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THE CLOSED RANGE PROPERTY FOR BANACH SPACE OPERATORS

THOMAS L. MILLER

Dept. of Mathematics and Statistics, Mississippi State University, Drawer MA, Mississippi State, MS 39762
e-mail: miller@math.msstate.edu

and VLADIMIR MÜLLER

Mathematical Institute, Czech Academy of Sciences, Zitna 25, 115 67 Prague 1, Czech Republic
e-mail: muller@math.cas.cz

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Abstract. Let T be a bounded operator on a complex Banach space X. Let V be an open subset of the complex plane. We give a condition sufficient for the mapping $f(z) \mapsto (T-z)f(z)$ to have closed range in the Fréchet space $H(V, X)$ of analytic X-valued functions on V. Moreover, we show that there is a largest open set U for which the map $f(z) \mapsto (T-z)f(z)$ has closed range in $H(V, X)$ for all $V \subseteq U$. Finally, we establish analogous results in the setting of the weak–$*$ topology on $H(V, X^*)$.

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Introduction. Let X be a complex Banach space and denote by $B(X)$ the algebra of bounded linear operators on X. For $T \in B(X)$, let $\sigma(T)$ denote the spectrum of T, and denote by $\mathrm{Lat}\,(T)$ the collection of closed T-invariant subspaces of X. If $M \in \mathrm{Lat}\,(T)$, we write the restriction of T to M as $T|_M$.

A basic notion in local spectral theory is that of decomposability. Given an open subset U of the complex plane $\mathbb{C}$, $T \in B(X)$ is said to be decomposable on U provided that for any open cover $\{V_1, \ldots, V_n\}$ of $\mathbb{C}$ with $\mathbb{C} \setminus U \subset V_1$, there exists $\{X_1, \ldots, X_n\} \subset \mathrm{Lat}\,(T)$ such that $X = X_1 + \cdots + X_n$ and $\sigma(T|_{X_k}) \subset V_k$ for each k, $1 \leq k \leq n$; see **[2]**, **[5]**, **[8]**, **[11]**, and **[12]**. That for each $T \in B(X)$ there exists a largest open set U on which T is decomposable was first shown by Nagy, **[11]**.

An alternative characterization of decomposability may be given in terms of a property introduced by E. Bishop, **[3]**. For an open subset V of $\mathbb{C}$, let $H(V, X)$ denote the space of all analytic X-valued functions on V. Then $H(V, X)$ is a Fréchet space with generating semi-norms given by $p_K(f) := \sup\{\|f(\lambda)\| : \lambda \in K\}$, where K runs through the compact subsets of V. Every operator $T \in B(X)$ induces a continuous linear mapping T_V on $H(V, X)$, defined by $T_V f(\lambda) := (T - \lambda)f(\lambda)$ for all $f \in H(V, X)$ and $\lambda \in V$. An operator T is said to possess Bishop's property (β) on an open set $U \subset \mathbb{C}$ if for each open subset V of U, the operator T_V is injective with range ran T_V

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closed in $H(V, X)$; see [**6**, Prop. 1.2.6]. Clearly there exists a largest open set $\rho_\beta(T)$ on which T has property (β).

Fundamental work by Albrecht and Eschmeier established that an operator $T \in B(X)$ has property (β) on U precisely when there exists an operator $S \in B(Y)$ such that S is decomposable on U, $X \in \mathrm{Lat}\,(S)$ and $T = S|_X$, [**2**, Theorem 10]. Moreover, [**2**, Theorems 8 and 21], T is decomposable on U if and only if T and its adjoint T^* share property (β) on U. Thus Nagy's largest open set on which T is decomposable is the set $\rho_\beta(T) \cap \rho_\beta(T^*)$.

