\Omega_R A \ni a_0da_1\ldots da_m &\mapsto \sum_{i_0} eq_{i_0}a_0da_1\ldots da_m p_{i_0}e \in e(\Omega_R A)e \\
&\mapsto \sum_{i_0,i_1,\ldots,i_m} eq_{i_0}a_0p_{i_1}ed(eq_{i_1}a_1p_{i_2}e)\ldots d(eq_{i_m}a_m p_{i_0}e) \in \Omega_{eRe}(eAe).
\end{aligned}
$$

The last part of the claim follows from lemma 3.4.   □

3.3. Functoriality

We now apply the constructions from the previous section 3.2 to the idempotent $\epsilon = 1 - e_2$ of R^+ (see definition 3.1), where $1 = \epsilon\epsilon + e_{21}\epsilon e_{12}$. Precomposing with the algebra morphism $A \to A^+$, we get maps $\Omega_R A \to \Omega_{R^f} A^f$ and $\mathrm{DR}_R(A) \to \mathrm{DR}_{R^f} A^f$ that we denote by $(-)^f$. Since $\epsilon e_{12} = e_{12}$ and $e_{21}\epsilon = e_{21}$, we have $\mathrm{Tr}(a) = \epsilon a\epsilon + e_{12}ae_{21}$ for all $a \in A^+$. Actually, the trace map in this situation also has a simpler expression on forms.

Lemma 3.7. *On Ω_{A^+/R^+}, we have*

$$
\mathrm{Tr}(adb) = \epsilon adb\epsilon + e_{12}adbe_{21},
$$

and dually, we have a trace map on double derivations

$$
\mathrm{Tr} : D_{R^+}A^+ \to D_{R^f} A^f , \quad \delta \mapsto \epsilon\delta\epsilon + e_{12}\delta e_{21}.
$$

More generally, if $\omega \in \Omega_{R^+}A^+$, we have $\mathrm{Tr}(\omega) = \epsilon\omega\epsilon + e_{12}\omega e_{21}$.

Proof. Thanks to lemma 3.6, we have on 1-forms

$$
\begin{aligned}
\mathrm{Tr}(adb) &= \epsilon a\epsilon d(\epsilon b\epsilon) + e_{12}a\epsilon d(\epsilon be_{21}) + e_{12}ae_{21}d(e_{12}be_{21}) + \epsilon ae_{21}d(e_{12}b\epsilon) \\
&= \epsilon a\epsilon db\epsilon + e_{12}a\epsilon dbe_{21} + e_{12}ae_2dbe_{21} + \epsilon ae_2db\epsilon.
\end{aligned}
$$

If $a \in Ae_2$ and $b \in e_2A$, as $\epsilon e_2 = e_2\epsilon = 0$, we get

$$
\mathrm{Tr}(adb) = e_{12}adbe_{21} + \epsilon adb\epsilon.
$$

If $a \in Ae_i$ and $b \in e_i A$ for some $i \neq 2$, as $\epsilon e_i = e_i \epsilon = e_i$, we again have

$$\mathrm{Tr}(adb) = \epsilon adb\epsilon + e_{12} adb e_{21}.$$

It generalizes to all forms. □

We go back to the context of a dg-category $\mathcal{C}$ with a finite set of objects I and set $A := A_{\mathcal{C}}$. We define idempotents $e_i = \mathrm{id}_i$ and set $R = \oplus_{i \in I} k e_i$, a subalgebra of A. Recall that $R^f \simeq \oplus_{i \neq 2} k e_i$ and consider the k-linear map $C_*(\mathcal{C}) \to \Omega_R^* A$ given by

$$a_0 \otimes a_1 \otimes \cdots \otimes a_m \mapsto a_0 da_1 \ldots da_m.$$

Since there is a functor $\mathcal{C} \to \mathcal{C}^f$, we have a natural map $\nu : C_*(\mathcal{C}) \to C_*(\mathcal{C}^f)$.

Lemma 3.8. *The following diagram commutes:*

$$
\begin{array}{ccc}
C_*(\mathcal{C}) & \longrightarrow & \Omega_R^*(A) \\
{\scriptstyle \nu} \downarrow & & \downarrow {\scriptstyle (-)^f} \\
C_*(\mathcal{C}^f) & \longrightarrow & \Omega_{R^f}^*(A^f).
\end{array}
$$

Proof. Thanks to lemma 3.6, the map $\Omega_R^*(A) \to \Omega_{R^f}^*(A^f)$ is given by

$$(a_0 da_1 \cdots a_m)^f = \sum_{i_0, \ldots, i_m} q_{i_0} a_0 p_{i_1} d(q_{i_1} a_1 p_{i_2}) \ldots d(q_{i_m} a_m p_{i_0}).$$

Since $p_{i_j} \epsilon = p_{i_j}$ and $\epsilon q_{i_j} = q_{i_j}$ in our situation, that is either $p_{i_j} = \epsilon = q_{i_j}$ or $p_{i_j} = e_{21}, q_{i_j} = e_{12}$. Now, if $a_0 \otimes \cdots a_m$ belongs to the Hochschild complex of $\mathcal{C}$, then these elements are completely determined by the a_j's. Indeed, if $a_j \in \mathcal{C}(x_{j+1}, x_j)$, then $q_{i_j} = \epsilon$ whenever $x_j \neq 2$ and $p_{i_{j+1}} = \epsilon$ whenever $x_{j+1} \neq 2$.

From the proof of proposition 3.3, we have that $\mathcal{C}(x, y) \to A^f$ is given by $a \mapsto qap$, with

○ $q = \epsilon$ if $y \neq 2$, and e_{12} otherwise.
○ $p = \epsilon$ if $x \neq 2$, and e_{21} otherwise.

Hence, the composed map $C_*(\mathcal{C}) \to C_*(\mathcal{C}^f) \to \Omega_{R^f}^*(A^f)$ is given by

$$a_0 \otimes \cdots \otimes a_m \mapsto q_{i_0} a_0 p_{i_1} \otimes q_{i_1} a_2 p_{i_2} \otimes \cdots \otimes q_{i_m} a_m p_{i_0},$$

with the same p_{i_j}'s and q_{i_j}'s as above, proving the commutativity. □

Lemma 3.9. *Let $\omega \in \Omega_R^2(A)$. Then ω induces a map $\iota(\omega) : D_{A/R} \to \Omega_{A/R}$. Under the fusion process, the following diagram commutes:*

$$
\begin{array}{ccccc}
D_{A/R} & \longrightarrow & D_{A^+/R^+} & \longrightarrow & D_{A^f/R^f} \\
{\scriptstyle \iota(\omega)} \downarrow & & {\scriptstyle \iota(\omega^+)} \downarrow & & \downarrow {\scriptstyle \iota(\mathrm{Tr}(\omega^+))=\iota(\omega^f)} \\
\Omega_{A/R} & \longrightarrow & \Omega_{A^+/R^+} & \longrightarrow & \Omega_{A^f/R^f}.
\end{array}
$$

Proof. The commutativity of the left-hand side square follows immediately from definitions, and the commutativity of the right-hand side square means that

$$\iota_{\mathrm{Tr}(\delta)}(\mathrm{Tr}(\omega)) = \mathrm{Tr}(\iota_\delta(\omega))$$

for all $\omega \in \Omega^2_{R^+}(A^+)$ and $\delta \in D_{A^+/R^+}$. We prove this now. Recall that the bimodule structure on $D_{A/R}$ is induced by the inner one on $A \otimes_R A$. We know from the proof of [9, Lemma 2.8.6] that $\iota_{a\delta b} = a\iota_\delta b$. Thanks to lemma 3.7, we thus have

$$
\begin{aligned}
\iota_{\mathrm{Tr}(\delta)}(\mathrm{Tr}(\omega)) &= \iota_{\epsilon\,\delta\,\epsilon + e_{12}\,\delta\,e_{21}}(\mathrm{Tr}(\omega)) \\
&= \epsilon\iota_\delta(\mathrm{Tr}(\omega))\epsilon + e_{12}\iota_\delta(\mathrm{Tr}(\omega))e_{21} \\
&= \epsilon\iota_\delta(\epsilon\omega\epsilon + e_{12}\omega e_{21})\epsilon + e_{12}\iota_\delta(\epsilon\omega\epsilon + e_{12}\omega e_{21})e_{21} \\
&= \epsilon\iota_\delta(\omega)\epsilon + e_{12}\iota_\delta(\omega)e_{21} \\
&= \mathrm{Tr}(\iota_\delta(\omega)),
\end{aligned}
$$

as wished.   □

3.4. Fusion and 1-smoothness

We start with the following notion simply called 'smoothness' in [9] or [30].

Definition 3.10. We call an R-algebra A 1-*smooth* if it is finitely generated over R and formally smooth in the sense of [13, §19], meaning that $\Omega_{A/R}$ is a projective A-bimodule.

It implies that A has a projective dimension at most 1 and that we may (and will) use short resolutions. Note that it implies smoothness of associated representation schemes, but we call it 1-smooth in order to emphasize that it is way more demanding than the notion of (homological) smoothness we use in previous works [3, 4] for dg-categories (see also section 4.1), following, for example, [18].

In the sequel, assume that $A = A_\mathcal{C}$, where $\mathcal{C}$ has a finite number of objects, and $R = \oplus_{e \in Ob(\mathcal{C})} ke$.

Proposition 3.11. *If A is 1-smooth over R, then so is A^f over R^f.*

Proof. Recall that $A^+ = A \otimes_R R^+$. By definition, Ω_{A^+/R^+} is the kernel of the multiplication map $m^+ : A^+ \otimes_{R^+} A^+ \to A^+$ which can be identified with

$$
R^+ \otimes_R A \otimes_R A \otimes_R R^+ \xrightarrow{\ \mathrm{id} \otimes m \otimes \mathrm{id}\ } R^+ \otimes_R A \otimes_R R^+.
$$

Since R-modules are $Ob(\mathcal{C}) \times Ob(\mathcal{C})$-graded k-vector space, R^+ is flat over R and

$$
\Omega_{A^+/R^+} \simeq (R^+)^e \otimes_{R^e} \Omega_{A/R} \simeq (R^+)^e \otimes_{R^e} A^e \otimes_{A^e} \Omega_{A/R} \simeq (A^+)^e \otimes_{A^e} \Omega_{A/R}.
$$

Since $\Omega_{A/R}$ is a projective A-bimodule, Ω_{A^+/R^+} is a projective A^+-bimodule.

Then, we know that $\Omega_{A^f/R^f} = e\Omega_{A^+/R^+}e$ from lemma 3.6. Since Ω_{A^+/R^+} is a projective A^+-bimodule, there exists $r \in \mathbb{N}$ such that Ω_{A^f/R^f} is a direct summand of $e(A^+ \otimes_{R^+} A^+)^r e = (eA^+ \otimes_{R^+} A^+e)^r \simeq (A^f \otimes_{R^f} A^f)^r$ by lemma 3.5. Hence, Ω_{A^f/R^f} is a projective A^f-bimodule.   □

4. Calabi–Yau versus bisymplectic structures

In this section, we recall the notion of Calabi–Yau structures for dg-categories as in [5, 25] and bisymplectic structures on algebras as in [9]. We then introduce the fusion process for bisymplectic structures in analogy with the fusion for double Poisson structures from [30]. We show that a relative Calabi–Yau structure on $\coprod_{c \in Ob(\mathcal{C})} k[x_c] \to \mathcal{C}$, $\mathcal{C}$ a k-linear category, gives rise to a bisymplectic one on the path algebra $A_\mathcal{C}$ associated to $\mathcal{C}$. Finally, we prove that the composition with the 'additive pair-of-pants' Calabi–Yau cospan induces fusion for the corresponding bisymplectic structures on $A_\mathcal{C}$.

