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THE DOUBLE B-DUAL OF AN INNER PRODUCT MODULE OVER A C^*-ALGEBRA B

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1. Introduction. The principal result of this paper states that if X is a pre-Hilbert B-module over an arbitrary C^*-algebra B, then the B-valued inner product on X can be lifted to a B-valued inner product on X'' (the B-dual of the B-dual X' of X). Appropriate identifications allow us to regard X as a submodule of X'' and the latter in turn as a submodule of X'. In this sense, the inner product on X'' is an extension of that on X. As an example (and application) of this result, we consider the special case in which X is a right ideal of B and give a topological description of X'' when in addition B is commutative.

We begin by recalling some definitions and facts from [**3**]. Let B be a C^*-algebra and X a right B-module. We denote the right action of $b \in B$ on $x \in X$ by $x \cdot b$; it is assumed that X has a vector space structure compatible with that of B in the sense that $\lambda(x \cdot b) = (\lambda x) \cdot b = x \cdot (\lambda b)$ for all $x \in X, b \in B, \lambda \in \mathbf{C}$ (the complex field). A *B-valued inner product* on X is a conjugate-bilinear map $\langle \cdot , \cdot \rangle : X \times X \to B$ satisfying:

(i) $\langle x, x \rangle \geqq 0$;
(ii) $\langle x, x \rangle = 0$ only if $x = 0$;
(iii) $\langle x, y \rangle = \langle y, x \rangle^*$;
(iv) $\langle x \cdot b, y \rangle = \langle x, y \rangle b$ for $x, y \in X, b \in B$.

A *pre-Hilbert B-module* is a right B-module equipped with a B-valued inner product. Any pre-Hilbert B-module X has a natural norm $||\cdot||_X$ defined by $||x||_X = ||\langle x, x \rangle||^{1/2} (x \in X)$ with respect to which X is a normed B-module (i.e. the map $(x, b) \to x \cdot b$ of $X \times B$ into X is jointly continuous) [**3**, 2.3].

If Y is a normed B-module, we let Y' (the *B-dual* of Y) denote the set of all bounded module maps (i.e. B-linear maps) of Y into B. Y' becomes a vector space if we define scalar multiplication on Y' by $(\lambda F)(y) = \bar{\lambda} F(y)$ ($\lambda \in \mathbf{C}$, $F \in Y', y \in Y$) and add maps elementwise. We make Y' into a right B-module by setting $(F \cdot b)(y) = b^* F(y)$ ($F \in Y', b \in B, y \in Y$). For $F \in Y'$, $||F||_{Y'}$ will denote the norm of F as a bounded linear map from Y to B.

If X is a pre-Hilbert B-module, then by [**3**, 2.8] X' is precisely the set of (complex) linear maps $\tau : X \to B$ such that for some real $K \geqq 0$, $\tau(x)^* \tau(x) \leqq K \langle x, x \rangle$ for all $x \in X$. Moreover, $||\tau||_{X'}$ for such a map τ is the infimum of the square roots of all such constants K. Each $x \in X$ gives rise to a map $\hat{x} \in X'$ defined by $\hat{x}(y) = \langle y, x \rangle (y \in X)$. The map $x \to \hat{x}$ is an isometric module map of X into X'. We may thus regard X as a submodule of X' by identifying X with $\hat{X}$.

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In case $B = \mathbf{C}$ (so X is a pre-Hilbert space), X' is of course just the Hilbert space completion of X, but in general the relationship between X and X' is less simple. For example, let X denote the set of all sequences $\mathbf{b} = (b_1, b_2, \ldots)$ of elements of B such that $\sum_{n=1}^{\infty} b_n{}^*b_n$ converges in norm. X is a right B-module under coordinatewise right multiplication by elements of B. For $\mathbf{a}, \mathbf{b} \in X$, it is easy to see that $\sum_{n=1}^{\infty} b_n{}^*a_n$ converges in norm; if we set

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{n=1}^{\infty} b_n{}^*a_n,$$

then $\langle \cdot , \cdot \rangle$ is a B-valued inner product on X. One checks that X is a Hilbert B-module, i.e. is complete with respect to $||\cdot||_X$. It also turns out, however, that X' may be identified with the right B-module of all sequences $\mathbf{b}$ such that the sequence $\{\sum_{k=1}^{n} b_k{}^*b_k\}_{n=1}^{\infty}$ is norm-bounded, normed by setting $||\mathbf{b}||_{X'} = \sup\{||\sum_{k=1}^{n} b_k{}^*b_k||^{1/2} : n = 1, 2, \ldots\}$. In general, then, X' may be quite a bit larger than X even when X is complete.