An operator $T \in B(X)$ is said to have the single-valued extension property (SVEP) at a point $\lambda \in \mathbb{C}$ provided that, for every open disc V centered at λ, the mapping T_V is injective on $H(V, X)$. If $U \subset \mathbb{C}$ is open, then T is said to have SVEP on U if T has SVEP at every $\lambda \in U$, equivalently, if T_V is injective for each open set $V \subseteq U$. Let $\rho_{\mathrm{SVEP}}(T)$ denote the largest open set on which T has SVEP.

Recently, M. Neumann, V. Miller and the first author of the current paper showed, [**9**, Theorem 2.5], that T_V has closed range in $H(V, X)$ for every open subset V of the "Kato-type" resolvent set of T, an open set that contains the semi-Fredholm region of T, thus extending a result of Eschmeier, [**5**]. Following Neumann, we say that an operator has the closed range property (CR) on an open set $U \subset \mathbb{C}$ provided $\mathrm{ran}\,(T_V)$ is closed in $H(V, X)$ for every open subset V of U. Thus T has property (β) on U if and only if T has both SVEP and (CR) on U.

In this note, we give a more general condition that suffices for $T \in B(X)$ to have (CR) on an open set U and prove that there is in fact a largest open set $\rho_{\mathrm{CR}}(T)$ on which T has the closed range property. Thus $\rho_\beta(T) = \rho_{\mathrm{SVEP}}(T) \cap \rho_{\mathrm{CR}}(T)$. In the last section we establish corresponding results in the setting of the weak–$*$ topology on $H(V, X^*)$.

Main results. We denote the kernel of $T \in B(X)$ by $\ker(T)$ and define $N^\infty(T) := \bigcup_{n\geq 0} \ker(T^n)$ and $R^\infty(T) := \bigcup_{n\geq 0} \mathrm{ran}\,(T^n)$. If $T \in B(X)$ is such that $\mathrm{ran}\,(T)$ is closed and $N^\infty(T) \subseteq R^\infty(T)$, then T is said to be a Kato operator. A systematic exposition of this class, also referred to as semi-regular operators, may be found in [**10**, Section II.12]; also see [**1**, Section 1.2] and [**6**, Section 3.1]. In particular, an equivalent condition may be given in terms of the reduced minimum modulus function: for $S \in B(X)$, define $\gamma(S) := \inf\{\|Sx\| : \mathrm{dist}\,(x, \ker(S)) = 1\}$. Then an operator T is Kato if and only if $\gamma(T) > 0$ and the mapping $z \to \gamma(T - z)$ is continuous at 0, [**10**, II.12 Theorem 2]. Denote by $\sigma_K(T)$ the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda$ is not Kato. Then $\sigma_K(T)$ is a nonempty compact set, $z \mapsto R^\infty(T - z)$ is constant on each component of $\rho_K(T) := \mathbb{C} \setminus \sigma_K(T)$, $R^\infty(T - \lambda)$ is closed and $(T - \lambda)R^\infty(T - \lambda) = R^\infty(T - \lambda)$ for each $\lambda \in \rho_K(T)$, [**10**, II.12, Theorem 15 and Cor. 19]. Moreover, if G is a component of $\rho_K(T)$ and $S \subset G$ has an accumulation point in G, then $\bigcap_{z\in S} \mathrm{ran}\,(T - z) = R^\infty(T - \lambda)$ for each $\lambda \in G$, [**6**, 3.1.11].

For each closed subset F of $\mathbb{C}$, define the "glocal" analytic spectral subspace $\mathfrak{X}_T(F) := \{x \in X : x \in \mathrm{ran}\, T_{\mathbb{C}\setminus F}\}$. These spaces are T-invariant, but generally not closed. If $M \in \mathrm{Lat}\,(T)$ and $V \subset \mathbb{C}$ is such that $(T - z)M = M$ for all $z \in V$, then $M \subset \mathfrak{X}_T(\mathbb{C} \setminus V)$ by a theorem of Leiterer, [**6**, Theorem 3.2.1]. It follows from above that if G is a component of $\rho_K(T)$ and $V \subset G$ is open, then $\mathfrak{X}_T(\mathbb{C} \setminus V) = R^\infty(T - \lambda)$ for all $\lambda \in G$; in particular, $\mathfrak{X}_T(\mathbb{C} \setminus V)$ is closed. Also, it is easily seen that if T has (CR) on an open set U, then $\mathfrak{X}_T(\mathbb{C} \setminus V)$ is closed for every open $V \subset U$.