4.1. Calabi–Yau structures, absolute and relative

Our notation follows [3, 4]. A dg-category $\mathcal{A}$ is called *(homologically) smooth* if $\mathcal{A}$ is a perfect $\mathcal{A}^e$-module. In this case, we have the following equivalence:

$$(-)^\flat : \mathrm{HH}(\mathcal{A}) \xrightarrow{\sim} \mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{Mod}_{\mathcal{A}^e}}(\mathcal{A}^\vee, \mathcal{A}),$$

where $\mathcal{A}^\vee$ is the dualizing bimodule.

Definition 4.1. Let $\mathcal{A}$ be a smooth dg-category. An *n-Calabi–Yau structure* on $\mathcal{A}$ is a negative cyclic class $c = c_0 + uc_1 + \cdots : k[n] \to \mathrm{HC}^-(\mathcal{A})$ such that the underlying Hochschild class $c^\natural = c_0 : k[n] \to \mathrm{HH}(\mathcal{A})$ is non-degenerate, in the sense that $c_0^\flat : \mathcal{A}^\vee[n] \to \mathcal{A}$ is an equivalence.

Relative Calabi–Yau structures on morphisms and cospans of dg-categories where introduced by Brav–Dyckerhoff [5] following Toën [25, §5.3].

Definition 4.2. An *n-Calabi–Yau structure* on a cospan $\mathcal{A} \xrightarrow{f} \mathcal{C} \xleftarrow{g} \mathcal{B}$ of smooth dg-categories is a homotopy commuting diagram

$$
\begin{array}{ccc}
k[n] & \xrightarrow{c_{\mathcal{B}}} & \mathrm{HC}^-(\mathcal{B}) \\
{\scriptstyle c_{\mathcal{A}}}\downarrow & & \downarrow \\
\mathrm{HC}^-(\mathcal{A}) & \longrightarrow & \mathrm{HC}^-(\mathcal{C})
\end{array}
$$

whose image under $(-)^\natural$ is non-degenerate in the following sense: $c_{\mathcal{A}}^\natural$ and $c_{\mathcal{B}}^\natural$ are non-degenerate, and the homotopy commuting square

$$
\begin{array}{ccc}
\mathcal{C}^\vee[n] & \xrightarrow{\quad g^\vee \quad} & (\mathcal{B}^\vee[n]) \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{C}^e \overset{(c_{\mathcal{B}}^\natural)^\flat \otimes \mathrm{id}}{\simeq} \mathcal{B} \overset{\mathbb{L}}{\underset{\mathcal{B}^e}{\otimes}} \mathcal{C}^e \\
{\scriptstyle f^\vee}\downarrow & & \downarrow {\scriptstyle g \otimes \mathrm{id}} \\
(\mathcal{A}^\vee[n]) \overset{\mathbb{L}}{\underset{\mathcal{A}^e}{\otimes}} \mathcal{C}^e \overset{(c_{\mathcal{A}}^\natural)^\flat \otimes \mathrm{id}}{\simeq} \mathcal{A} \overset{\mathbb{L}}{\underset{\mathcal{A}^e}{\otimes}} \mathcal{C}^e & \xrightarrow{\quad f \otimes \mathrm{id} \quad} & \mathcal{C}
\end{array}
$$

is cartesian. We say that a morphism $g : \mathcal{A} \longrightarrow \mathcal{C}$ is *relative n-Calabi–Yau* if the copsan $\mathcal{A} \xrightarrow{f} \mathcal{C} \longleftarrow \varnothing$ is *n*-Calabi–Yau.

We will also use the fact that by [5, Theorem 6.2], *n*-Calabi–Yau cospans compose. It is immediate with the above definitions that an *n*-Calabi–Yau structure on $\varnothing \to \mathcal{C} \leftarrow \varnothing$ is the same as an $(n + 1)$-Calabi–Yau structure on $\mathcal{C}$. Finally, recall (see, for example, [4, Proposition 2.3]) that a non-degenerate Hochschild class on a smooth dg-category $\mathcal{A}$ concentrated in degree zero admits a unique cyclic lift, making $\mathcal{A}$ a Calabi–Yau category.

Example 4.3.

○ The algebra $k[x]$ carries a 1-Calabi–Yau structure. We call the Calabi–Yau structure induced by $1 \otimes x \in \mathrm{HH}_1(k[x])$ the natural Calabi–Yau structure.

○ Let $Q = (I, E)$ be a finite quiver, where I is the set of vertices and E the set of arrows. Denote by $\overline{Q}$ the double quiver obtained by adding for every arrow $a \in E$ an arrow a^* in the opposite direction. Consider the path algebra of the double quiver $A := k\overline{Q}$. There is a relative 1-Calabi–Yau structure on the moment map $k[x] \to kA$, $x \mapsto \sum_{a \in E}[a, a^*]$, which is compatible with the natural one on $k[x]$; see [3, 5.3.2].

○ The algebra $k[x^{\pm 1}]$ carries a natural 1-Calabi–Yau structure induced by $\frac{1}{2}(x^{-1} \otimes x - x \otimes x^{-1}) \in$ $\mathrm{HH}_1(k[x^{\pm}])$. This has been shown in [4] Section 3.1. See also section 2.4 for the cyclic lift.

The next example of a Calabi–Yau cospan was investigated thoroughly in Section 3.3 of [4] and related to the pair-of-pants.

Example 4.4 (Pair-of-pants). The cospan

$$k[x^{\pm 1}] \amalg k[y^{\pm 1}] \longrightarrow k\langle x^{\pm 1}, y^{\pm 1} \rangle \longleftarrow k[z^{\pm 1}] \qquad (4.1)$$

where the rightmost map is $z \mapsto xy$, is a relative 1-Calabi–Yau cospan with the Calabi–Yau structures $\alpha_1(x) + \alpha_1(y) - \alpha_1(z) = b(\beta_1) \sim 0$ and homotopy $\beta_1 := y^{-1} \otimes x^{-1} \otimes xy - y \otimes y^{-1}x^{-1} \otimes x$.

We prove here the additive version of the previous example which we refer to as the additive pair-of-pants, as opposed to the multiplicative pair-of-pants of the previous example.

Lemma 4.5. *There exists a relative* 1-*Calabi–Yau structure on*

$$k[x] \coprod k[y] \longrightarrow k\langle x, y \rangle \longleftarrow k[z], \qquad (4.2)$$

where the rightmost map is $z \mapsto x + y$, *such that the underlying absolute* 1-*Calabi–Yau structures on* $k[x]$, $k[y]$ *and* $k[z]$ *are the natural ones.*

Proof. The algebra $\mathcal{B} := k\langle x, y \rangle$ has a small resolution as a $\mathcal{B}$-bimodule:

$$(\mathcal{B}^e)^{\oplus 2}[1] \oplus \mathcal{B}^e$$

with differential sending $(1 \otimes 1, 0)$ to $x \otimes 1 - 1 \otimes x$, and $(0, 1 \otimes 1)$ to $y \otimes 1 - 1 \otimes y$. Therefore,

$$\mathcal{B}^\vee \simeq \mathcal{B}^e \oplus (\mathcal{B}^e)^{\oplus 2}[-1]$$

with differential sending $1 \otimes 1$ to $(x \otimes 1 - 1 \otimes x, y \otimes 1 - 1 \otimes y)$.

The canonical Calabi–Yau structures on $\mathcal{A} := k[x]$ are given by $\alpha_1(x) = 1 \otimes x \in \mathrm{HH}_1(\mathcal{A})$. Note that α_1 has a unique cyclic lift by Proposition 2.3 of [4] which we denote α. The following diagram induced by the natural Calabi–Yau structures on $\mathcal{A}$ is strictly commutative:

$$
\begin{array}{ccc}
\mathcal{B}^\vee[1] & \xrightarrow{\hspace{3cm}} & \mathcal{A}^\vee \underset{\mathcal{A}^e}{\otimes} \mathcal{B}^e[1] \xrightarrow[\simeq]{\alpha_1(x+y)} \mathcal{A} \underset{\mathcal{A}^e}{\otimes} \mathcal{B}^e \\
\downarrow & & \downarrow \\
(\mathcal{A}^{\oplus 2})^\vee \underset{\mathcal{A}^e}{\otimes} \mathcal{B}^e[1] \xrightarrow[\simeq]{\alpha_1(x)+\alpha_1(y)} \mathcal{A}^{\oplus 2} \underset{\mathcal{A}^e}{\otimes} \mathcal{B}^e & \xrightarrow{\hspace{3cm}} & \mathcal{B}.
\end{array}
$$

Using the small resolution of $\mathcal{A}$, we find $\mathcal{A} \underset{\mathcal{A}^e}{\otimes} \mathcal{B}^e \simeq \mathcal{B}^e[1] \oplus \mathcal{B}^e$, with differential sending $1 \otimes 1$ to $x \otimes 1 - 1 \otimes x$. Hence, we get that the diagram is cartesian. The zero homotopy is the unique lift in cyclic homology between $\alpha(z)$ and $\alpha(x) + \alpha(y)$. Therefore, the cospan (4.2) carries a relative 1-Calabi–Yau structure. $\qquad \square$

4.2. Bisymplectic structures and fusion

Let A be an R-algebra, where $R = ke_1 \oplus \cdots \oplus ke_n$ is based on pairwise orthogonal idempotents as usual. We define gauge elements $E_i = (a \mapsto ae_i \otimes e_i - e_i \otimes e_i a) \in D_{A/R}$ and recall notions introduced in [9].

Definition 4.6. We call $\omega \in \Omega^2_R(A)$ a *bisymplectic* structure on A if

○ ω is closed; that is, $d\omega = 0 \in \mathrm{DR}_R(A)$,
○ ω is non-degenerate that is, $\iota(\omega) : D_{A/R} \to \Omega_{A/R}, \delta \mapsto \iota_\delta(\omega)$ is an isomorphism.

An element $\mu = (\mu_i) \in \oplus_i e_i A e_i$ is a *moment map* for a bisymplectic algebra (A, ω) if

$$d\mu_i = \iota_{E_i}(\omega)$$

for all $i \in I$.

A moment map always exists; see [30, A.7]. Now we discuss fusion of bisymplectic structures and aim to prove [30, Proposition 2.6.6]. We use the notation of section 3. Recall that we have trace maps $A \to A^f, a \mapsto a^f = \epsilon a \epsilon + e_{12} a e_{21}, \Omega_R^*(A) \to \Omega_{R^f}^*(A^f)$ and $D_R^*(A) \to D_{R^f}^*(A^f)$. Let A be an algebra equipped with a bisymplectic structure ω, with moment map μ. We define $\mu_i^{ff} = \mu_i^f = \mu_i$ for $i \geq 3$ and

$$\mu_1^{ff} = \mu_1 + e_{12}\mu_2 e_{21} = \mu_1^f + \mu_2^f.$$

Lemma 4.7. *The form $\omega^f \in \Omega_{R^f}^2(A^f)$ is a bisymplectic structure on A^f, with moment map μ^{ff}.*

Proof. By definition, $\omega^f \in \Omega_{R^f}^2 A^f$ is a closed form. We need to show that $\iota(\omega^f) : D_{A^f/R^f} \to \Omega_{A^f/R^f}$ is an isomorphism. Recall from lemma 3.9 that we have the following commutative diagram:

$$
\begin{array}{ccccc}
D_{A/R} & \longrightarrow & D_{A^+/R^+} & \longrightarrow & D_{A^f/R^f} \\
\downarrow {\scriptstyle \iota(\omega)} & & \downarrow {\scriptstyle \iota(\omega^+)} & & \downarrow {\scriptstyle \iota(\omega^f)} \\
\Omega_{A/R} & \longrightarrow & \Omega_{A^+/R^+} & \longrightarrow & \Omega_{A^f/R^f}.
\end{array}
$$

Now, $\iota(\omega^+)$ is an isomorphism as it is obtained from $\iota(\omega)$ by an extension of rings $- \otimes_R R^+$, where R is semi-simple.