It was shown in [**3**] that if B is a W^*-algebra, then the B-valued inner product on any pre-Hilbert B-module X lifts to a B-valued inner product on X' satisfying $\langle \hat{x}, \tau \rangle = \tau(x)$ for all $x \in X$, $\tau \in X'$. The proposition below shows that this extension cannot be carried out for even the simplest sort of pre-Hilbert B-module unless B is at least an AW^*-algebra. (Notice that any right ideal J of B is a pre-Hilbert B-module with B-valued inner product given by $\langle x, y \rangle = y^*x (x, y \in J)$.)

1.1 Proposition. *Let B be a C^*-algebra with the property that for every right ideal J of B, there is a B-valued inner product $\langle \cdot , \cdot \rangle$ on J' satisfying $\langle \hat{x}, \tau \rangle = \tau(x)$ for all $x \in J$, $\tau \in J'$. Then B is an AW^*-algebra.*

Proof. Let J be a right ideal of B. It will suffice to show that $L(J) = Bp$ for some projection $p \in B$, where $L(J)$ is the left annihilator of J. For $a \in B$, define $\tilde{a} \in J'$ by $\tilde{a}(x) = a^*x (x \in J)$ and let $\tau_i \in J'$ denote the inclusion map of J into B. Notice $\tau_i \cdot a = \tilde{a}$ for $a \in B$ and that $\tilde{x} = \hat{x}$ for $x \in J$. Set $q = \langle \tau_i, \tau_i \rangle$. Then $q = q^*$ and, for $x \in J$, $qx = \tilde{q}(x) = \langle \tau_i \cdot x, \tau_i \rangle = \langle \tilde{x}, \tau_i \rangle = \langle \hat{x}, \tau_i \rangle = \tau_i(x) = x$. (In the case $J = B$, this reasoning shows that B necessarily has 1.) We thus have $\tilde{q} = \tau_i$, so $q^2 = \langle \tau_i \cdot q, \tau_i \rangle = \langle \tilde{q}, \tau_i \rangle = \langle \tau_i, \tau_i \rangle = q$, i.e., q is a projection. Set $p = 1 - q$, so p is a projection in $L(J)$. For $a \in L(J)^*$, we have $\tilde{a} = 0$, so $qa = \langle \tau_i \cdot a, \tau_i \rangle = \langle \tilde{a}, \tau_i \rangle = 0$, which shows that $ap = a$ for all $a \in L(J)$. Hence $L(J) = Bp$, as required.

For an arbitrary C^*-algebra B and an arbitrary pre-Hilbert B-module X, then, we cannot expect to be able to extend the inner product on X to an inner product on X'. The next best thing would be to lift the inner product to the right B-module X'' (the B-dual of the normed B-module X'). We will show in the next section that this can be done in general.

Before embarking on the construction of the lifted inner product, we establish some more notation. For $x \in X$, define $\dot{x} \in X''$ by $\dot{x}(\tau) = \tau(x)^* (\tau \in X')$.

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The map $x \to \dot{x}$ is an isometric module map of X into X''. For $\Gamma \in X''$, define $\tilde{\Gamma} \in X'$ by $\tilde{\Gamma}(x) = \Gamma(\hat{x})$ $(x \in X)$. If we identify X with $\hat{X}$, $\tilde{\Gamma}$ is just the restriction of Γ to X. Notice that $(\dot{x})^{\sim} = \hat{x}$ for all $x \in X$. It is clear that the map $\Gamma \to \tilde{\Gamma}$ is a module map of X'' into X' and that $||\tilde{\Gamma}||_{X'} \leqq ||\Gamma||_{X''}$ for $\Gamma \in X''$. We will show among other things that this map is in fact an isometry. After making all permissible identifications, we can then array the modules X, X', and X'' as $X \subseteq X'' \subseteq X'$.

2. The inner product on X''. Throughout this section, B will denote an arbitrary C^*-algebra (not assumed to possess a unit), and X an arbitrary pre-Hilbert B-module. Our candidate for a B-valued inner product on X'' is a fairly obvious one; we define $\langle \cdot , \cdot \rangle : X'' \times X'' \to B$ by $\langle \Gamma, \Phi \rangle = \Phi(\tilde{\Gamma})$ $(\Gamma, \Phi \in X'')$. This map is conjugate-bilinear and satisfies $\langle \Gamma \cdot b, \Phi \rangle = \langle \Gamma, \Phi \rangle b$ for Γ, $\Phi \in X''$, $b \in B$. For $x, y \in X$, we have $\langle \dot{x}, \dot{y} \rangle = \dot{y}((\dot{x})^{\sim}) = \dot{y}(\hat{x}) = \hat{x}(y)^* = \langle x, y \rangle$, so $\langle \cdot , \cdot \rangle$ is an extension of the original inner product on X in the appropriate sense. Our problem is to show that $\langle \Gamma, \Gamma \rangle \geqq 0$ for all $\Gamma \in X''$ and that $\langle \Gamma, \Gamma \rangle = 0$ only if $\Gamma = 0$. (It will follow easily from this that $\langle \Gamma, \Phi \rangle^* = \langle \Phi, \Gamma \rangle$ for all $\Gamma, \Phi \in X''$.)