A consequence of Theorem 5 below is that the converse holds under the additional assumption that $\mathrm{ran}\,(T - z)$ is closed for all but countably many $z \in V$. Some additional

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assumption beyond closeness of the glocal spectral subspaces is seen to be necessary for (CR) by the facts that, on one hand, T has property (β) on all of $\mathbb{C}$ precisely when T has (CR) on $\mathbb{C}$, [**6**, Prop. 3.3.5], while on the other hand, there is an operator $T \in B(X)$ without property (β) but for which $\mathfrak{X}_T(F)$ is closed for all closed $F \subset \mathbb{C}$, [**7**].

If (X, d) is a metric space, let $B(x, r)$ denote the open ball in X with radius $r > 0$ and center $x \in X$.

Lemma 1. *Let $T \in B(X)$ and let V be an open subset of $\mathbb{C}$. Let $(D_i)_{i\in A}$ be a cover of V consisting of simply connected open sets D_i such that $\mathfrak{X}_T(\mathbb{C} \setminus D_i)$ is closed for each $i \in A$ and $D_i \setminus D_j \neq \emptyset$ if $\mathfrak{X}_T(\mathbb{C} \setminus D_i) \neq \mathfrak{X}_T(\mathbb{C} \setminus D_j)$.*

Let $M = \bigcap_{i\in A} \mathfrak{X}_T(\mathbb{C} \setminus D_i)$. Then M is closed, $TM \subset M$ and

(i) *if $x \in M$ and $g_j \in H(D_j, X)$ are such that $T_{D_j} g_j = x$, then $g_j(D_j) \subset M$;*

(ii) *$\ker T_{D_j} \subset H(D_j, M)$;*

(iii) *$(T - z)M = M$ for all $z \in V$;*

(iv) *if $\widetilde{T} : X/M \to X/M$ is the quotient map induced by T then $\widetilde{T}_{D_j}$ is injective on $H(D_j, X/M)$.*

Proof. Clearly M is a closed subspace of X and $TM \subset M$.

(i) Let $x \in M$ and $g_j \in H(D_j, X)$ be such that $T_{D_j} g_j = x$.

We show first that $g_j(D_j) \subset \mathfrak{X}_T(\mathbb{C} \setminus D_j)$. Let $z \in D_j$, and define $h_j : D_j \to X$ by $h_j(\omega) = (g_j(\omega) - g_j(z))/(\omega - z)$ if $\omega \in D_j \setminus \{z\}$ and $h_j(z) = g_j'(z)$. Then $h_j \in H(D_j, X)$ and if $\omega \neq z$, then

$$(T - \omega)h_j(\omega) = \frac{1}{\omega - z}(x - ((T - z) + (z - \omega))g_j(z)) = g_j(z).$$

By continuity, $(T - z)h_j(z) = g_j(z)$ as well. Hence $g_j(z) \in \mathfrak{X}_T(\mathbb{C} \setminus D_j)$ and so $g_j(D_j) \subset \mathfrak{X}_T(\mathbb{C} \setminus D_j)$.

If $i \in A$ is such that $\mathfrak{X}_T(\mathbb{C} \setminus D_i) \neq \mathfrak{X}_T(\mathbb{C} \setminus D_j)$, let $g_i \in H(D_i, X)$ be such that $T_{D_i} g_i = x$, let $z \in D_j \setminus D_i$ and define $h_i : D_i \to X$ by $h_i(\omega) = (g_i(\omega) - g_j(z))/(\omega - z)$. Then $h_i \in H(D_i, X)$ and again

$$\begin{aligned}(T - \omega)h_i(\omega) &= \frac{1}{\omega - z}((T - \omega)g_i(\omega) - ((T - z) + (z - \omega))g_j(z)) \\ &= \frac{1}{\omega - z}(x - x + (\omega - z)g_j(z)) = g_j(z).\end{aligned}$$

Thus $g_j(z) \in \mathfrak{X}_T(\mathbb{C} \setminus D_i)$ and $g_j(D_j \setminus D_i) \subset \mathfrak{X}_T(\mathbb{C} \setminus D_i)$.