We observe that the map $\mathrm{Tr} : \Omega_{A^+/R^+} \to \Omega_{A^f/R^f}$ is surjective. As $\iota(\omega^+)$ is surjective, $\iota(\omega^f)$ is also surjective by lemma 3.9. Furthermore, the kernel of $\mathrm{Tr} : \Omega_{A^+/R^+} \to \Omega_{A^f/R^f}$ is given by $\epsilon \Omega_{A/R} e_2 + e_2 \Omega_{A/R} \epsilon$ and the kernel of $\mathrm{Tr} : D_{A^+/R^+} \to D_{A^f/R^f}$ is $\epsilon D_{A^+/R^+} e_2 + e_2 D_{A^+/R^+} \epsilon$. The morphism $\iota(\omega^+)$ maps the two kernels bijectively to each other as it is an $A^+ \otimes_{R^+} A^+$-linear isomorphism. Furthermore, $\mathrm{Tr} : D_{A^+/R^+} \to D_{A^f/R^f}$ is surjective. As a consequence, $\iota(\omega^f)$ is also an isomorphism proving that ω^f is non-degenerate. This shows that ω^f is a bisymplectic structure o, A^f. The moment map $\mu := (\mu_i)_i$ associated to ω is determined by the condition $d\mu_i = \iota_{E_i}(\omega)$. Denote by F_i for $i \neq 2$ the gauge elements in A^f. By lemma 3.9,

$$d(\mu_i^f) = (d\mu_i)^f = (\iota_{E_i}(\omega))^f = \iota_{E_i^f}(\omega^f) = \iota_{F_i}(\omega^f)$$

for $i \neq 1, 2$. We know from [30, Lemma 5.3.3] that $F_1 = E_1^f + E_2^f$, so

$$d(\mu_1^{ff}) = d(\mu_1^f + \mu_2^f) = \iota_{E_1^f}(\omega^f) + \iota_{E_2^f}(\omega^f) = \iota_{F_1}(\omega^f),$$

as expected. $\qquad \square$

4.3. From Calabi–Yau structures to bisymplectic structures

Let $\mathcal{C}$ be a k-linear category with set of objects $I = \{1, \ldots, n\}$ (in particular, we assume that $\mathcal{C}$ is concentrated in degree 0). Set $e_i = \mathrm{id}_i$, $R = \oplus_{i \in I} k e_i$, $\hat{\mathcal{R}} = \coprod_{i \in I} k[x_i]$ and $A = A_{\mathcal{C}}$. Note that $\hat{R} := A_{\hat{\mathcal{R}}} \simeq \bigoplus_{i \in I} k[x_i]$. We assume that we are given an endomorphism of each object i. This amounts to having a k-linear functor $\mu : \hat{\mathcal{R}} \to \mathcal{C}$ or, equivalently, an R-algebra morphism $\hat{\mathcal{R}} \to A$. Let us set $\mu_i := \mu(x_i) \in e_i A e_i$.

Theorem 4.8. *Assume we have a relative* 1-*Calabi–Yau structure on* $\mu : \hat{\mathcal{R}} \to \mathcal{C}$ *inducing the natural Calabi–Yau structure on each* $k[x_i]$, *and assume that* $A_{\mathcal{C}}$ *is* 1-*smooth. Then* $A_{\mathcal{C}}$ *is bisymplectic with moment map* $\sum_{i=1}^{n} \mu_i$.

Proof. The 1-Calabi–Yau structure gives a homotopy $0 \sim \mu(\sum_{i=1}^{n} 1 \otimes x_i) = \sum_{i=1}^{n} 1 \otimes \mu_i$ which yields, thanks to section 2.3, an element $\omega_1 \in \Omega_R^2(A)$ satisfying $\iota_E(\omega_1) = \sum_{i=1}^{n} d\mu_i$. Hence, μ is a moment map for ω_1.

It remains to show that ω_1 is closed and non-degenerate. First, note that $\gamma := \sum_{i=1}^{n} 1 \otimes x_i \in \Omega_R^1 \hat{R}$ trivially lifts in negative cyclic homology as $B(\gamma) = 0$. Then the Calabi–Yau structure is given by a family $\omega_k \in \bar{\Omega}_R^{2k} A$, satisfying

$$(\iota_E - ud)\left(\sum_{k \geq 0} u^k \omega_{k+1}\right) = \mu(\gamma),$$

which implies $d\omega_1 = \iota_E(\omega_2) = 0 \in \overline{\mathrm{DR}}_R A$. This proves the closedness of ω_1.

The (Calabi–Yau) non-degeneration property yields the homotopy fiber sequence

$$A^\vee[1] \to R^\vee[1] \otimes_{R^e} A^e \overset{\gamma_1}{\simeq} R \otimes_{R^e} A^e \to A.$$

Using short resolutions (thanks to the 1-smoothness of A), we get the homotopy commuting diagram

$$
\begin{array}{ccccccc}
A^e & \overset{\mathrm{id}}{\longrightarrow} & A^e & \overset{\mathrm{id}}{\longrightarrow} & A^e & \overset{d\mu}{\longrightarrow} & \Omega_{A/R} \\
{\scriptstyle E}\downarrow & & {\scriptstyle \mu \otimes \mathrm{id} - \mathrm{id} \otimes \mu}\downarrow & {\scriptstyle \mu \otimes \mathrm{id} - \mathrm{id} \otimes \mu} & \downarrow & & \downarrow \\
D_{A/R} & \overset{\mathrm{ev}_\mu}{\longrightarrow} & A^e & \overset{\mathrm{id}}{\longrightarrow} & A^e & \overset{\mathrm{id}}{\longrightarrow} & A^e
\end{array}
$$

The homotopy is given by $\iota(\omega_1) : D_{A/R} \to \Omega_{A/R}$

Now, as the Calabi–Yau structure is non-degenerate, we have

$$A^\vee[1] \simeq \mathrm{hofib}\left(R^\vee[1] \otimes_{R^e} A^e \overset{\gamma_1}{\simeq} R \otimes_{R^e} A^e \to A\right).$$

In short resolutions, this yields a quasi-isomorphism between the vertical complexes

$$
\begin{array}{ccc}
A^e & \overset{\mathrm{id}}{\longrightarrow} & A^e \\
{\scriptstyle E}\downarrow & & \downarrow {\scriptstyle d\mu} \\
D_{A/R} & \underset{\iota(\omega_1)}{\longrightarrow} & \Omega_{A/R}
\end{array}
$$

which, in particular, gives an isomorphism $\iota(\omega_1) : D_{A/R} \to \Omega_{A/R}$. $\qquad\square$

Example 4.9. Let $Q = (I, E)$ be a finite quiver where I is the set of vertices and E the set of arrows. Denote by $\overline{Q}$ the double quiver obtained by adding for every arrow $a \in E$ an arrow a^* in the opposite direction. Consider the path algebra of the double quiver $A := k\overline{Q}$. We have

○ a relative 1-Calabi–Yau structure on $\mu : k[x] \to A$, $x \mapsto \sum_{a \in E}[a, a^*]$ from example 4.3;
○ a bisymplectic structure $\omega = \sum_{a \in E} dada^* \in \overline{\mathrm{DR}}_R^2 A$ on A given in [9, Proposition 8.1.1], with moment map μ.

We claim that the first structure implies (twice) the second one under theorem 4.8. Indeed, the homotopy between 0 and $\mu(1 \otimes x)$ is given by $\sum_{a \in E}(1 \otimes a \otimes a^* - 1 \otimes a^* \otimes a)$ which corresponds to $2\sum_{a \in E} dada^*$.

We next investigate the relationship between fusion of bisymplectic structures and relate them to the compositions of Calabi–Yau cospans. Consider a dg-category $\mathcal{C}$ with object set I, along with a relative 1-Calabi–Yau structure $\mu : \hat{\mathcal{R}} \to \mathcal{C}$ that induces natural absolute Calabi–Yau structures on each $k[x_i]$. Set $\hat{\mathcal{R}}_{\geq 3} = \coprod_{i \geq 3} k[x_i]$. We can consider the composition of cospans
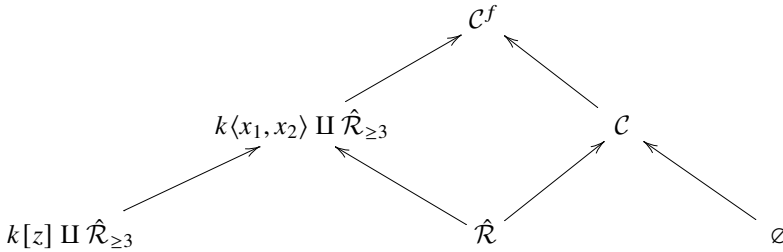
defining $\mathcal{C}^f$, where z is mapped to $x_1 + x_2$. This yields a relative Calabi–Yau structure on

$$k[z] \amalg \hat{\mathcal{R}}_{\geq 3} \to \mathcal{C}^f . \tag{4.3}$$

Theorem 4.10. *Assume that $A_{\mathcal{C}}$ is 1-smooth. Let $(A_{\mathcal{C}}, \omega)$ be the bisymplectic structure induced by the relative 1-Calabi–Yau structure μ, thanks to theorem 4.8. Then the fusion bisymplectic structure $(A_{\mathcal{C}}^f, \omega^f)$ obtained from fusing the two objects 1 and 2 is induced by the relative 1-Calabi–Yau structure (4.3).*

Proof. Set $A = A_{\mathcal{C}}$. We know, thanks to proposition 3.3, that $A^f \simeq A_{\mathcal{C}^f}$. As the bisymplectic structure is compatible with the relative 1-Calabi–Yau structure, we have that the image of z under this isomorphism is $\mu(x_1)^f + \mu(x_2)^f$. Hence, the moment map of the fusion bisymplectic structure is induced from the Calabi–Yau cospan. Let $\omega \in \Omega_R^2(A)$ denote the homotopy $\mu(1 \otimes (\sum_{i \in I} x_i)) \sim 0$ of the Calabi–Yau structure which induces by assumption the bisymplectic structure on A. Since the homotopy between the 1-forms in the cospan

$$k[z] \amalg \hat{\mathcal{R}}_{\geq 3} \longrightarrow k\langle x_1, x_2 \rangle \amalg \hat{\mathcal{R}}_{\geq 3} \longleftarrow \hat{\mathcal{R}}$$

is trivial, the zero-homotopy of the composition of Calabi–Yau cospans is given by the image of ω under the map ν from lemma 3.8. But it is proven there that this image is ω^f, which is precisely what we want. □

To summarize, we have proven that the following diagram commutes, with $R^f \simeq \oplus_{i \in I \setminus \{2\}} k e_i$ and $\hat{\mathcal{R}}^f \simeq \amalg_{i \in I \setminus \{2\}} k[x_i]$.
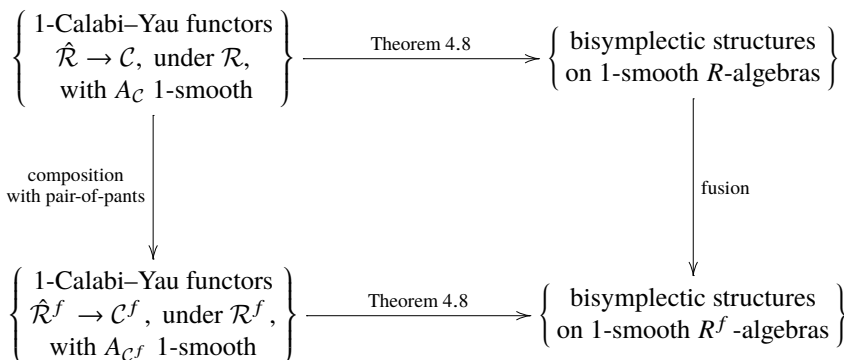
5. Calabi–Yau versus quasi-bisymplectic structures

We prove in this section that relative Calabi–Yau structures on $k[x^{\pm 1}] \to \mathcal{C}$, $\mathcal{C}$ a k-linear dg-category, induces quasi-bisymplectic ones on $A_{\mathcal{C}}$, in the sense of [31]. We prove again that fusion of quasi-bisymplectic structures on $A_{\mathcal{C}}$ is induced by the composition of Calabi–Yau cospans with the multiplicative pair-of-pants.

