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KNOT PROJECTIONS AND COXETER GROUPS

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Abstract

Every knot admits a special projection with the property that under the projection discs in the canonical Seifert surface project disjointly. Under an isotopy, such a projection can be turned into a connected sum of what we call inseparable projections. The main result is that if there is no band in an inseparable projection with half-twisting number $+1$ or -1, then the projection is not a projection of the trivial knot. To prove this a non-cyclic Coxeter group is constructed as a quotient of the knot group. The construction is possibly of interest in itself. The techniques developed are applied to give a criterion to decide when an inseparable projection with 3 discs comes from the trivial knot.

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1. Introduction

A projection of a knot induces a canonical Seifert surface (see, for example, [6]). It is not hard to show that every knot admits a projection with the property that in this projection the discs project disjointly. Such projections were called special by Murasugi [3]. For the rest of the paper it is assumed, unless otherwise stated, that a knot projection is special.

Given a projection, we associate a weighted graph on the 2-sphere via the canonical surface as follows. A Seifert surface consists of discs joined by rectangular half-twists (± 1) at crossings of the projection. Firstly, a sequence of half-twists and discs, with the property that each disc is connected only to the disc before it and after it, is combined into what we call a *band*. Since the projection is special, distinct bands do not intersect. (The word band is used interchangeably below for both the corresponding parts of the Seifert surface and of the projection.) Secondly, the following two operations are used to simplify the projection and the corresponding Seifert surface.

(1) If there are half-twists of opposite sign in a band, eliminate them in pairs, by an isotopy, to yield a band with half-twists of a single sign. A band with 0 half-twists is eliminated by amalgamating the two discs in question.

(2) If there is a band which intersects at most one disc, delete the band.

Note that operation (2) can reduce a projection to a single disc; the knot involved is the trivial knot.

DEFINITION. A knot projection is *reduced* if operations (1) and (2) cannot be applied any further to the projection.

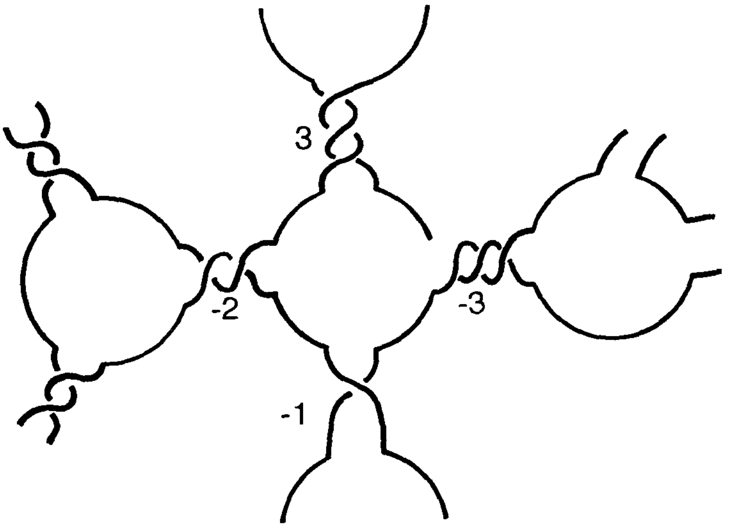
The canonical Seifert surface of a reduced projection consists of disjoint discs connected by non-intersecting bands. We associate a connected weighted graph to a reduced projection: vertices are the discs, edges are the bands connecting discs. Each edge of the graph is assigned a weight which is the half-twisting number of the corresponding band (see Figure 1). Note that the graph is a deformation retract of the Seifert surface. Such a weighted graph is called the *graph of the projection*. The graph is considered a subset of S^2, the one point compactification of the plane.

REMARK 1. Given an arbitrary weighted graph, it is possible to construct a corresponding surface. However this does not have to be the Seifert surface of a knot. Some necessary conditions for this are:

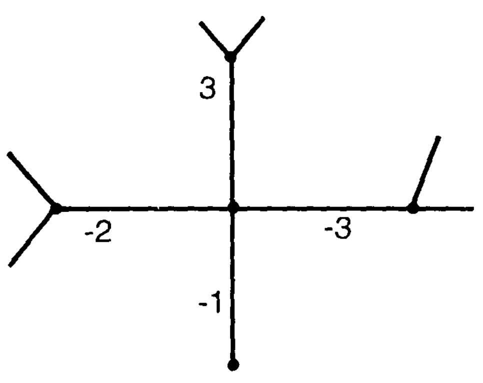
(i) Each vertex can be given a positive or negative label which corresponds to the orientation of the associated discs on the surface. Then bands joining vertices with the same sign have an even number of half-twists; bands joining vertices with opposite sign have an odd number of half-twists.

(ii) Two vertices with the same sign cannot be joined by more than one band. This is required so that the boundary has just one component.