Consider the right B-module $B \times X$. This module possesses a natural B-valued inner product $\{\cdot , \cdot\}$ defined by $\{(a, x), (b, y)\} = b^*a + \langle x, y \rangle$ $(a, b \in B, x, y \in X)$. It will be useful for us to define some other inner products on $B \times X$ in the following manner. Take $\tau \in X'$ $(\tau \neq 0)$ and $t > ||\tau||_{X'}$. For $(a, x), (b, y) \in B \times X$, set

$$[(a, x), (b, y)]_{\tau, t} = t^2 b^* a + b^* \tau(x) + \tau(y)^* a + \langle x, y \rangle.$$

The map $[\cdot , \cdot]_{\tau, t} : (B \times X) \times (B \times X) \to B$ is clearly conjugate-bilinear and satisfies (iii) and (iv) of the definition of a B-valued inner product. To check that (i) and (ii) hold also, take $(a, x) \in B \times X$ and observe that

$$\begin{aligned} [(a, x), (a, x)]_{\tau, t} &= t^2 a^* a + a^* \tau(x) + \tau(x)^* a + \langle x, x \rangle \\ &\geqq t^2 a^* a + a^* \tau(x) + \tau(x)^* a + ||\tau||_{X'}^{-2} \tau(x)^* \tau(x) \\ &\geqq t^2 a^* a + a^* \tau(x) + \tau(x)^* a + t^{-2} \tau(x)^* \tau(x) \\ &= (ta + t^{-1}\tau(x))^* (ta + t^{-1}\tau(x)) \geqq 0, \end{aligned}$$

where we have used the fact that $\tau(x)^* \tau(x) \leqq ||\tau||_{X'}^2 \langle x, x \rangle$ to obtain the first inequality. Hence, (i) holds. If $[(a, x), (a, x)]_{\tau, t} = 0$, then we have equality at each step above and in particular $(||\tau||_{X'}^{-2} - t^{-2}) \tau(x)^* \tau(x) = 0$, so $\tau(x) = 0$, so $t^2 a^* a + \langle x, x \rangle = 0$, so $a = 0$ and $x = 0$, which establishes (ii). The map $[\cdot , \cdot]_{\tau, t}$ is thus a B-valued inner product on X.

Let $||\cdot||_{\tau, t}$ be the norm on $B \times X$ gotten from this inner product. Observe that $||x||_X = ||(0, x)||_{\tau, t}$ for all $x \in X$. For $x, y \in X$, $b \in B$, we have

$$\begin{aligned} ||(\tau \cdot b + \hat{y})(x)|| &= ||b^* \tau(x) + \langle x, y \rangle|| \\ &= ||[(0, x), (b, y)]_{\tau, t}|| \\ &\leqq ||(0, x)||_{\tau, t} ||(b, y)||_{\tau, t} \\ &= ||x||_X ||(b, y)||_{\tau, t}, \end{aligned}$$

the inequality holding by virtue of [**3**, 2.3]. We conclude that $||\tau \cdot b + \hat{y}||_{X'} \leqq ||(b, y)||_{\tau, t}$ for all $y \in X$, $b \in B$.

This construction is the main ingredient in the proof of our next proposition.

2.1. PROPOSITION. *Let Y be a submodule of X' containing $\hat{X}$. For any $F \in Y'$, we have* $||F||_{Y'} = ||F|_{\hat{X}}||$.

Proof. We may assume without loss of generality that $||F||_{Y'} = 1$. Define $\tau \in X'$ by $\tau(x) = F(\hat{x})$ $(x \in X)$. We have $||\tau||_{X'} \leqq 1$ and must establish the reverse inequality.

Take $\psi \in Y$ with $||\psi||_{X'} < 1$ and set $c = \Gamma(\psi)$. For brevity, let $[\cdot\,,\cdot]$ denote the B-valued inner product $[\cdot\,,\cdot]_{\psi,1}$ on $B \times X$ defined above and let $||\cdot||$ be the corresponding norm on $B \times X$. For $a \in B$, $x \in X$, we have

$$||ca + \tau(x)|| = ||F(\psi \cdot a + \hat{x})|| \leqq ||\psi \cdot a + \hat{x}||_{X'} \leqq ||(a, x)||,$$

so the map $(a, x) \to ca + \tau(x)$ of $B \times X$ into B is a bounded module map of norm $\leqq 1$ with respect to the inner product $[\cdot\,,\cdot]$. By [**3**, 2.8], we have $(ca + \tau(x))^*(ca + \tau(x)) \leqq [(a, x), (a, x)]$ for all $a \in B$, $x \in X$. That is,

$$a^*c^*ca + a^*c^*\tau(x) + \tau(x)^*ca + \tau(x)^*\tau(x) \leqq a^*a + a^*\psi(x) + \psi(x)^*a + \langle x, x\rangle.$$