5.1. Quasi-bisymplectic structures

Consider an R-algebra A.

Definition 5.1 [31]. A *quasi-bisymplectic algebra* is a triple (A, ω, Φ), where $\omega \in \mathrm{DR}_R^2 A$ and $\Phi \in A^*$, satisfying the following conditions:

(𝔹1) $d\omega = \frac{1}{6}(\Phi^{-1}d\Phi)^3 \mod [-, -]$.
(𝔹2) $\iota_E \omega = \frac{1}{2}(\Phi^{-1}d\Phi + d\Phi\Phi^{-1})$
(𝔹3) The map

$$D_{A/R} \oplus A d\Phi A \to \Omega_A : (\delta, \eta) \mapsto \iota(\omega)(\delta) + \eta$$

is surjective.

Recall from [31, Theorem 7.1] the $A \otimes_R A$-linear map $T : \Omega_{A/R} \xrightarrow{e} AE^*A \xrightarrow{T^0} A d\Phi A \xrightarrow{c} \Omega_{A/R}$, where c denotes the canonical embedding, e denotes the adjoint of c and T^0 is uniquely determined by $T^0(E^*) = \Phi^{-1}d\Phi - d\Phi\Phi^{-1}$.

Definition 5.2. We say that a triple $(\omega, P, \Phi) \in \Omega_R^2(A) \times D_R^2(A) \times A^*$ is *compatible* if $\iota(\omega)\iota(P) = 1 - \frac{1}{4}T$.

What is proved by [31, Theorem 7.1] is that each quasi-bisymplectic structure of $\mathrm{DR}_R^2(A)$ corresponds to a unique non-degenerate double quasi-Poisson bracket in $(D_R A / [D_R A, D_R A])_2$. We will not recall the definition of the latter here.

Lemma 5.3. *Let (ω, P, Φ) be a compatible triple on A such that (ω, Φ) is quasi-bisymplectic. Then (ω^+, Φ^+) is quasi-bisymplectic on A^+ and (ω^+, P^+, Φ^+) is also compatible.*

Proof. The compatibility condition is given by $\iota(\omega)\iota(P) = 1 - \frac{1}{4}T$. Since R is semi-simple, $- \otimes_R R^+$ is exact. Recall also that $\Omega_{A^+/R^+} \simeq \Omega_{A/R} \otimes_R R^+$ and $D_{A^+/R^+} \simeq D_{A/R} \otimes_R R^+$. From this, it follows immediately that (ω^+, Φ^+) is a quasi-bisymplectic structure. Now by functoriality of the extension of scalar functor $- \otimes_R R^+$, we obtain that $\iota(\omega^+)\iota(P^+) = 1 - \frac{1}{4}T^+$. □

Assume that $R = \oplus_{i \in I} k e_i$ is based on pairwise orthogonal idempotents. Let (ω, P, Φ) be a compatible triple on A such that (ω, Φ) is quasi-bisymplectic and assume that $\Phi = (\Phi_i)_{i \in I} \in \oplus_{i \in I} e_i A^* e_i$. Set $\Phi_1^{ff} = \Phi_1^f \Phi_2^f$ and $\Phi_i^{ff} = \Phi_i^f = \Phi_i$ if $i > 2$. The following rather computational result is the noncommutative analog of [1, Proposition 10.7].

Proposition 5.4. *Set $\omega_{\mathrm{cor}} = \frac{1}{2}(\Phi_1^f)^{-1}d\Phi_1^f d\Phi_2^f(\Phi_2^f)^{-1}$. Then $\omega^{ff} := \omega^f - \omega_{\mathrm{cor}}$ is compatible with $P^{ff} := P^f + \frac{1}{2}E_1^f E_2^f$.*

Proof. We need to prove that $\iota(\omega^{ff})\iota(P^{ff}) = 1 - \frac{1}{4}T^{ff}$, which is equivalent to

$$\underbrace{\iota(\omega^f)\iota(P^f)}_{(I)} - \frac{1}{2}\underbrace{\iota(\omega_{\mathrm{cor}})\iota(E_1^f E_2^f)}_{(II)} - \underbrace{\iota(\omega_{\mathrm{cor}})\iota(P^f)}_{(III)} + \frac{1}{2}\underbrace{\iota(\omega^f)\iota(E_1^f E_2^f)}_{(IV)} = 1 - \frac{1}{4}\underbrace{T^{ff}}_{(V)}. \tag{5.1}$$

Note that $A^+ \to A^f$, $a \mapsto \mathrm{Tr}(a)$ is surjective. Hence, it is sufficient to show compatibility on all images of $da \in \Omega_{A^+/R^+}$. We will systematically use the notation $(-)^f = \mathrm{Tr}(-)$ in the rest of this proof.

We have $\Phi_1^{ff} = \Phi_1^f \Phi_2^f = \Phi_1^+ e_{12}\Phi_2^+ e_{21}$ and $\Phi_i^{ff} = \Phi_i^f = \Phi_i$ if $i > 2$. We abusively note $\Phi_i = \Phi_i^+$, as they do not involve e_{ij}'s, so that $\Phi_i^f = \Phi_i$ when $i \neq 2$, $\Phi_2^f = e_{12}\Phi_2 e_{21}$ and we set $\Psi = \Phi_1^{ff}$. Then for any $a \in A^+$,

$$
\begin{aligned}
(\mathrm{V})(da^f) &= T^{ff}(da^f) \\
&= [a^f, (\Phi^{ff})^{-1}d\Phi^{ff} - d\Phi^{ff}(\Phi^{ff})^{-1}] \\
&= [a^f, \Psi^{-1}d\Phi_1\Phi_2^f + (\Phi_2^f)^{-1}d\Phi_2^f - d\Phi_1\Phi_1^{-1} - \Phi_1 d\Phi_2^f \Psi^{-1}] \\
&\quad + \sum_{i>2}[a^f, \Phi_i^{-1}d\Phi_i - d\Phi_i\Phi_i^{-1}] \\
&= [a^f, \Psi^{-1}d\Phi_1\Phi_2^f + (\Phi_2^f)^{-1}d\Phi_2^f - d\Phi_1\Phi_1^{-1} - \Phi_1 d\Phi_2^f \Psi^{-1}] \\
&\quad + \sum_{i>2}\epsilon[a, \Phi_i^{-1}d\Phi_i - d\Phi_i\Phi_i^{-1}]\epsilon,
\end{aligned}
$$

whereas, thanks to lemma 3.9,

$$
\begin{aligned}
(\mathrm{I})(da^f) &= \iota(\omega^f)\iota(P^f)(da^f) \\
&= \iota(\omega^f)\big(\iota(P)(da)\big)^f \\
&= \big(\iota(\omega)\iota(P)(da)\big)^f \\
&= \big(a - \tfrac{1}{4}T(da)\big)^f \\
&= a^f - \tfrac{1}{4}\epsilon[a, \Phi_1^{-1}d\Phi_1 - d\Phi_1\Phi_1^{-1}]\epsilon - \tfrac{1}{4}e_{12}[a, \Phi_2^{-1}d\Phi_2 - d\Phi_2\Phi_2^{-1}]e_{21} \\
&\quad - \tfrac{1}{4}\sum_{i>2}\epsilon[a, \Phi_i^{-1}d\Phi_i - d\Phi_i\Phi_i^{-1}]\epsilon \\
&= a^f - \tfrac{1}{4}[\epsilon a\epsilon, \Phi_1^{-1}d\Phi_1 - d\Phi_1\Phi_1^{-1}]\epsilon - \tfrac{1}{4}[e_{12}ae_{21}, (\Phi_2^f)^{-1}d\Phi_2^f - d\Phi_2^f(\Phi_2^f)^{-1}] \\
&\quad - \tfrac{1}{4}\sum_{i>2}\epsilon[a, \Phi_i^{-1}d\Phi_i - d\Phi_i\Phi_i^{-1}]\epsilon.
\end{aligned}
$$

Recall that for every $\delta \in D_{A^f}$,

$$
\begin{aligned}
2\iota(\omega_{\mathrm{cor}})(\delta) &= {}^\circ i_\delta(\Phi_1^{-1}d\Phi_1 d\Phi_2^f(\Phi_2^f)^{-1}) \\
&= {}^\circ(\Phi_1^{-1}\delta\Phi_1 d\Phi_2^f(\Phi_2^f)^{-1} - \Phi_1^{-1}d\Phi_1\delta\Phi_2^f(\Phi_2^f)^{-1}) \\
&= \delta(\Phi_1)''d\Phi_2^f \Psi^{-1}\delta(\Phi_1)' - \delta(\Phi_2^f)''\Psi^{-1}d\Phi_1\delta(\Phi_2^f)',
\end{aligned}
$$

and that for every $a \in A$, we have $\iota(P)(da) = H_a$, the Hamiltonian vector field which satisfies

$$
H_a(\Phi) = -\tfrac{1}{2}(\Phi E + E\Phi)(a)^\circ.
$$

It implies (recall that the bimodule structure on double derivations is induced by the inner one on $A \otimes_R A$)

$$
\begin{aligned}
2H_a^f(\Phi_1^f) &= 2(\epsilon H_a \epsilon + e_{12} H_a e_{21})(\Phi_1) \\
&= -(\epsilon \Phi_1 E_1 \epsilon + \epsilon E_1 \Phi_1 \epsilon + e_{12}\Phi_1 E_1 e_{21} + e_{12}E_1\Phi_1 e_{21})(a)^\circ \\
&= -(\Phi_1 E_1 \epsilon + \epsilon E_1 \Phi_1)(a)^\circ \\
&= -(a\epsilon \otimes \Phi_1 - \epsilon \otimes \Phi_1 a + a\Phi_1 \otimes \epsilon - \Phi_1 \otimes \epsilon a)^\circ \\
&= -\Phi_1 \otimes a\epsilon + \Phi_1 a \otimes \epsilon - \epsilon \otimes a\Phi_1 + \epsilon a \otimes \Phi_1 \\
&= -\Phi_1 \otimes \epsilon a\epsilon + \Phi_1 \epsilon a\epsilon \otimes e_1 - e_1 \otimes \epsilon a\epsilon\Phi_1 + \epsilon a\epsilon \otimes \Phi_1
\end{aligned}
$$