(iii) The number of edges minus the number of vertices in the graph must be odd. This can easily be seen by computing the genus of the surface.



Projection



Weighted graph

FIGURE 1

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DEFINITION. A reduced knot projection (or the graph of the projection) is *inseparable* if the graph of the projection cannot be separated into two components, either by (1) deleting a vertex from the graph, or (2) deleting the interiors of one or two edges from the graph.

REMARK 2. Any knot projection can be isotoped to a connected sum of inseparable projections.

THEOREM 1. *If, in the graph of an inseparable projection, no weight is equal to* $+1$ *or* -1, *then the knot is not trivial.*

The proof of this theorem involves the construction of a non-cyclic Coxeter group as the quotient of the knot group. This construction is possibly of interest in itself.

The detailed proof of Theorem 1 is given in Section 2 together with a refinement of the theorem. In Section 3, as an application of the theorem and the ideas involved, we establish a criterion for when an inseparable projection with 3 discs represents the trivial knot.

The authors thank H. Trotter for helpful communications, and in particular, for originally bringing the Reidemeister quotient to their attention.

2. Reidemeister and Coxeter quotients

Let K be a reduced projection for a knot with associated weighted graph Γ. Let G denote the fundamental group of the complement of the knot represented by K in S^3. In this section we use K to construct the Reidemeister quotient and a Coxeter quotient of G.

We begin with the Wirtinger presentation for G computed from K (see [6]). To find this representation, K is considered as a union of disjoint arcs in the usual way; associated with each arc is a meridianal generator of G represented as an arrow drawn under the arc. Generators are chosen so that arrows point into the canonical Seifert surface (see Figure 2).

Since all crossings of K are contained within bands, the Wirtinger relations are computed by looking at these. In Figure 2 we compute, and then simplify, the part of the Wirtinger presentation arising from a band e of t half-twists. A similar calculation takes place if half-twists of the opposite sign are used. We deduce that for any t,

$$
\begin{aligned}
R(e):\quad x' &= \left(yx^{-1}\right)^{t/2} x \left(yx^{-1}\right)^{-t/2}\\
y' &= \left(yx^{-1}\right)^{t/2} y \left(yx^{-1}\right)^{-t/2} \qquad (t \text{ even}),
\end{aligned}
$$

$$
\begin{aligned}
R(e):\quad x' &= \left(yx^{-1}\right)^{(t-1)/2} y^{-1} \left(yx^{-1}\right)^{-(t-1)/2}\\
y' &= \left(yx^{-1}\right)^{(t+1)/2} x^{-1} \left(yx^{-1}\right)^{-(t+1)/2} \qquad (t \text{ odd}).
\end{aligned}
$$

LEMMA 1. *Suppose x, y are Wirtinger generators associated with the beginning and x', y' with the end of a band e of t half-twists. On imposing the relations $x^2 = y^2 = 1$, the relations $R(e)$ are equivalent to $x' = (yx)^t x$, $y' = (yx)^t y$.*

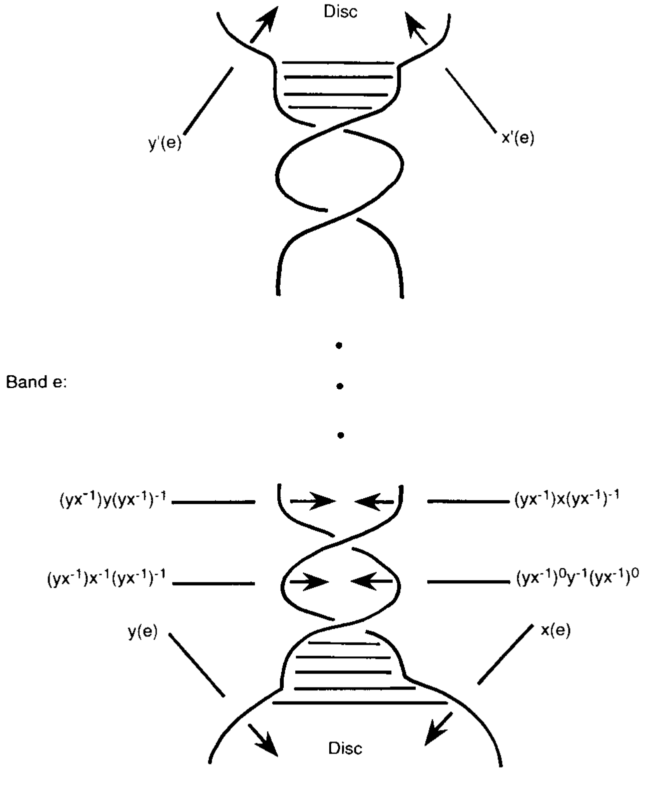

FIGURE 2

PROOF. Suppose $x^2 = y^2 = 1$ and t odd. We see that

$$\begin{aligned} x' &= (yx)^{(t-1)/2} y (xy)^{(t-1)/2} yy \\ &= (yx)^{(t-1)/2} (yx)^{(t-1)/2} yxx \\ &= (yx)^t x \end{aligned}$$

and

$$\begin{aligned} y' &= (yx)^{(t+1)/2} x (xy)^{(t+1)/2} xx \\ &= (yx)^{(t+1)/2} (yx)^{(t+1)/2} x \\ &= (yx)^t yxx = (yx)^t y. \end{aligned}$$

The proof for t even is similar.