Setting $a = -2\psi(x)$ and collecting terms, we obtain

$$4\psi(x)^*c^*c\psi(x) + \tau(x)^*\tau(x) \leqq \langle x, x\rangle + 2(\psi(x)^*c^*\tau(x) + \tau(x)^*c\psi(x))$$

for $x \in X$. But

$$\psi(x)^*c^*\tau(x) + \tau(x)^*c\psi(x) \leqq \psi(x)^*c^*c\psi(x) + \tau(x)^*\tau(x),$$

so

$$\begin{aligned} 2\psi(x)^*c^*c\psi(x) &\leqq \langle x, x\rangle + \tau(x)^*\tau(x) \\ &\leqq (1 + ||\tau||_{X'}^2)\langle x, x\rangle \end{aligned}$$

for all $x \in X$. Hence $||\psi \cdot c^*||_{X'} \leqq 2^{-1/2}(1 + ||\tau||_{X'}^2)^{1/2}$ and consequently

$$||F(\psi \cdot c^*)|| = ||cc^*|| = ||c||^2 \leqq 2^{-1/2}(1 + ||\tau||_{X'}^2)^{1/2}.$$

This holds for any $\psi \in Y$ with $||\psi||_{X'} < 1$; since $||F||_{Y'} = 1$, we must therefore have $1 \leqq 2^{-1/2}(1 + ||\tau||_{X'}^2)^{1/2}$, which forces $||\tau||_{X'} \geqq 1$. This completes the proof.

Notice that if we take $Y = X'$, then 2.1 shows in particular that the map $\Gamma \to \tilde{\Gamma}$ is an isometry of X'' into X'.

We will need the following lemma to show that $\langle \Gamma, \Gamma\rangle \geqq 0$ for $\Gamma \in X''$.

2.2. LEMMA. *Suppose $c \in B$ is such that for every $a \in B$ with $a \geqq 0$, there is a state f of B such that $f(aca) = ||aca||$. Then $c \geqq 0$.*

Proof. Write $c = h + ik$ with $h = h^*$ and $k = k^*$. We first claim that $||aca|| = ||aha||$ for every $a \in B$ with $a \geqq 0$. To see this, let f be a state of B

such that $f(aca) = ||aca||$. Since aha and aka are self-adjoint, we must have $f(aka) = 0$, so $||aca|| = f(aha) \leqq ||aha||$. Now let g be a state of B such that $|g(aha)| = ||aha||$. We have

$$\begin{aligned} |g(aca)|^2 &= |g(aha) + ig(aka)|^2 \\ &= g(aha)^2 + g(aka)^2 \\ &\leqq ||aca||^2 \leqq ||aha||^2 = g(aha)^2. \end{aligned}$$

This forces $g(aka) = 0$ and $||aca|| = ||aha||$.

Now write $k = k^+ - k^-$, where $k^+, k^- \geqq 0$ and $k^+k^- = k^-k^+ = 0$. We claim that $k^+ = 0$. For this, let g be a state of B such that $|g(k^+hk^+)| = ||k^+hk^+||$. As in the reasoning above, we must have $g(k^+kk^+) = g((k^+)^3) = 0$, so k^+ belongs to the left kernel of g, so $g(k^+hk^+) = 0$, so $k^+hk^+ = 0$. But we know that $||k^+hk^+|| = ||k^+ck^+||$, so $0 = k^+ck^+ = k^+hk^+ + ik^+kk^+ = i(k^+)^3$, so $k^+ = 0$.

The hypothesis of the lemma is satisfied by $c^* = h + ik^-$ as well as by c, so in like manner we have $k^- = 0$, so $k = 0$, i.e., $c = c^*$.

The lemma now follows by application of the functional calculus. If $\mathrm{sp}(c) \cap (-\infty, 0)$ were non-empty, we could find a non-zero, non-negative continuous function F on $\mathrm{sp}(c)$ such that $F([0, +\infty) \cap \mathrm{sp}(c)) = \{0\}$. Setting $a = F(c)$, we would then have $a \geqq 0$ and $aca \leqq 0$, a contradiction.

2.3 PROPOSITION. $\langle \Gamma, \Gamma \rangle \geqq 0$ *and* $||\langle \Gamma, \Gamma \rangle|| = ||\Gamma||_{X''}^2$ *for all* $\Gamma \in X''$.