and

$$
\begin{aligned}
2H_a^f(\Phi_2^f) &= 2e_{12}(\epsilon H_a \epsilon + e_{12} H_a e_{21})(\Phi_2)e_{21} \\
&= -\big(e_{12}(\epsilon \Phi_2 E_2 \epsilon + \epsilon E_2 \Phi_2 \epsilon + e_{12}\Phi_2 E_2 e_{21} + e_{12}E_2\Phi_2 e_{21})(a)e_{21}\big)^\circ \\
&= -\big(e_{12}(e_{12}\Phi_2 E_2 e_{21} + e_{12}E_2\Phi_2 e_{21})(a)e_{21}\big)^\circ \\
&= -\big(e_{12}ae_{21} \otimes e_{12}\Phi_2 e_{21} - e_{12}e_{21} \otimes e_{12}\Phi_2 ae_{21} \\
&\qquad + e_{12}a\Phi_2 e_{21} \otimes e_{12}e_2 e_{21} - e_{12}\Phi_2 e_{21} \otimes e_{12}e_2 ae_{21}\big)^\circ \\
&= -\big(e_{12}ae_{21} \otimes \Phi_2^f - e_1 \otimes \Phi_2^f e_{12}ae_{21} + e_{12}ae_{21}\Phi_2^f \otimes e_1 - \Phi_2^f \otimes e_{12}ae_{21}\big)^\circ \\
&= -\Phi_2^f \otimes e_{12}ae_{21} + \Phi_2^f e_{12}ae_{21} \otimes e_1 - e_1 \otimes e_{12}ae_{21}\Phi_2^f + e_{12}ae_{21} \otimes \Phi_2^f .
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
4(\text{III})(da^f) &= 4\iota(\omega_{\text{cor}})\iota(P^f)(da^f) \\
&= 4\iota(\omega_{\text{cor}})(H_a^f) \\
&= 2H_a^f(\Phi_1)'' d\Phi_2^f \Psi^{-1} H_a^f(\Phi_1)' - 2H_a^f(\Phi_2^f)'' \Psi^{-1} d\Phi_1 H_a^f(\Phi_2^f)' \\
&= -e_1 a d\Phi_2^f \Psi^{-1}\Phi_1 + d\Phi_2^f \Psi^{-1}\Phi_1 ae_1 - e_1 a\Phi_1 d\Phi_2^f \Psi^{-1} + \Phi_1 d\Phi_2^f \Psi^{-1}ae_1 \\
&\quad + e_{12}ae_{21}\Psi^{-1}d\Phi_1\Phi_2^f - \Psi^{-1}d\Phi_1\Phi_2^f e_{12}ae_{21} + e_{12}ae_{21}\Phi_2^f \Psi^{-1}d\Phi_1 - \Phi_2^f \Psi^{-1}d\Phi_1 e_{12}ae_{21} \\
&= -\epsilon a\epsilon d\Phi_2^f (\Phi_2^f)^{-1} + d\Phi_2^f (\Phi_2^f)^{-1}\epsilon a\epsilon - \epsilon a\epsilon\Phi_1 d\Phi_2^f \Psi^{-1} + \Phi_1 d\Phi_2^f \Psi^{-1}\epsilon a\epsilon \\
&\quad + e_{12}ae_{21}\Psi^{-1}d\Phi_1\Phi_2^f - \Psi^{-1}d\Phi_1\Phi_2^f e_{12}ae_{21} + e_{12}ae_{21}\Phi_1^{-1}d\Phi_1 - \Phi_1^{-1}d\Phi_1 e_{12}ae_{21} \\
&= -[\epsilon a\epsilon, d\Phi_2^f (\Phi_2^f)^{-1} + \Phi_1 d\Phi_2^f \Psi^{-1}] + [e_{12}ae_{21}, \Psi^{-1}d\Phi_1\Phi_2^f + \Phi_1^{-1}d\Phi_1].
\end{aligned}
$$

Also,

$$
\begin{aligned}
2\iota(\omega_{\text{cor}})\iota(E_1^f E_2^f)(da^f) &= 2\iota(\omega_{\text{cor}})^\circ(i_{da^f}(E_1^f)E_2^f - E_1^f i_{da^f}(E_2^f)) \\
&= 2\iota(\omega_{\text{cor}})^\circ(E_1^f(a^f)E_2^f - E_1^f E_2^f(a^f)) \\
&= 2\iota(\omega_{\text{cor}})(e_1 E_2^f \epsilon ae_1 - e_1 a\epsilon E_2^f e_1 - e_1 E_1^f e_{12}ae_{21} + e_{12}ae_{21}E_1^f e_1) \\
&= e_1\iota(2\omega_{\text{cor}})(E_2^f)\epsilon ae_1 - e_1 a\epsilon\iota(2\omega_{\text{cor}})(E_2^f)e_1 \\
&\quad - e_1\iota(2\omega_{\text{cor}})(E_1^f)e_{12}ae_{21} + e_{12}ae_{21}\iota(2\omega_{\text{cor}})(E_1^f)e_1.
\end{aligned}
$$

But

$$
\begin{aligned}
E_1^f(a^f) &= \epsilon E_1^+(a)\epsilon + e_{12}E_1^+(a)e_{21} \\
&= \epsilon ae_1 \otimes e_1\epsilon - \epsilon e_1 \otimes e_1 a\epsilon + e_{12}ae_1 \otimes e_1 e_{21} - e_{12}e_1 \otimes e_1 ae_{21} \\
&= \epsilon a\epsilon \otimes e_1 - e_1 \otimes \epsilon a\epsilon
\end{aligned}
$$

and

$$E_2^f(a^f) = \epsilon(e_{12}E_2^+e_{21})(a)\epsilon + e_{12}(e_{12}E_2^+e_{21})(a)e_{21}$$
$$= \epsilon a e_{21} \otimes e_{12}\epsilon - \epsilon\epsilon_{21} \otimes e_{12}a\epsilon + e_{12}ae_{21} \otimes e_{12}e_{21} - e_{12}e_{21} \otimes e_{12}ae_{21}$$
$$= e_{12}ae_{21} \otimes e_1 - e_1 \otimes e_{12}ae_{21}$$

imply $E_1^f(\Phi_1) = E_1(\Phi_1)$, $E_1^f(\Phi_2^f) = 0$, $E_2^f(\Phi_1) = 0$, $E_2^f(\Phi_2^f) = E_1(\Phi_2^f)$ and

$$\iota(2\omega_{\mathrm{cor}})(E_1^f) = d\Phi_2^f(\Phi_2^f)^{-1} - \Phi_1 d\Phi_2^f\Psi^{-1}$$
$$\iota(2\omega_{\mathrm{cor}})(E_2^f) = -\Psi^{-1}d\Phi_1\Phi_2^f + \Phi_1^{-1}d\Phi_1.$$

Hence,

$$2(\mathrm{II})(da^f) = 2\iota(\omega_{\mathrm{cor}})\iota(E_1^f E_2^f)(da^f)$$
$$= e_1(-\Psi^{-1}d\Phi_1\Phi_2^f + \Phi_1^{-1}d\Phi_1)\epsilon a\epsilon - \epsilon a\epsilon(-\Psi^{-1}d\Phi_1\Phi_2^f + \Phi_1^{-1}d\Phi_1)e_1$$
$$- e_1(d\Phi_2^f(\Phi_2^f)^{-1} - \Phi_1 d\Phi_2^f\Psi^{-1})e_{12}ae_{21} + e_{12}ae_{21}(d\Phi_2^f(\Phi_2^f)^{-1} - \Phi_1 d\Phi_2^f\Psi^{-1})e_1$$
$$= [e_{12}ae_{21}, d\Phi_2^f(\Phi_2^f)^{-1} - \Phi_1 d\Phi_2^f\Psi^{-1}] + [\epsilon a\epsilon, \Psi^{-1}d\Phi_1\Phi_2^f - \Phi_1^{-1}d\Phi_1].$$

Similarly, using $\iota(2\omega^f)(E_i^f) = (\Phi_i^{-1}d\Phi_i + d\Phi_i\Phi_i^{-1})^f$, one gets

$$2(\mathrm{IV})(da^f) = 2\iota(\omega^f)\iota(E_1^f E_2^f)(da^f)$$
$$= e_1\iota(2\omega^f)(E_2^f)\epsilon a\epsilon - \epsilon a\epsilon\iota(2\omega^f)(E_2^f)e_1$$
$$- e_1\iota(2\omega^f)(E_1^f)e_{12}ae_{21} + e_{12}ae_{21}\iota(2\omega^f)(E_1^f)e_1$$
$$= e_1(\Phi_2^{-1}d\Phi_2 + d\Phi_2\Phi_2^{-1})^f \epsilon a\epsilon - \epsilon a\epsilon(\Phi_2^{-1}d\Phi_2 + d\Phi_2\Phi_2^{-1})^f e_1$$
$$- e_1(\Phi_1^{-1}d\Phi_1 + d\Phi_1\Phi_1^{-1})^f e_{12}ae_{21} + e_{12}ae_{21}(\Phi_1^{-1}d\Phi_1 + d\Phi_1\Phi_1^{-1})^f e_1$$
$$= e_{12}(\Phi_2^{-1}d\Phi_2 + d\Phi_2\Phi_2^{-1})e_{21}\epsilon a\epsilon - \epsilon a\epsilon e_{12}(\Phi_2^{-1}d\Phi_2 + d\Phi_2\Phi_2^{-1})e_{21}$$
$$- (\Phi_1^{-1}d\Phi_1 + d\Phi_1\Phi_1^{-1})e_{12}ae_{21} + e_{12}ae_{21}(\Phi_1^{-1}d\Phi_1 + d\Phi_1\Phi_1^{-1})$$
$$= [e_{12}ae_{21}, \Phi_1^{-1}d\Phi_1 + d\Phi_1\Phi_1^{-1}] - [\epsilon a\epsilon, (\Phi_2^f)^{-1}d\Phi_2^f + d\Phi_2^f(\Phi_2^f)^{-1}].$$

Putting everything together yields (5.1) as expected. □

5.2. From Calabi–Yau structures to quasi-bisymplectic structures

Again, let $\mathcal{C}$ be a k-linear category with objects set $I = \{1, \ldots, n\}$. Set $e_i = \mathrm{id}_i$, $R = \oplus_{i \in I} k e_i$ and $\mathcal{T} := \coprod_{i \in I} k[x_i^{\pm 1}]$.