Given an edge e of Γ let $u(e)$ represent any one of $x(e)$, $x'(e)$, $y(e)$ or $y'(e)$. For two edges e and f of Γ, we say that $u(e) \sim u(f)$ if the arrows corresponding to $u(e)$ and $u(f)$ pass under the same arc contained in the boundary of a disc in the canonical Seifert surface (see Figure 3). In G, $u(e) = u(f)$ if $u(e) \sim u(f)$. Now a presentation for G is $\langle u(e);\ R(e),\ u(e) = u(f) \text{ if } u(e) \sim u(f)\rangle$, where in writing this presentation it is understood that e, f range over the edges of Γ.

The *Reidemeister quotient* of G (see [5]) is the quotient group

$$R = \langle u(e);\ (u(e))^2 = 1,\ R(e),\ u(e) = u(f) \text{ if } u(e) \sim u(f)\rangle$$

where each meridian has order 2. By Lemma 1 the relations $R(e)$ are equivalent to $x' = (yx)^t x$, $y' = (yx)^t y$ where $x = x(e)$, $x' = x'(e)$, $y = y(e)$ and $y' = y'(e)$ with t the weight of e.

A *Coxeter quotient* of G using Γ is obtained from the above presentation of R by further placing $x'(e) = x(e)$, $y'(e) = y(e)$ for each edge e of Γ.

The Reidemeister quotient does not depend on the projection chosen. However, the Coxeter quotient does (see Remark 3(iii)). Let Γ' be the dual graph of Γ where Γ is thought of as a graph on S^2. We consider Γ' as a weighted graph by assigning to each edge the weight of the dual edge in Γ. In the construction of the Coxeter quotient the relations imply there is a unique generator which can be associated with each component of $S^2 - \Gamma$ (which corresponds to a vertex of Γ') (see Figure 3).

The following proposition summarizes our construction.

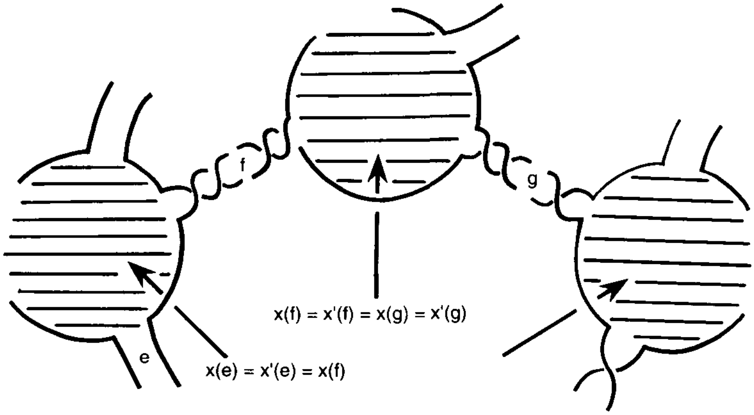

FIGURE 3

PROPOSITION 1. The Coxeter quotient of G corresponding to a graph of a projection Γ has a presentation with generators $\{v_i\}$ where v_i ranges over the vertices of Γ'. The relators are $v_i^2 = (v_i v_j)^{t_{ij}} = 1$ where t_{ij} is the weight of an edge between the vertices v_i and v_j.

We use the notation $C(\Gamma')$ for the (Coxeter) presentation as given in Proposition 1 obtained from a weighted graph Γ'.

REMARK 3.

(i) Two vertices v_i, v_j may be the endpoints of more than one edge, so there may be more than one relation of the type $(v_i v_j)^{t_{ij}} = 1$ involving the same two vertices. Also, there may be relations of the type $(v_i v_i)^{t_{ii}} = 1$. There are no relations of the type $(v_i v_j)^{t_{ij}} = 1$ if there are no edges between v_i and v_j. In this case formally set $t_{ij} = \infty$.

(ii) If K represents the trivial knot then G is infinite cyclic and $R \cong Z_2 \cong C(\Gamma')$.

(iii) In Section 3 it will be seen that the knots with projection $K(3, 5, 1; -2; 1, 3)$ and projection $K(3, 5, 5)$ (see Section 3 for notation) are equivalent. However, the Coxeter quotient arising from the first projection is $\mathbb{Z}_2$, whereas the Coxeter quotient from the second projection is isomorphic to an infinite extended triangle group.