Proof. Take $\Gamma \in X''$ $(\Gamma \neq 0)$ and set $c = \Gamma(\tilde{\Gamma})$, $D = ||\Gamma||_{X''}$ $(= ||\tilde{\Gamma}||_{X'}$ by 2.1). We first show that $D^2 \in \mathrm{sp}(c)$. For $t > D$, consider the B-valued inner product $[\cdot, \cdot]_{\tilde{\Gamma}, t}$ on $B \times X$. The map $(a, x) \rightarrow \Gamma(\tilde{\Gamma} \cdot a + \hat{x}) = ca + \tilde{\Gamma}(x)$ is a bounded module map of $B \times X$ into B of norm $\leqq D$ with respect to this inner product (since $||\tilde{\Gamma} \cdot a + \hat{x}||_{X'} \leqq ||(a, x)||_{\tilde{\Gamma}, t}$ for $(a, x) \in B \times X$). Hence

$$(ca + \tilde{\Gamma}(x))^*(ca + \tilde{\Gamma}(x)) \leqq D^2[(a, x), (a, x)]_{\tilde{\Gamma}, t}$$

for $a \in B$, $x \in X$ by [**3**, 2.8]. This holds for every $t < D$, so we have

$$(ca + \tilde{\Gamma}(x))^*(ca + \tilde{\Gamma}(x)) \leqq D^2(D^2a^*a + \tilde{\Gamma}(x)^*a + a^*\tilde{\Gamma}(x) + \langle x, x \rangle)$$

for $a \in B$, $x \in X$. Setting $a = -D^{-2}\tilde{\Gamma}(x)$, we obtain

$$\tilde{\Gamma}(x)^*(D^{-2}c - 1)^*(D^{-2}c - 1)\tilde{\Gamma}(x) \leqq D^2(-D^{-2}\tilde{\Gamma}(x)^*\Gamma(x) + \langle x, x \rangle)$$

and hence

$$\tilde{\Gamma}(x)^*((D^{-2}c - 1)^*(D^{-2}c - 1) + 1)\tilde{\Gamma}(x) \leqq D^2\langle x, x \rangle, \qquad x \in X.$$

Now if $D^2 \notin \mathrm{sp}(c)$, we can find a $\delta > 0$ such that

$$\tilde{\Gamma}(x)^*((D^{-2}c - 1)^*(D^{-2}c - 1))\tilde{\Gamma}(x) \geqq \delta\tilde{\Gamma}(x)^*\Gamma(x), \qquad x \in X.$$

We would then have

$$\tilde{\Gamma}(x)^*\tilde{\Gamma}(x) \leqq D^2(1 + \delta)^{-1}\langle x, x \rangle, \qquad x \in X,$$

forcing

$$D^2 = ||\tilde{\Gamma}||_{X'}{}^2 \leqq D^2(1 + \delta)^{-1},$$

a contradiction. Hence $D^2 \in \operatorname{sp}(c)$.

But $||c|| = ||\Gamma(\tilde{\Gamma})|| \leqq ||\Gamma||_{X''}||\tilde{\Gamma}||_{X'} = D^2$, so $||c|| = D^2$, i.e., $||\langle \Gamma, \Gamma \rangle|| = ||\Gamma||_{X''}{}^2$ and $||\langle \Gamma, \Gamma \rangle|| \in \operatorname{sp}(\langle \Gamma, \Gamma \rangle)$. For $a \in B$, $a \geqq 0$, we have

$$||aca|| = ||a^*\Gamma(\tilde{\Gamma} \cdot a)|| = ||\langle \Gamma \cdot a, \Gamma \cdot a \rangle|| \in \operatorname{sp}(\langle \Gamma \cdot a, \Gamma \cdot a \rangle) = \operatorname{sp}(aca),$$

so $c \geqq 0$ by 2.2. This completes the proof.

We have shown in 2.3 that the map $\langle \cdot\,, \cdot \rangle$ satisfies (i) and (ii) of the definition of a B-valued inner product. Property (iii) now follows routinely from the fact that $\langle \Gamma + \Phi, \Gamma + \Phi \rangle \geqq 0$ and $\langle \Gamma + i\Phi, \Gamma + i\Phi \rangle \geqq 0$ for all $\Gamma, \Phi \in X''$. X'' is a Hilbert B-module with respect to the inner product we have introduced since (by 2.3) the norm on X'' gotten from this inner product coincides with the operator norm $||\cdot||_{X''}$. For $F \in (X'')'$, define $\tau_F \in X'$ by $\tau_F(x) = F(\dot{x})$ $(x \in X)$ and for $\tau \in X'$ define $F_\tau \in (X'')'$ by $F_\tau(\Gamma) = \Gamma(\tau)^*$ $(\Gamma \in X'')$. The maps $F \rightarrow \tau_F$ and $\tau \rightarrow F_\tau$ are module maps; using 2.1, one checks that they are isometries and inverses of each other. We thus have $(X'')' = X'$ and $(X'')'' = X''$. We summarize the results of this section in the theorem below.

2.4. THEOREM. *The map* $\langle \cdot\,, \cdot \rangle : X'' \times X'' \rightarrow B$ *defined by* $\langle \Gamma, \Phi \rangle = \Phi(\tilde{\Gamma})$ $(\Gamma, \Phi \in X'')$ *is a B-valued inner product on X''. The norm obtained from this inner product coincides with the operator norm on X''. The map* $\Gamma \rightarrow \tilde{\Gamma}$ *is an isometry of X'' into X'.*

3. Right ideals of B**.** In this section we investigate the double B-dual of a right ideal J of a C^*-algebra B, where J is considered as a pre-Hilbert B-module with B-valued inner product defined by $\langle x, y \rangle = y^*x (x, y \in J)$.