Theorem 5.5. *Assume that we have a relative 1-Calabi–Yau structure on a k-linear functor $\mu : \mathcal{T} \to \mathcal{C}$ which induces the natural 1-Calabi–Yau structure on each $k[x_i^{\pm 1}]$. If $A = A_\mathcal{C}$ is 1-smooth, then it is quasi-bisymplectic with multiplicative moment map $\sum_{i=1}^n \mu(x_i)$.*

Proof. Define $\Phi : k[x^{\pm 1}] \to A$ by $\Phi(x) = \sum_{i=1}^n \mu(x_i) \in \oplus_{i \in I} e_i A^* e_i$. Since μ is 1-Calabi–Yau, using the notation of section 2.4, we know that there exists $\omega_k \in \bar\Omega_R^{2k} A$ for all k such that

$$(\iota_E - ud)\left(\sum_{k \geq 0} u^k \omega_{k+1}\right) = \Phi(\gamma),$$

or equivalently,

$$\iota_E \omega_1 = \Phi(\gamma_1) = \frac{1}{2}(\Phi^{-1}d\Phi + d\Phi\Phi^{-1}) \tag{B2}$$

$$\iota_E \omega_2 - d\omega_1 = -\frac{1}{6}\Phi(\gamma_2) \Rightarrow d\omega_1 = \frac{1}{6}(\Phi^{-1}d\Phi)^3 \bmod [-,-] \tag{B1}$$

$$\iota_E \omega_3 - d\omega_2 = \frac{2!}{5!}\Phi(\gamma_3)$$

$$\vdots$$

$$\iota_E \omega_{k+1} - d\omega_k = (-1)^k \frac{k!}{(2k+1)!}\Phi(\gamma_{k+1}) \qquad\qquad k \geq 1.$$

For (B3), set $T = k[x^{\pm 1}]$ and write the relative 1-pre-Calabi–Yau structure

$$A^\vee[1] \rightarrow T^\vee[1] \otimes_{T^e} A^e \overset{\gamma}{\simeq} T \otimes_{T^e} A^e \rightarrow A$$

with short resolutions (thanks to our 1-smoothness assumption) to get the homotopy commuting diagram

where the homotopy $D_{A/R} \rightarrow \Omega_{A/R}$ gives $\iota_E \omega_1 = (\Phi^{-1}d\Phi + d\Phi\Phi^{-1})/2$.

Now assume that our Calabi–Yau structure is non-degenerate; that is,

$$A^\vee[1] \simeq \mathrm{hofib}\left(T^\vee[1] \otimes_{T^e} A^e \overset{\gamma}{\simeq} T \otimes_{T^e} A^e \rightarrow A\right).$$

In short resolutions, this yields a quasi-isomorphism (between vertical complexes)

which, in particular, gives a surjection $D_{A/R} \rightarrow \Omega_{A/R}/\langle d\Phi\rangle$, that is (B3). □

5.3. Fusion

Set $\mathcal{T}_{\geq 3} = \amalg_{i \geq 3} k[x_i^{\pm 1}]$ and consider the following composition of 1-Calabi–Yau cospans:

$$
\begin{array}{ccccc}
 & & \mathcal{C}^f & & \\
 & \nearrow & & \nwarrow & \\
 & k\langle x^{\pm 1}, y^{\pm 1}\rangle \amalg \mathcal{T}_{\geq 3} & & & \mathcal{C} \\
 \nearrow & & \nwarrow & \nearrow & & \nwarrow \\
 k[z^{\pm 1}] \amalg \mathcal{T}_{\geq 3} & & & \mathcal{T} & & \varnothing
\end{array}
\tag{5.2}
$$

where the leftmost one is induced by the pair-of-pants. We want to prove the following multiplicative analog of theorem 4.10.

Theorem 5.6. *Consider a 1-Calabi–Yau functor $\mathcal{T} \to \mathcal{C}$ inducing the natural 1-Calabi–Yau structure on each $k[x_i^{\pm 1}]$, and assume that $A_\mathcal{C}$ is 1-smooth. Then the quasi-bisymplectic structure on $\mathcal{C}^f$ induced, thanks to theorem 5.5, by the 1-Calabi–Yau functor*

$$
k[z^{\pm 1}] \amalg \mathcal{T}_{\geq 3} \to \mathcal{C}^f
$$

is the one obtained by fusion of 1 and 2 from the quasi-bisymplectic structure of $A_\mathcal{C}$ induced by theorem 5.5.

Proof. Denote by Φ_1^f, Φ_2^f the images of $x = x_1, y = x_2$ in the pushout $\mathcal{C}^f$. The extra difficulty here with respect to the proof of theorem 4.10 is that the homotopy β_1 involved in the pair-of-pants cospan is nontrivial; see example 4.4. This non-degenerate homotopy

$$
\beta_1 = \frac{1}{2}\left(y^{-1} \otimes x^{-1} \otimes xy - y \otimes y^{-1}x^{-1} \otimes x\right) \in \overline{\mathrm{HH}}_2 k\langle x^{\pm 1}, y^{\pm 1}\rangle
$$

is mapped in $\overline{\mathrm{DR}}^2 k\langle x^{\pm 1}, y^{\pm 1}\rangle$ to

$$
\begin{aligned}
\omega &= \frac{1}{4}\left(y^{-1}dx^{-1}d(xy) - yd(y^{-1}x^{-1})dx\right) \\
&= \frac{1}{4}\left(-y^{-1}x^{-1}dxx^{-1}(xdy + dxy) + dyy^{-1}x^{-1}dx + x^{-1}dxx^{-1}dx\right) \\
&= \frac{1}{4}\left(-y^{-1}x^{-1}dxdy - y^{-1}x^{-1}dxx^{-1}dxy + dyy^{-1}x^{-1}dx + x^{-1}dxx^{-1}dx\right) \\
&\equiv -\frac{1}{2}x^{-1}dxdyy^{-1} \quad \mathrm{mod} \; [-, -],
\end{aligned}
$$

which is mapped to

$$
-\frac{1}{2}(\Phi_1^f)^{-1}d\Phi_1^f d\Phi_2^f (\Phi_2^f)^{-1} \in \overline{\mathrm{DR}}_{R^f}^2 \mathcal{C}^f .
$$

The proposition 5.4 allows us to conclude, thanks to the uniqueness [31, Theorem 7.1] of compatibility and [31, Theorem 8.2.1]. □

To summarize, we have proven that the following diagram commutes, where $R^f = \oplus_{i \in I \setminus \{2\}} ke_i$ and $\mathcal{T}^f = \amalg_{i \in I \setminus \{2\}} k[x_i^{\pm 1}]$.

$$
\begin{Bmatrix} \text{1-Calabi–Yau functors} \\ \mathcal{T} \to \mathcal{C}, \text{ over } \mathcal{R}, \\ \text{with } A_{\mathcal{C}} \text{ 1-smooth} \end{Bmatrix} \xrightarrow{\text{Theorem 5.5}} \begin{Bmatrix} \text{quasi-bisymplectic structures} \\ \text{on 1-smooth } R\text{-algebras} \end{Bmatrix}
$$

<div style="text-align:center">

composition
with multiplicative
pair-of-pants

fusion

</div>

$$
\begin{Bmatrix} \text{1-Calabi–Yau functors} \\ \mathcal{T}^f \to \mathcal{C}^f, \text{ over } \mathcal{R}^f, \\ \text{with } A_{\mathcal{C}^f} \text{ 1-smooth} \end{Bmatrix} \xrightarrow{\text{Theorem 5.5}} \begin{Bmatrix} \text{quasi-bisymplectic structures} \\ \text{on 1-smooth } R^f\text{-algebras} \end{Bmatrix}
$$
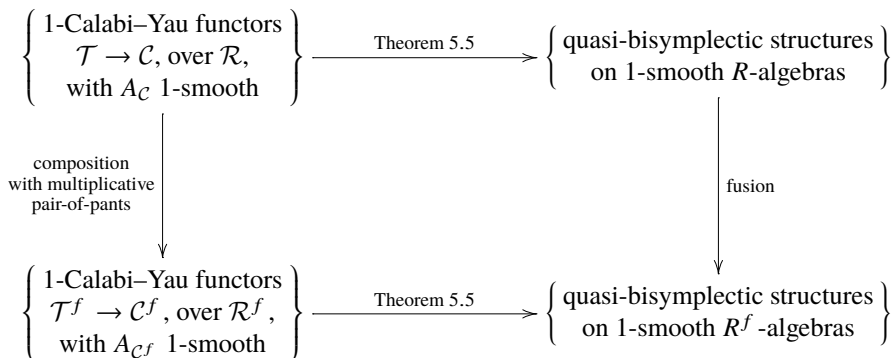
5.4. Examples

5.4.1. An elementary quiver

Consider the quiver $A_2 = (V = \{1, 2\}, E = \{e : 1 \to 2\})$, with orthogonal idempotents e_1 and e_2 satisfying $1 = e_1 + e_2$, $R = k e_1 \oplus k e_2$, and set

$$
a_1 = e_1 + e^* e \quad \text{and } a_2 = e_2 + e e^*.
$$

Let us denote by A the localization $(k\overline{A_2})_{a_1, a_2}$. Recall that we have given in [4] a relative 1-Calabi–Yau structure on $\Phi : k[x^{\pm 1}] \to A$ defined by

$$
\Phi_1(x_1) = a_1^{-1} \quad \text{and} \quad \Phi_2(x_2) = a_2.
$$

Define $\partial/\partial e$ and $\partial/\partial e^*$ in $D_R A$ by $\partial e/\partial e = e_2 \otimes e_1$, $\partial e^*/\partial e = 0$, $\partial e^*/\partial e^* = e_1 \otimes e_2$ and $\partial e/\partial e^* = 0$.

In the previous section, we proved that this Calabi–Yau structure induces a quasi-bisymplectic one $\omega_1 \in \overline{\mathrm{DR}}_R^2 A$ on A. We want to prove the following.

Proposition 5.7. *The double quasi-Poisson bracket compatible with ω_1 through [31, Theorem 7.1] is the one described in [31, §8.3]:*

$$
P = \frac{1}{2}\left((1 + e e^*)\frac{\partial}{\partial e^*}\frac{\partial}{\partial e} - (1 + e^* e)\frac{\partial}{\partial e}\frac{\partial}{\partial e^*} \right) \in (D_R A/[D_R A, D_R A])_2.
$$

Note that we use the convention regarding concatenation of paths opposite to the one in [30]; that is, $e = e_2 e e_1$.

Proof. In [3], one homotopy $\phi(\gamma_1) \sim 0$ is given by

$$
\begin{aligned}
\beta_1 = \frac{1}{2}\big(& e^* \otimes e \otimes \Phi + \Phi \otimes e^* \otimes e - e^* \otimes \Phi^{-1} \otimes e - \Phi^{-1} \otimes e \otimes e^* \\
& + 1 \otimes e^* \otimes e\Phi - 1 \otimes e\Phi \otimes e^* \big),
\end{aligned} \tag{5.3}
$$

where $\Phi = \Phi_1(x_1) + \Phi_2(x_2)$. It yields an element ($1/4$ appears because of the degree operator)

$$
\omega_1 = \frac{1}{4}\big(e^* de\, d\Phi + \Phi de^* de - e^* d\Phi^{-1} de - \Phi^{-1} de\, de^* + de^* d(e\Phi) - d(e\Phi)de^* \big)
$$

in $\overline{\mathrm{DR}}^2 A = \left(\overline{\Omega}A/[\overline{\Omega}A, \overline{\Omega}A] \right)_2$. We can heavily simplify this expression working modulo $[\overline{\Omega}A, \overline{\Omega}A]$.