We call a weighted graph a *Coxeter graph* if there are no loops, and if two

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vertices are connected by at most one edge.

LEMMA 2. *If* Γ *is an inseparable graph then* Γ' *is a Coxeter graph.*

PROOF. Suppose there is a loop e' in Γ'. Then $\Gamma - \{\text{interior } e\}$ has two components (one inside and the other outside e') contradicting the assumption that Γ is inseparable. Suppose now that there are two edges e' and f' in Γ' joining the same two vertices. Then $\Gamma - \{\text{interior } e \text{ and } f)\}$ has two components (one inside and the other outside the circle $e'f'$), again a contradiction.

Suppose now that in addition Γ' is a Coxeter graph with $|t_{ij}| \geq 2$ for $i \neq j$. Then, by setting $t_{ij} = t_{ji}$, and $t_{ii} = 1$, the matrix (t_{ij}) becomes a *Coxeter matrix* and $C(\Gamma')$ is a *Coxeter group* (see Brown [1]).

PROOF OF THEOREM 1. Let Γ be the graph of an inseparable projection K with no weight $+1$ or -1. By Lemma 2, Γ' is a Coxeter graph so that $C(\Gamma')$ is a Coxeter group. It is shown in Brown [1, Theorem A, p. 54] that each pair of generators $\{v_i, v_j\}$ generates a dihedral group of order exactly $2|t_{ij}|$. Thus $C(\Gamma') \ncong \mathbb{Z}_2$ and so K is not the projection of the trivial knot.

The remainder of this section is devoted to refining Theorem 1 in the situation where the graph contains weights ± 1.

Let Γ be a weighted graph (not necessarily a Coxeter graph). We introduce three operations on weighted graphs Γ which correspond to changes which can be made in the presentation of $C(\Gamma)$. Figure 4 gives an illustration of these operations.

(1) Delete the interior points of all edges which form loops at vertices.

(2) Suppose there is an edge with weight ± 1 and two distinct endpoints v_1, v_2. Then delete the interior points of all edges whose endpoints are v_1 and v_2, and identify v_1 and v_2.

(3) Suppose there are edges e (with weight t) and f (with weight s) between the same two vertices. Then delete the interior of e and replace f with weight GCD (t, s).

DEFINITION. Given a graph Γ, a *reduction* $\widetilde{\Gamma}$ of Γ is defined to be a graph obtained from Γ using a sequence of operations (1), (2) and (3) which cannot be further simplified using these operations.

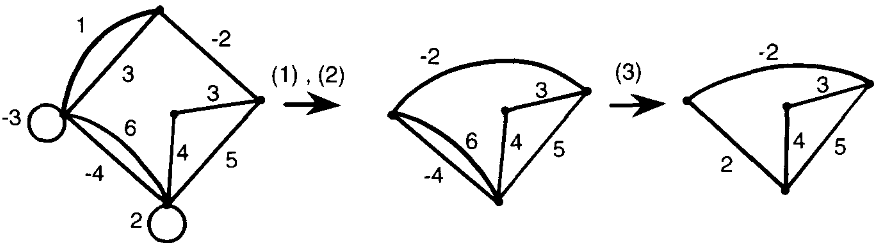

FIGURE 4

The following two assertions are easy consequences of the construction of a Coxeter quotient, the definitions, and Theorem 1.

PROPOSITION 2. For any graph Γ, $C(\Gamma) \cong C(\widetilde{\Gamma})$. Furthermore, $C(\Gamma) \cong \mathbb{Z}_2$ if and only if $\widetilde{\Gamma}$ is a single point.

PROPOSITION 3. Let Γ be the graph of a projection of the trivial knot. Then any reduction of Γ' is a single point.

3. Knots with 2 or 3 discs

There are two obvious isotopies which change an inseparable projection of a knot into another inseparable projection of the same knot.

T_1. Any band with weight ± 1 between two discs can be permuted with an adjacent band between the same two discs. We call this *permutation* (of ± 1 bands) (see Trotter [8, p. 54]).

T_2. Adjacent $+1$ and -1 bands between two discs can be cancelled. We call this *cancellation* (of ± 1 bands).

Let Γ be the weighted graph of an inseparable knot projection K. Then Γ has at least two vertices. In case K has two vertices, $K = K(t_1, \dots, t_{n+1})$ is the generalized pretzel knot in Figure 5. By Remark 1, n is even and each t_i is odd.

THEOREM 2. *Let $K = K(t_1, \dots, t_{n+1})$ be a generalized pretzel knot. If K is equivalent to the trivial knot then K can be reduced to a single band between two discs by using permutation and cancellation of ± 1 bands.*

This result is implicit in Trotter [7] in the case of pretzel knots and Parris [4] for generalized pretzel knots. The result has also been discovered by others.