Let $\tau_i \in J'$ denote the inclusion map of J into B and set

$$\tilde{J} = \{\Gamma(\tau_i)^* : \Gamma \in J''\}.$$

$\tilde{J}$ is clearly a linear subspace of B and in fact a right ideal, since for $b \in B$, $\Gamma \in J''$, we have $\Gamma(\tau_i)^*b = (b^*\Gamma(\tau_i))^* = ((\Gamma \cdot b)(\tau_i))^*$. For $x \in J$, we have $\dot{x}(\tau_i)^* = x$, so $J \subseteq \tilde{J}$.

3.1. PROPOSITION. *$\tilde{J}$ is closed. J'' and $\tilde{J}$ are isomorphic as Hilbert B-modules via the map* $\Gamma \rightarrow \Gamma(\tau_i)^*$.

Proof. We have observed that the map in question is a module map; it is contractive since $||\tau_i||_{J'} = 1$. Observe that for $x, y \in J$, we have $(\tau_i \cdot x)(y) = x^*\tau_i(y) = x^*y = \hat{x}(y)$, so $\tau_i \cdot x = \hat{x}$ $(x \in J)$. Hence

$$\begin{aligned} ||\Gamma||_{J''} = ||\tilde{\Gamma}||_{J'} &= \sup\{||\tilde{\Gamma}(x)|| : x \in J, ||x|| \leqq 1\} \\ &= \sup\{||\Gamma(\tau_i \cdot x)|| : x \in J, ||x|| \leqq 1\} \\ &= \sup\{||\Gamma(\tau_i)x|| : x \in J, ||x|| \leqq 1\} \\ &\leqq ||\Gamma(\tau_i)|| = ||\Gamma(\tau_i)^*|| \end{aligned}$$

for all $\Gamma \in J''$. The map $\Gamma \to \Gamma(\tau_i)^*$ is thus an isometry of J'' onto $\tilde{J}$. By [**3**, 2.8], applied to this map and its inverse, we have $\langle \Gamma, \Gamma \rangle = \langle \Gamma(\tau_i)^*, \Gamma(\tau_i)^* \rangle = \Gamma(\tau_i)\Gamma(\tau_i)^*$, $\Gamma \in J''$. It now follows (just as for an isometry between Hilbert spaces) that $\langle \Gamma, \Phi \rangle = \Phi(\tau_i)\Gamma(\tau_i)^*$, for $\Gamma, \Phi \in J''$. $\tilde{J}$ is closed because J'' is complete with respect to $||\cdot||_{J''}$.

If X is a pre-Hilbert B-module, every map in X' lifts to a unique map in $(X'')'$, so in particular every map in J' extends uniquely to a map in $\tilde{J}'$. Suppose K is a right ideal of B containing J such that each $\tau \in J'$ extends uniquely to a map $\bar{\tau} \in K'$. Given $a \in K$, define $\Gamma \in J''$ by $\Gamma(\tau) = \bar{\tau}(a)^*$. By uniqueness, $\bar{\tau}_i$ is the inclusion map of K into B, so for this Γ we have $\Gamma(\tau_i)^* = \bar{\tau}_i(a) = a$, and we conclude that $K \subseteq \tilde{J}$. We may thus describe $\tilde{J}$ as the unique largest right ideal K of B such that every bounded module map of J into B extends uniquely to a bounded module map of K into B.

Since $(X'')'' = X''$ for any pre-Hilbert B-module X, we have $(\tilde{J})^\sim = \tilde{J}$ for any right ideal J of B. If J and K are two right ideals of B with $J \subseteq K$, then for any $\Gamma \in J''$, the map $\psi \to \Gamma(\psi|_J)$ of K' into B belongs to K'', whence it follows that $\tilde{J} \subseteq \tilde{K}$. $\tilde{J}$ might thus be thought of as a "closure" of J, albeit in a rather restricted sense. In fact, if B is a W^*-algebra, $\tilde{J}$ is precisely the ultraweak closure of J. (This follows from the fact that every map in J' can be realized as right multiplication by a unique element b^*, where b belongs to the ultraweak closure of J–which fact in turn is easily proved by making use of a bounded left approximate unit for J.)

We will shortly examine the topological relationship between J and $\tilde{J}$ in the commutative case. When B is not assumed to be commutative, we can at least obtain some rudimentary information about the relationship between the open projections in B^{**} corresponding to J and $\tilde{J}$. (See [1] for a discussion of open and closed projections in the second dual of a C^*-algebra. For a projection $p \in B^{**}$, $\bar{p}$ denotes the smallest closed projection in B^{**} majorizing p.)