First, note that (again, dab stands for $(da)b$)

$$d\Phi = -a_1^{-1}(de^*e + e^*de)a_1^{-1} + dee^* + ede^* = -\Phi(de^*e + e^*de)\Phi + dee^* + ede^*$$
$$d\Phi^{-1} = de^*e + e^*de - a_2^{-1}(dee^* + ede^*)a_2^{-1} = de^*e + e^*de - \Phi^{-1}(dee^* + ede^*)\Phi^{-1}.$$

Thus, using $\Phi e\Phi = e$ and $\Phi e^*\Phi = e^*$ (cf [4, (4.3)]),

$$4\omega_1 = \Phi de^*de - \Phi^{-1}dede^* + e^*ded\Phi - e^*d\Phi^{-1}de + 2de^*d(e\Phi)$$
$$= \Phi de^*de - \Phi^{-1}dede^* - e^*de\Phi(de^*e + e^*de)\Phi$$
$$+ e^*\Phi^{-1}(dee^* + ede^*)\Phi^{-1}de + 2de^*de\Phi - 2de^*e\Phi(de^*e + e^*de)\Phi$$
$$= \Phi de^*de - \Phi^{-1}dede^* - e^*de\Phi de^*e\Phi$$
$$\underbrace{-e^*de\Phi e^*de\Phi + e^*\Phi^{-1}dee^*\Phi^{-1}de}_{\equiv 0} + e^*\Phi^{-1}ede^*\Phi^{-1}de$$
$$+ 2de^*de\Phi - 2\underbrace{de^*e\Phi de^*e\Phi}_{\equiv 0} - 2de^*e\Phi e^*de\Phi$$
$$\equiv 3\Phi de^*de - \Phi^{-1}dede^* - e\Phi e^*de\Phi de^* + e^*\Phi^{-1}ede^*\Phi^{-1}de + 2de^*e\Phi e^*de\Phi$$
$$= 3\Phi de^*de - \Phi^{-1}dede^* - ee^*\Phi_2^{-1}de\Phi de^* + e^*e\Phi_1 de^*\Phi^{-1}de + 2de^*ee^*\Phi_2^{-1}de\Phi$$
$$= 3\Phi de^*de - \Phi^{-1}dede^* - de\Phi de^* + \Phi^{-1}de\Phi de^*$$
$$+ de^*\Phi^{-1}de - \Phi de^*\Phi^{-1}de - 2de^*de\Phi + 2de^*\Phi^{-1}de\Phi$$
$$\equiv 2\Phi de^*de - 2\Phi^{-1}dede^*.$$

We now need to prove that P and ω_1 are compatible, meaning that

$$\iota(\omega_1)\iota(P) = 1 - \frac{1}{4}T \tag{5.4}$$

with $T(dp) = [p, \Phi^{-1}d\Phi - d\Phi\Phi^{-1}]$. For $p = e$, the LHS is

$$\iota(\omega_1)\iota(P)(de) = \frac{1}{2}\iota(\omega_1)\left(\frac{\partial}{\partial e^*}(1 + e^*e) + (1 + ee^*)\frac{\partial}{\partial e^*}\right)$$
$$= \frac{1}{2}({}^\circ i_{\partial/\partial e^*}(\omega_1)(1 + e^*e) + (1 + ee^*){}^\circ i_{\partial/\partial e^*}(\omega_1)),$$

where

$$i_\delta(pdqdr) = p\delta(q)' \otimes \delta(q)''dr - pdq\delta(r)' \otimes \delta(r)'' \in A \otimes \Omega^1 + \Omega^1 \otimes A,$$

as stated earlier. Note that above we have used, for $\pi, \nu \in A$ and $\delta \in D_{A/R}$,

$${}^\circ i_{\pi\delta\nu}(pdqdr) = {}^\circ(p\delta(q)'\nu \otimes \pi\delta(q)''dr - pdq\delta(r)'\nu \otimes \pi\delta(r)'')$$
$$= \pi{}^\circ i_\delta(pdqdr)\nu$$

since the bimodule structure on $D_{A/R}$ is induced by the inner one on A^e, as explained in the proof of [9, 2.8.6]. We have

$${}^\circ i_{\partial/\partial e^*}(2\omega_1) = {}^\circ(\Phi \otimes de + \Phi^{-1}de \otimes e_2) = de\Phi + \Phi^{-1}de.$$

Thus,

$$
\begin{aligned}
4\iota(\omega_1)\iota(P)(da) &= (de\Phi + \Phi^{-1}de)(1 + e^*e) + (1 + ee^*)(de\Phi + \Phi^{-1}de) \\
&= 2de + \Phi^{-1}de\Phi^{-1} + \Phi de\Phi,
\end{aligned}
$$

whereas 4 times the RHS of (5.4) evaluated at de is

$$
\begin{aligned}
4de - [e, \Phi^{-1}d\Phi - d\Phi\Phi^{-1}] &= 4de - e\Phi^{-1}(-\Phi(de^*e + e^*de)\Phi + dee^* + ede^*) \\
&\quad + e(-\Phi(de^*e + e^*de)\Phi + dee^* + ede^*)\Phi^{-1} \\
&\quad + \Phi^{-1}(-\Phi(de^*e + e^*de)\Phi + dee^* + ede^*)e \\
&\quad - (-\Phi(de^*e + e^*de)\Phi + dee^* + ede^*)\Phi^{-1}e \\
&= 4de + ede^*e\Phi + ee^*de\Phi - e\Phi de^*e - e\Phi e^*de \\
&\quad + \Phi^{-1}dee^*e + \Phi^{-1}ede^*e - dee^*\Phi^{-1}e - ede^*\Phi^{-1}e \\
&= 4de + ee^*de\Phi - \Phi^{-1}ee^*de + \Phi^{-1}dee^*e - dee^*e\Phi \\
&= 4de + \Phi de\Phi - de\Phi - de + \Phi^{-1}de \\
&\quad + \Phi^{-1}de\Phi^{-1} - \Phi^{-1}de - de + de\Phi \\
&= 2de + \Phi^{-1}de\Phi^{-1} + \Phi de\Phi,
\end{aligned}
$$

as wished. Computations are similar to prove eq. (5.4) evaluated at de^*. □

5.4.2. Arbitrary quivers

Let us go back to the proof [4, Theorem 4.8] of the 1-Calabi–Yau structure on the multiplicative moment map $\mu_Q : \coprod_{v \in V} k[z_v^{\pm 1}] \to k\overline{Q}_{loc} := k\overline{Q}[(1 + ee^*)^{-1}]_{e \in \overline{E}}$ defined by

$$
z_v \longmapsto \prod_{e \in E \cap t^{-1}(v)} (1 + ee^*) \times \prod_{e \in E \cap s^{-1}(v)} (1 + e^*e)^{-1}.
$$

It is done by realizing this functor as successive compositions of Calabi–Yau cospans. Let us specify an order that better suits our purpose. As usual, we denote by Q^{sep} the quiver with same edge set E but vertex set $\overline{E} = \{v_e = s(e), v_{e^*} = t(e)\}$. It is the disjoint union of $|E|$ copies of A_2. We have a 1-Calabi–Yau morphism

$$
\mu_{Q^{\text{sep}}} : \coprod_{e \in E} (k[x_e^{\pm 1}] \amalg k[y_e^{\pm 1}]) \longrightarrow k\overline{Q^{\text{sep}}}_{loc} \tag{5.5}
$$

given by $x_e \mapsto (e_{s(e)} + e^*e)^{-1}$ and $y_e \mapsto e_{t(e)} + ee^*$. We know, thanks to the previous section, that the quasi-bisymplectic structure on $k\overline{Q^{\text{sep}}}_{loc}$ induced by this 1-Calabi–Yau multiplicative moment map matches the one described by Van den Bergh in [31].

We want to prove the same for Q by fusing pairs of vertices (v_e, v_f) any time $s(e) = s(f)$ in $\overline{Q}$. Precisely, pick a finite sequence of fusion of pairs of vertices that takes us from Q^{sep} to Q, and consider an intermediary step Q°. Assume that the quasi-bisymplectic structure induced by the 1-Calabi–Yau one on μ_{Q° matches Van den Bergh's, and proceed to the next fusion in our sequence. Assume that we fuse 1 and 2 in the vertex set I of Q°. By that, we mean that we precisely proceed to the composition (5.2), where $\mathcal{C} = k\overline{Q^\circ}_{loc}$. By induction, and using theorem 5.6, we get the following.

Theorem 5.8. *The quasi-bisymplectic structure on $k\overline{Q}_{loc}$ induced by the 1-Calabi–Yau one on μ_Q matches the one given by Van den Bergh.*

6. Representation spaces

As before, assume that A is a 1-smooth R-algebra with $R = \oplus_{i \in I} k e_i$, where the e_i are pairwise orthogonal idempotents and $I := \{1, \cdots, n\}$. For any I-graded finite dimensional space V, define A_V by

$$\mathrm{Hom}_{\mathrm{Alg}/R}(A, \mathrm{End}(V)) = \mathrm{Hom}_{\mathrm{CommAlg}/k}(A_V, k).$$

Thanks to [9, (6.2.2)], setting $X_V = \mathrm{Spec}(A_V)$, we have a map

$$\underline{\mathrm{tr}} : \mathrm{DR}^* A \longrightarrow \Omega^*(X_V)^{\mathrm{GL}_V} \tag{6.1}$$

given by $\alpha \mapsto \mathrm{tr}(\hat{\alpha})$, where $\hat{\alpha}$ is induced by the evaluation

$$A \to (A_V \otimes \mathrm{End}(V))^{\mathrm{GL}_V} \quad ; \quad a \mapsto \hat{a}.$$

Thanks to [31, Proposition 6.1], there is a quasi-Hamiltonian structure on $(X_V, \underline{\mathrm{tr}}(\omega), \hat{\Phi})$ when (A, ω, Φ) is quasi-bisymplectic. Now, $\hat{\Phi} : X_V \to \mathrm{GL}_V$ induces a lagrangian structure on $[X_V / \mathrm{GL}_V] \to [\mathrm{GL}_V / \mathrm{GL}_V]$.

However, thanks to [6], if Φ carries a 1-Calabi–Yau structure, it yields a lagrangian structure on $\mathrm{Perf}_A \to \mathrm{Perf}_{k[x^{\pm 1}]}$, and thus considers substacks on $[X_V / \mathrm{GL}_V] \to [\mathrm{GL}_V / \mathrm{GL}_V]$ again.

In both cases, we know that the induced 1-shifted symplectic structure on $[\mathrm{GL}_V / \mathrm{GL}_V]$ is the standard one, thanks to [4, §5.1] for the latter.

Now, assume that the 1-Calabi–Yau structure on Φ induces the quasi-bisymplectic structure (A, ω, Φ); that is, ω_1 in the proof of theorem 5.5 is ω. The current section is devoted to the proof of the following.

Theorem 6.1. *These two lagrangian structures are identical.*

6.1. Lagrangian morphisms and quasi-hamiltonian spaces

Let X be a smooth algebraic variety. Since we will apply the following results to $X = X_V$, we assume X to be affine for simplicity, but these results can be extended to the non-affine case. Assume that a reductive group G acts on X and consider a G-equivariant morphism $\mu : X \to G$, which induces $[\mu] : [X/G] \to [G/G]$. Consider the standard 1-shifted symplectic structure on $[G/G]$ given by $\underline{\omega} = \underline{\omega}_0 + \underline{\omega}_1$, where $\underline{\omega}_0 \in (\Omega^1(G) \otimes \mathfrak{g}^*)^G$ and $\underline{\omega}_1 \in \Omega^3(G)^G$.

We refer to [3, §3] for a precise definition of the space $\mathcal{A}^{p,(\mathrm{cl})}(X, n)$ of (closed) p-forms of degree n on X. When $\alpha \in \Omega^2(X)^G$, we say that (α, μ) satisfies the multiplicative moment condition if

$$\forall u \in \mathfrak{g}, \ i_{\vec{u}} \alpha = \langle \mu^* \underline{\omega}_0, u \rangle. \tag{$\mathbb{M}$}$$

This is condition (B2) in [31].