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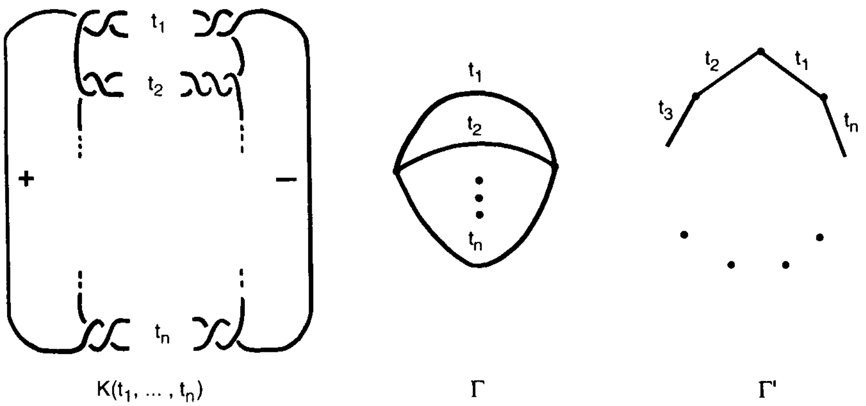

FIGURE 5

Below we establish similar criteria for the triviality of knots with 3 discs. A proof of the above theorem can be given as follows. Assume K is trivial, that $n \neq 0$, and that all possible reductions using T_1 and T_2 have taken place. Thus ± 1 bands take the same sign. Suppose now that K is not a single band. Then Γ' is a polygon with at least 3 sides. Since Γ' is reducible to a point, Proposition 2 implies that at most two t_i, say t_1, t_2 are not ± 1, and that there exists a t_i, say t_3, with $t_3 = \pm 1$. Since K is trivial, the determinant (see Trotter [8, p. 56]) of a Seifert matrix associated with the canonical Seifert surface of K is 0: that is,

$$\prod_{i=1}^{n}(t_i + 1) - \prod_{i=1}^{n}(t_i - 1) = 0.$$

Since $t_3 = +1$ or -1, we have $(t_1 + 1)(t_2 + 1) = 0$ or $(t_1 - 1)(t_2 - 1) = 0$, in the respective cases. In this way we obtain a contradiction, as K is assumed reduced under T_1 and T_2. Thus K is a single band.

We now turn to knots having an inseparable projection with 3 discs. The conditions given in Remark 1 imply that up to isotopy of S^2 we may take $K = K(t_1, \dots, t_{n+1}; u; s_1, \dots, s_{m+1})$ as given in Figure 6. Here m is odd ≥ 1, n is even ≥ 2, each t_i, s_i is odd, and u is even.

Strictly speaking, when $u = 0$ the projection has two discs and then Theorem 2 can be applied. In addition to T_1 and T_2 above, there is the following special isotopy which we call an *unwinding* isotopy:

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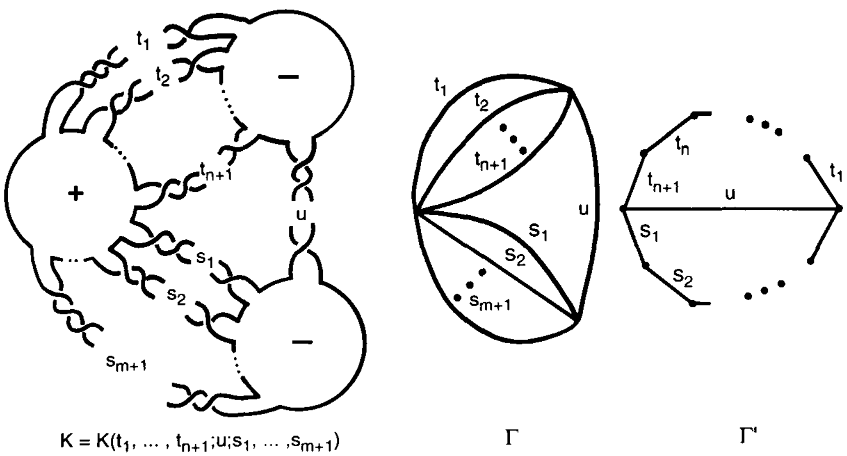

FIGURE 6

T_3. If $u = 2$ (respectively -2), $m = 1$ and $t_{n+1} = s_1 = -1$ (respectively $+1$), then there is the isotopy indicated in Figure 7 which changes K to the pretzel knot $K(t_1, \ldots, t_n, s_2 - 2)$ (respectively $K(t_1, \ldots, t_n, s_2 + 2)$). (The two cases are mirror images of each other).