Let p be an open projection in B^{**} and set $J = pB^{**} \cap B$ (so J is the unique norm-closed right ideal of B whose w^{**}-closure in B^{**} is pB^{**}). Let $\tilde{p}$ be the (open) projection in B^{**} which generates the w^{**}-closure of $\tilde{J}$.

3.2. Proposition. *$p \leqq \tilde{p} \leqq \bar{p}$ and $||pa|| = ||\tilde{p}a||$ for all $a \in B$.*

Proof. We have $p \leqq \tilde{p}$ because $J \subseteq \tilde{J}$. The projection $1 - \bar{p}$ is open, so there is a net $\{a_\alpha\}$ consisting of positive elements of B majorized by $1 - \bar{p}$ and converging w^{**} to $1 - \bar{p}$. We have $\bar{p}a_\alpha = a_\alpha\bar{p} = 0$ for all α and $\bar{p}x = x$, $x \in J$. As before, we let $\tau_i \in J'$ denote the inclusion map of J into B. We have $(\tau_i \cdot a_\alpha)(x) = a_\alpha\tau_i(x) = a_\alpha x = a_\alpha\bar{p}x = 0$ for all $x \in J$, α, so $\tau_i \cdot a_\alpha = 0$ for all α. Thus $\Gamma(\tau_i)a_\alpha = 0$ and hence $\Gamma(\tau_i)(1 - \bar{p}) = 0$, for $\Gamma \in J''$. This shows that $\tilde{J} \subseteq \bar{p}B^{**}$, so $\tilde{p} \leqq \bar{p}$.

For the second part of the proposition, let $\{b_\alpha\}$ be a net of positive elements of B majorized by $\tilde{p}$ (and hence belonging to $\tilde{J}$) and converging w^{**} to $\tilde{p}$. For each b_α let $\Gamma_\alpha \in J''$ be such that $\Gamma_\alpha(\tau_i) = b_\alpha$. Notice that $||\Gamma_\alpha||_{J''} = ||b_\alpha||$.

Take $a \in B$. For each α we have

$$\begin{aligned} ||b_\alpha a|| &= ||\Gamma_\alpha(\tau_i \cdot a)|| \leqq ||b_\alpha||\,||\tau_i \cdot a||_{J'} \\ &\leqq ||\tau_i \cdot a||_{J'} = \sup\{||a^*x|| : x \in J, ||x|| \leqq 1\} \\ &= \sup\{||a^*px|| : x \in J, ||x|| \leqq 1\} \\ &\leqq ||a^*p|| = ||pa||. \end{aligned}$$

Hence $||\tilde{p}a|| \leqq ||pa||$. Since $p \leqq \tilde{p}$, the reverse inequality holds also and we have $||\tilde{p}a|| = ||pa||$.

Let Ω be a locally compact space. We denote by $C_0(\Omega)$ the algebra of all complex-valued continuous functions on Ω which vanish at infinity. For a subset S of Ω, $C(S)$ will denote the algebra of all bounded complex-valued continuous functions on S, and we will write $k(S)$ for the ideal of $C_0(\Omega)$ consisting of those functions in $C_0(\Omega)$ which vanish identically on S. Given an open set W in Ω, it is not hard to see that there is a unique largest open set $\tilde{W}$ containing W such that every function in $C(W)$ extends uniquely to a function in $C(\tilde{W})$. (Indeed, our proof of 3.4 will show this in a roundabout fashion.) Clearly $\tilde{W} \subseteq \bar{W}$. If Ω is a metric space, it can be shown without too much difficulty that $\tilde{W} = W$ for every open subset W of Ω. In general, though, it can happen that W is a proper subset of $\tilde{W}$. For instance, if Ω is a Stonian space and W is a dense open subset of Ω, it follows from [**2**, 4.2] that $\tilde{W} = \Omega$.

Let $B = C_0(\Omega)$. Let E be a closed subset of Ω and let $J = k(E)$, $U = \Omega\backslash E$. We will show in 3.4 that $\tilde{J} = k(\Omega\backslash\tilde{U})$ but before we can do that we must describe J' in this setting. Let Y be the space of all bounded complex-valued functions on Ω which vanish identically on E and whose restrictions to U are continuous on U, normed with the uniform norm. Notice that Y is naturally a B-module (under pointwise multiplication) and that the product of any function in Y with any function in J belongs to J. For $f \in Y$, define $\tau_f \in J'$ by $\tau_f(x) = \bar{f}x$ $(x \in J)$.