Lemma 6.2. *The space of homotopies between $[\mu]^* \underline{\omega}_0$ and 0 in $\mathcal{A}^{2,\mathrm{cl}}([X/G], 1)$ is discrete. It is the space of invariant 2-forms $\alpha \in \Omega^2(X)^G$ satisfying ($\mathbb{M}$).*

Proof. The *cochain* complex of 2-forms on $[X/G]$ is given by

$$\Omega^2(X)^G \overset{\partial}{\longrightarrow} (\Omega^1(X) \otimes \mathfrak{g}^*)^G \longrightarrow (\mathcal{O}(X) \otimes S^2 \mathfrak{g}^*)^G .$$

The result follows from the fact that, by definition, ∂ is given by $\langle \partial \alpha, u \rangle = i_{\vec{u}} \alpha$ for every $u \in \mathfrak{g}$. $\square$

This can be extended to the following, where we recognize the extra condition (B1) of [31].

Lemma 6.3. *The space of homotopies between $[\mu]^* \underline{\omega}$ and 0 in $\mathcal{A}^{2,\mathrm{cl}}([X/G], 1)$ is discrete. It is the space of 2-forms $\alpha \in \Omega^2(X)^G$ satisfying ($\mathbb{M}$) and*

$$d_{\mathrm{dR}} \alpha = \mu^* \underline{\omega}_1.$$

Proof. The de Rham (cochain) complex of $[X/G]$ in weight ≥ 2 is the total (cochain) complex of the bicomplex

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega^3(X)^G & \longrightarrow & (\Omega^2(X) \otimes \mathfrak{g}^*)^G & \longrightarrow & (\Omega^2(X) \otimes S^2\mathfrak{g}^*)^G & \longrightarrow & (\mathcal{O}(X) \otimes S^3\mathfrak{g}^*)^G \\
{\scriptstyle d_{\mathrm{dR}}}\uparrow & & \uparrow & & \uparrow & & \\
\Omega^2(X)^G & \stackrel{\partial}{\longrightarrow} & (\Omega^1(X) \otimes \mathfrak{g}^*)^G & \longrightarrow & (\mathcal{O}(X) \otimes S^2\mathfrak{g}^*)^G & &
\end{array}
$$

The space of 2-forms $\alpha \in \Omega^2(X)^G$ mapped on $\mu^*\omega \in \Omega^3(X)^G \oplus (\Omega^1(X) \otimes \mathfrak{g}^*)^G$ by $d_{\mathrm{dR}} \oplus \partial$ has the expected description. $\qquad\square$

Now thanks to [21], the non-degeneracy condition (that is, (B3) in [31]) defines a union of connected components in the space of (closed) 2-forms. Therefore, we have the following result (which is already implicit in [7, 23]).

Theorem 6.4. *The space of lagrangian structures on $[\mu]$ is discrete; it is the set of 2-forms $\alpha \in \Omega^2(X)^G$ such that ($\mathbb{M}$).*

In particular, the space of lagrangian structures on $[\mu]$ (or, equivalently, the set of quasi-hamiltonian structures on X with group valued moment map μ) is a subset of $\Omega^2(X)$.

Corollary 6.5. *Two lagrangian structures on $[\mu]$ coincide if and only if the associated 2-forms on X are the same.*

Remark 6.6. Here is how we understand geometrically the 2-form on X we get from an α satisfying ($\mathbb{M}$). The pull-back of $\underline{\omega}_0$ along the quotient $G \to [G/G]$ is zero. As $[\mu]^*\underline{\omega}_0 \sim 0$ via α, we get a self-homotopy of 0 in the space 2-forms of degree 1 on the fiber product

$$
[X/G] \underset{[G/G]}{\times} G \simeq X.
$$

Such a self-homotopy is a 2-form of degree 0 on X, which is nothing but α.

6.2. Identifying two lagrangian structures: proof of theorem 6.1

Consider the composition

$$
\mathrm{Spec}(A_V) = X_V \twoheadrightarrow [X_V/\mathrm{GL}_V] \hookrightarrow \mathrm{Perf}_A.
$$

It is given by an $A - A_V$-bimodule M which induces a chain

$$
\overline{\mathrm{HH}}A \longrightarrow \overline{\mathrm{HH}}(\mathrm{Mod}^{\mathrm{perf}}_{A_V}) \xleftarrow{\ \sim\ } \overline{\mathrm{HH}}(\mathrm{End}_{A_V}(M)) \xrightarrow{\ \mathrm{tr}\ } \overline{\mathrm{HH}}A_V \simeq \Omega^*A_V
$$

$$
\underset{a \mapsto \hat{a}}{\underbrace{\hspace{4cm}}}
$$

given by

$$
a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto \mathrm{tr}(\hat{a}_0)d\mathrm{tr}(\hat{a}_1)\dots d\mathrm{tr}(\hat{a}_n)
$$

(that is, $\underline{\mathrm{tr}}$ again, *cf* (6.1)). Thus, the 2-forms match on X_V, and therefore the associated lagrangian structures as well thanks to the previous subsection.

Example 6.7.

(i) Let us get back to section 5.4.1, where A is a localization of the path algebra of the A_2 quiver and Φ denotes the associated multiplicative moment map. Thanks to the computations in section 5.4.1, theorem 6.1 applies and the 1-Calabi–Yau structure on Φ exhibited in [4] induces the same lagrangian structure on

$$\left[\hat{\Phi}\right] : \left[\mathrm{Rep}(A, \vec{n})/GL_{\vec{n}}\right] \longrightarrow \left[GL_{\vec{n}}/GL_{\vec{n}}\right],$$

for some dimension vector $\vec{n} = (n_1, n_2)$, as the one induced by Van den Bergh's quasi-Hamiltonian $GL_{\vec{n}}$-structure in [31].

(ii) Similarly, using section 5.4.2, we finally prove the conjecture raised in [4, §5.3], which is the identical statement for an arbitrary quiver Q.

References

[1] A. Alekseev, Y. Kosmann-Schwarzbach and E. Meinrenken, 'Quasi-Poisson manifolds', *Canad. J. Math.* **54**(1) (2002), 3–29.

[2] A. Alekseev, A. Malkin and E. Meinrenken, 'Lie group valued moment maps', *J. Differential Geom.* **48**(3) (1998), 445–495.

[3] T. Bozec, D. Calaque and S. Scherotzke, 'Relative critical loci and quiver moduli', Preprint, arXiv:2006.01069v2.

[4] T. Bozec, D. Calaque and S. Scherotzke, 'Calabi–Yau structures for multiplicative preprojective algebras', *J. Noncommut. Geom.* **17**(3) (2023), 783–810.

[5] C. Brav and T. Dyckerhoff, 'Relative Calabi–Yau structures', *Compositio Math.* **155** (2019), 372–412.

[6] C. Brav and T. Dyckerhoff, 'Relative Calabi–Yau structures II: shifted Lagrangians in the moduli of objects', *Sel. Math. New Ser.* **27**(63) (2021).

[7] D. Calaque, 'Lagrangian structures on mapping stacks and semi-classical TFT's, *Contemp. Math.* **643** (2015).

[8] D. Calaque, T. Pantev, B. Toën, M. Vaquié and G. Vezzosi, 'Shifted Poisson structures and deformation quantization', *J. Topol.* **10**(2) (2017), 483–584.

[9] W. Crawley-Boevey, P. Etingof and V. Ginzburg, 'Noncommutative geometry and quiver algebras', *Adv. Math.* **209**(1) (2007), 274–336.

[10] W. Crawley-Boevey and P. Shaw, 'Multiplicative preprojective algebras, middle convolution and the Deligne–Simpson problem', *Adv. Math.* **201** (2006), 180–208.

[11] J. Cuntz and D. Quillen, 'Algebra extensions and nonsingularity', *J. Amer. Math. Soc.* **8** (1995), 251–289.

[12] M. Fairon, 'Double quasi-Poisson brackets: fusion and new examples', *Algebr. Represent. Theory* **24** (2021), 911–958.

[13] V. Ginzburg, 'Lectures on noncommutative geometry', Preprint, arXiv:math/0506603v1.

[14] V. Ginzburg and T. Schedler, 'Free products, cyclic homology, and the Gauss–Manin connection', *Adv. Math.* **231**(3–4) (2012), 2352–2389.

[15] V. Ginzburg and T. Schedler, 'A new construction of cyclic homology', *Proc. Lond. Math. Soc.* **112**(3) (2016), 549–587.

[16] M. Hoyois, 'The homotopy fixed points of the circle action on Hochschild homology', Preprint, arXiv:1506.07123v2.

[17] M. Karoubi, 'Homologie cyclique et K-théorie', *Astérisque* **149** (1987), 156 p.

[18] B. Keller, 'On differential graded categories', in *Proc. ICM* vol. 2 (Eur. Math. Soc., Zürich, 2006), 151–190.

[19] J.-L. Loday, *Cyclic Homology (Grundlehren der mathematischen Wissenschaften)* vol. 301 (Springer Verlag, Berlin, 1992).

[20] J. Lurie, *Higher Topos Theory* (Princeton University Press, 2009).

[21] T. Pantev, B. Toën, M. Vaquié and G. Vezzosi, 'Shifted symplectic structures', *Publications mathématiques de l'IHÉS* **117**(1) (2013), 271–328.

[22] J. P. Pridham, 'Shifted bisymplectic and double Poisson structures on non-commutative derived prestacks', Preprint, arXiv:2008.11698v1.

[23] P. Safronov, 'Quasi-Hamiltonian reduction via classical Chern–Simons theory', *Adv. Math.* **287** (2016), 733–773.

[24] B. Toën, 'Lectures on dg-categories', in *Topics in Algebraic and Topological K-Theory* (Lecture Notes in Mathematics) (Springer, Berlin, Heidelberg, 2008).

[25] B. Toën, 'Derived algebraic geometry', *EMS Surv. Math. Sci.* **1**(2) (2014), 153–240.

[26] B. Toën, 'Structures symplectiques et de Poisson sur les champs en catégories', Preprint, arXiv:1804.10444v1.

[27] B. Toën and M. Vaquié, 'Moduli of objects in dg-categories', *Annales de l'ENS* **40** (2007), 387–444.

[28] B. Toën and G. Vezzosi, 'Homotopical algebraic geometry II: geometric stacks and applications', *Mem. Amer. Math. Soc.* **904** (2008), 224 p.

[29] B. Toën and G. Vezzosi, 'Algèbres simpliciales S^1-équivariantes, théorie de de Rham et théorèmes HKR multiplicatifs', *Compos. Math.* **147**(6) (2011), 1979–2000.

[30] M. Van den Bergh, 'Double Poisson algebras', *Trans. Amer. Math. Soc.* **360**(11) (2008), 5711–5769.

[31] M. Van den Bergh, 'Non-commutative quasi-Hamiltonian spaces', in *Poisson Geometry in Mathematics and Physics* (Contemp. Math.) vol. 450 (American Mathematical Society, 2008), 273–300.

[32] D. Yamakawa, 'Geometry of multiplicative preprojective algebra', *International Mathematics Research Papers* **2008** (2008), rpn008.

[33] W.-K. Yeung, 'A higher Hodge extension of the Feigin–Tsygan theorem', Preprint, arXiv:2202.09499v1.