THEOREM 3. *Let $K = K(t_1, \ldots, t_{n+1} ; u ; s_1, \ldots, s_{m+1})$ be an inseparable projection of a knot with 3 discs. If K is equivalent to the trivial knot then K can be reduced to a single band using isotopies* T_1, T_2 *and* T_3.

The proof of this result occupies the remainder of the paper. The strategy is similar to that used in the proof of Theorem 2. Again, we utilize a Coxeter quotient (via Proposition 3), but immediately afterwards we pass to a close analysis of the corresponding Reidemeister quotient.

In what follows now, it is supposed that K is equivalent to the trivial knot and $u \neq 0$.

LEMMA 3. *Using* T_1 *and* T_2 *it can be assumed one of the following cases holds:*

(1) *all $|t_i|$, $|s_i|$ are 1 except possibly t_{n+1}, s_1 and s_2.*

(2) *all $|t_i|$, $|s_i|$ are 1 except possibly t_n, t_{n+1} and s_1.*

PROOF. By Proposition 3, a reduction of Γ' is a single point. The conclusion follows by analyzing Γ' in Figure 6.

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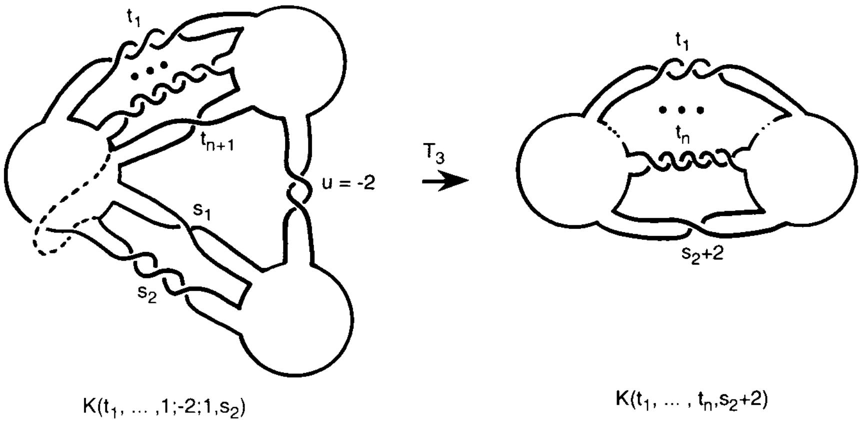

FIGURE 7

To proceed further, we assume that T_1 and T_2 have been applied so as to simplify K as much as possible. Consequently *all* t_i *with* $|t_i| = 1$ *have the same sign, and all* s_i *with* $|s_i| = 1$ *have the same sign.*

Take case (1) of Lemma 3. Case (2) can be treated in an exactly analogous way, and we do not include an argument. Place $t = t_{n+1}$. We compute a presentation for the Reidemeister quotient in Figure 8 using the indicated generators.

Since all s_i (respectively t_i) except possibly s_1, s_2 (respectively t_{n+1}) have the same sign these have been replaced with a band of $m-1$ (respectively n) half-twists. In addition, as we have now altered our notation, $m-1$ (respectively n) now stands for a possibly negative integer with $|m-1| \geq 0$ (respectively $|n| \geq 2$).

All generators have order 2. Further relators are

$$\begin{array}{rclcrcl}
(yb)^n y & = & (da)^{m-1}a & \quad & w & = & (xy)^t x \\
(da)^{m-1}d & = & (wc)^u c & & b & = & (xy)^t y \\
(zx)^{s_2}x & = & (wc)^u w & & c & = & (yb)^n b \\
(zx)^{s_2} & = & (az)^{s_1} & & d & = & (az)^{s_1}a
\end{array}$$

We successively eliminate generators. At each step, the generators eliminated are conjugates of remaining generators (of order 2) so no other relations arise.