3.3. PROPOSITION. *Y and J' are isometrically isomorphic as normed B-modules via the map $f \to \tau_f$.*

Proof. It is clear that the map in question is an isometric module map of Y into J'. Given $\tau \in J'$, we must show that $\tau = \tau_f$ for some $f \in Y$. For simplicity, we may assume that $||\tau||_{J'} \leqq 1$. We then have $\overline{\tau(x)}\tau(x) \leqq \bar{x}x$ $(x \in J)$, whence $|\tau(x)(t)| \leqq |x(t)|$ for $x \in J$, $t \in \Omega$. For each $t \in U$, select $x_t \in J$ such that $x_t(t) = 1$. Define $g: U \to \mathbf{C}$ by $g(t) = \tau(x_t)(t)$. Notice that $|g(t)| \leqq 1$ for all $t \in U$. For any $y \in J, t \in U$, we have $(yx_t - y)(t) = 0$, so $\tau(yx_t - y)(t) = 0$. Hence

$$\tau(y)(t) = \tau(yx_t)(t) = \tau(x_t)(t)y(t) = g(t)y(t) \quad , \qquad y \in J, t \in U.$$

Now since τ maps J into B, gy must be continuous on U for every $y \in J$. It follows that g is continuous on U. Let f be that function in Y whose restriction to U is $\bar{g}$. Then $\tau = \tau_f$ and the proof is complete.

We will thus identify J'' with the space of all bounded B-module maps of Y into B. Let $p \in Y$ be the characteristic function of U (so τ_p is the inclusion map of J into B). We then have $\tilde{J} = \{\Gamma(p) : \Gamma \in Y'\}$.

3.4. PROPOSITION. $\tilde{J} = k(\Omega \backslash \tilde{U})$.

Proof. $\tilde{J}$ is a closed ideal of B, so $\tilde{J} = k(F)$, where F is the intersection of the zero sets of all the functions $\Gamma(p)(\Gamma \in Y')$. Notice that $F \subseteq E$, since $J \subseteq \tilde{J}$. Set $V = \Omega \backslash F$. It follows from 3.2 (or by a simple direct argument) that $V \subseteq \bar{U}$. We must show that $V = \tilde{U}$, i.e. that every function in $C(U)$ extends uniquely to a function in $C(V)$, and that if W is an open set containing U such that every function in $C(U)$ extends uniquely to a function in $C(W)$, then $W \subseteq V$.

We begin by observing that $\Gamma(f)(t) = \Gamma(p)(t)f(t)$ for all, $\Gamma \in Y'$, $f \in Y$, $t \in U$. Indeed, take $x \in J$ such that $x(t) = 1$ and notice that $fx \in B$. Since also $fx = pfx$, we have $\Gamma(f)(t) = \Gamma(f)(t)x(t) = \Gamma(fx)(t) = \Gamma(p(fx))(t) = \Gamma(p)(t)(fx)(t) = \Gamma(p)(t)f(t)$.

Now consider $\varphi \in C(U)$; we show that φ can be extended to a bounded continuous function on V. (The extension will necessarily be unique, since $V \subseteq \bar{U}$.) Let f be the function in Y whose restriction to U is φ. Let W be a relatively compact open set whose closure is contained in V. We can find a $\Gamma \in Y'$ such that $\Gamma(p)(W) = \{1\}$ and $||\Gamma(p)||$ $(= ||\Gamma||_{Y'}$ by 3.1) $= 1$. Define $\tilde{\varphi}$ on W by $\tilde{\varphi}(t) = \Gamma(f)(t)$ $(t \in W)$, so $\tilde{\varphi}$ is continuous on W and $|\tilde{\varphi}| \leqq ||f|| = ||\varphi||$. For any $t \in W$ and any net $\{t_\alpha\}$ in U with $t_\alpha \to t$, we have $\Gamma(f)(t_\alpha) = \Gamma(p)(t_\alpha)f(t_\alpha) = \Gamma(p)(t_\alpha)\varphi(t_\alpha) = \varphi(t_\alpha)$ for sufficiently large α (the first equality holding by virtue of our observation above), so $\varphi(t_\alpha) \to \tilde{\varphi}(t)$. We may therefore define $\tilde{\varphi}(t)$ $(t \in W)$ unambiguously as

$$\lim_{\alpha} \varphi(t_\alpha),$$

where $\{t_\alpha\}$ is any net in U with $t_\alpha \to t$. Since every $t \in V$ is contained in some such neighborhood W, we may define $\tilde{\varphi}$ on all of V in this way. The function $\tilde{\varphi}$ is then the desired extension of φ.

Now let W be an open set containing U such that every function in $C(U)$ extends uniquely to a function in $C(W)$. We must show that $W \subseteq V$. Take $t \in W$ and let $b \in B$ be such that $b(t) = 1$ and $b(\Omega \backslash W) = \{0\}$. For $f \in Y$, let $\tilde{f}$ denote the unique bounded continuous extension of $f|U$ to W. Define $\Gamma: Y \to B$ by setting $\Gamma(f)(s) = \tilde{f}(s)b(s)$ for $s \in W$ and $\Gamma(f)(s) = 0$ for $s \notin W$. It is immediate that $\Gamma \in Y'$. We have $\Gamma(p)(t) = \tilde{p}(t)b(t) = 1$, so $t \notin F$, i.e., $t \in V$. This shows that $W \subseteq V$, which completes the proof.

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