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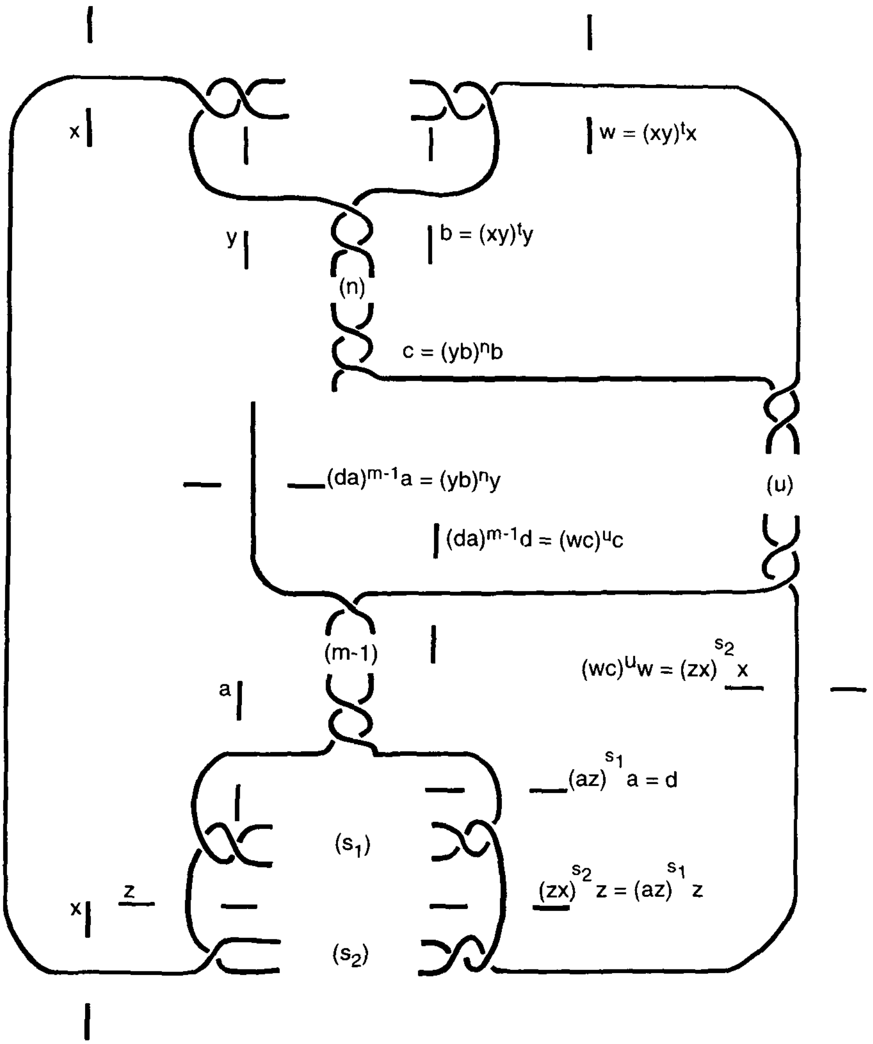

FIGURE 8

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(1) Eliminate b using $yb = (yx)^t$:

$$\begin{aligned} (yx)^{nt}y &= (da)^{m-1}a & w &= (xy)^t x \\ (da)^{m-1}d &= (wc)^u c & c &= (yx)^{nt-t}y \\ (za)^{s_2}x &= (wc)^u w & d &= (az)^{s_1}a \\ (zx)^{s_2} &= (az)^{s_1} \end{aligned}$$

(2) Eliminate w using $wx = (xy)^t$ (so $wc = (xy)^t x(yx)^{nt-t}y = (xy)^{nt+1}$) :

$$\begin{aligned} (yx)^{nt}y &= (da)^{m-1}a & c &= (yx)^{nt-t}y = (xy)^{-nt+t}y \\ (da)^{m-1}d &= (xy)^{u(nt+1)}c & d &= (az)^{s_1}a \\ (zx)^{s_2} &= (xy)^{u(nt+1)+t} & (zx)^{s_2} &= (az)^{s_1} \end{aligned}$$

(3) Eliminate $c = (yx)^{nt-t}y$ and $d = (az)^{s_1}a$:

$$\begin{aligned} (yx)^{nt}y &= (az)^{(m-1)s_1}a & (az)^{ms_1}a &= (xy)^{u(nt+1)-nt+t}y \\ (zx)^{s_2} &= (xy)^{u(nt+1)+t} & (zx)^{s_2} &= (az)^{s_1} \end{aligned}$$

One relation is redundant, and we obtain on simplifying

$$\left.\begin{aligned} a &= (yx)^{(m-1)u(nt+1)+(m-1)t+nt}y \\ (zx)^{s_2} &= (az)^{s_1} = (xy)^{u(nt+1)+t} \end{aligned}\right\} \tag{$*$}$$

We return to these relations below, but first things can be simplified.

LEMMA 4. *All* $|s_i| = |t_i| = 1$ *except possibly* s_1 *and* t_{n+1}.

PROOF. As mentioned above, we have taken case (1) of Lemma 2, since case (2) is analogous. Take a further quotient of the Reidemeister quotient:

$$\begin{aligned} Q = \langle a, x, y, z;\quad & a^2 = x^2 = y^2 = z^2 = (zx)^{s_2} = (az)^{s_1} = (xy)^{u(nt+1)+t} = 1, \\ & a = (yx)^{nt}y\rangle. \end{aligned}$$

Since $a = (yx)^{nt}y$, $ax = (yx)^{nt+1}$, so that $(ax)^u = (yx)^{-t}$ and $y = (ax)^{nu}a$. Thus, on eliminating y, we have

$$Q = \langle a, x, z\ ;\ a^2 = x^2 = z^2 = (zx)^{s_2} = (az)^{s_1} = (ax)^{u(nt+1)+t} = 1\rangle.$$

Now Q is an extended triangle (see Magnus [2]) and is not cyclic unless $|s_1| = 1$, $|s_2| = 1$ or $|u(n+1)+t| = 1$.

However, our conditions preclude $|u(nt+1)+t| = 1$. To see this, suppose $u(nt+1)+t = \pm 1$ so $\pm 1 - u = t(un+1)$. If $t = 1$ then $u(n+1) = 0, -2$.

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Since $u \neq 0$ and n is even, $u(n+1) = -2$; so $u = 2$, $n = -2$ which is not allowed, since the projection would not then be reduced. Similarly when $t = -1$. Finally, for $|t| \geq 3$,

$$1 + |u| \geq |\pm 1 - u| = |t|\,|un+1| \geq 3(|u|\,|n| - 1) \geq 3(2|u| - 1),$$

so $4 \geq 5|u|$ which contradicts $u \neq 0$.

It follows without loss in generality that we can take $|s_1| = 1$, and, up to taking a mirror image, assume $s_1 = 1$. Place $s = s_2$. The equations $(*)$ reduce to $az = (xy)^{u(nt+1)+t} = (zx)^s$ and $a = (yx)^{(n-1)u(nt+1)+(m-1)t+nt}y$. Now eliminate $a = (zx)^s z$ so as to obtain $(zx)^s = (xy)^{u(nt+1)+t}$ and $(zx)^s z = (yx)^{(m-1)u(nt+1)+(m-1)t+nt}y$. We then express $zx = (yx)^{mu(nt+1)+mt+nt+1}$ and eliminate z to obtain a dihedral group generated by x and y (with $x^2 = y^2 = 1$) of order 2ρ where

$$\begin{aligned}\rho &= ustmn + st(m+n) + u(sm+tn) + s + t + u\\ &= u(sm+1)(tn+1) + t(sm+1) + s(tn+1).\end{aligned}$$

Since K is trivial $|\rho| = 1$. The following lemma completes the proof of Theorem 3.

LEMMA 5. *Suppose that K is reduced using isotopies* T_1 *and* T_2. *Then* $K = K(t, 1, 1; -2; 1, -3)$ *or* $K = K(t, -1, -1; 2; -1, 3)$. *Thus K can be reduced to a single band of t half-twists using isotopies* T_1, T_2 *and* T_3.

PROOF. We have

$$\begin{aligned}|u| &= \frac{|t(sm+1) + s(tn+1) \pm 1|}{|sm+1|\,|tn+1|}\\ &\leq \frac{1}{\left|n + 1/t\right|} + \frac{1}{\left|m + 1/s\right|} + \frac{1}{|tn+1|\,|sm+1|}.\end{aligned}$$

Let $Q_1 = 1/|n + 1/t|$, $Q_2 = 1/|m + 1/s|$ and $Q_3 = 1/|tn+1|\,|sm+1|$.

Notice that when $|t| = 1$, $|nt| = nt$ and when $|s| = 1$, $|sm| = sm$ since K is assumed reduced under T_1 and T_2. Also, as $|m| \geq 1$ and $|n| \geq 2$ we see that $|sm+1| \geq 2$ and $|tn+1| \geq 3$, so $Q_3 \leq 1/6$. Now $|sm+1| \geq |s|\,|m| - 1 \geq |s| - 1$ so $3/2 \geq 1 + 1/|sm+1| \geq Q_2$ and $Q_2 \leq 3/2$. In addition it is easily seen that $Q_1 \leq 3/5$, the bound being achieved for $n = \pm 2$, $t = \mp 3$. It follows that $|u| \leq Q_1 + Q_2 + Q_3 = 34/15$ so $|u| = 2$.

From $2 \le Q_1 + Q_2 + Q_3 \le Q_1 + Q_2 + 1/6$ we have $11/6 \le Q_1 + Q_2$. Firstly, $11/6 \le Q_1 + 3/2$ gives $Q_1 \ge 1/3$ which forces $|n| = 2$. Secondly, $11/6 \le 3/5 + Q_2$ gives $Q_2 \ge 37/30$. Taking cases, we see this is possible only if $m = \pm 1$, $s = \mp 3$ or ∓ 5. However the cases $m = \pm 1$, $s = \mp 5$ can be eliminated since they would give $|ms + 1| = 4$ with $Q_3 \le 1/12$, and this would require $Q_2 = 5/4 \ge 2 - 3/5 - 1/12 = 79/60$.

Finally, testing $|\rho| = 1$ with the possible combinations of values of m, n, s and u we conclude that either (i) $m = 1$, $n = 2$, $s = -3$ and $u = -2$, or (ii) $m = -1$, $n = -2$, $s = 3$ and $u = 2$, thus proving the lemma.

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