Since the sets D_i and D_j are open, simply connected and $D_j \setminus D_i \neq \emptyset$, it is easy to see that $D_j \setminus D_i$ contains an accumulation point. Indeed, let $z_0 \in D_j \setminus D_i$. If $z_0 \notin \overline{D_i}$ then there is an open neighborhood of z_0 contained in $D_j \setminus \overline{D_i}$. If $z_0 \in \partial D_i$, then there is a sequence $(z_n) \subset D_j \setminus D_i$ such that $z_n \to z_0$.

Since $\mathfrak{X}_T(\mathbb{C} \setminus D_i)$ is closed and $g_j(D_j \setminus D_i) \subset \mathfrak{X}_T(\mathbb{C} \setminus D_i)$, it follows that $g_j : D_j \to \mathfrak{X}_T(\mathbb{C} \setminus D_i)$.

This proves (i).

(ii) is an immediate consequence of (i).

(iii) Let $z \in D_j$ and $x \in M \subset \mathfrak{X}_T(\mathbb{C} \setminus D_j)$. There is a function $g_j : D_j \to X$ such that $T_{D_j} g_j = x$. By (i), $g_j(z) \in M$ and so $x = (T - z)g_j(z) \in (T - z)M$.

(iv) If $\pi : X \to X/M$ is the canonical projection, then Gleason's theorem implies that the sequence $0 \to H(\Omega, M) \to H(\Omega, X) \xrightarrow{\pi} H(\Omega, X/M) \to 0$ is exact, [**6**, Prop. 2.1.5]. Thus, if $\widetilde{T}_{D_j} h = 0$ for some $h \in H(D_j, X/M)$, then there exists

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$f \in H(D_j, X)$ such that $h = \tilde{f}$, where $\tilde{f} = \pi \circ f$. Clearly $T_{D_j} f \in H(D_j, M)$ and (iii) together with Leiterer's theorem implies that there exists $g \in H(D_j, M)$ such that $T_{D_j} f = T_{D_j} g$. Thus $f - g \in \ker T_{D_j} \subset H(D_j, M)$ by (ii). Consequently, $f \in H(D_j, M)$ and therefore, $h = \tilde{f} = 0$. □

LEMMA 2. *Let V_1, V_2 be open subsets of $\mathbb{C}$ and suppose that Ω is an open subset of $V_1 \cup V_2$. Then there exist open sets Ω_1, Ω_2 so that $\Omega_j \subset V_j$, $\Omega = \Omega_1 \cup \Omega_2$ and an open cover $\mathcal{U}$ of Ω such that*

(i) *each $D \in \mathcal{U}$ is a simply connected subset of either V_1 or V_2;*
(ii) *if G is a component of $\Omega_1 \cap \Omega_2$, then there is a $D \in \mathcal{U}$ such that $D \subset G$;*
(iii) *$D \setminus D' \neq \emptyset$ whenever $D, D' \in \mathcal{U}$ are distinct.*

Proof. Let $U_j = V_j \cap \Omega$ for $j = 1, 2$ and define Ω_1 to be the union of all components G of U_1 such that $G \setminus U_2 \neq \emptyset$, and Ω_2 the union of components H of U_2 such that $H \setminus \Omega_1 \neq \emptyset$. Then, each Ω_j is open, and every component of Ω_j is a component of U_j. We may assume that each Ω_j is nonempty. Clearly, $\Omega = \Omega_1 \cup U_2$, and if H is a component of U_2, then either $H \subset \Omega_1$ or $H \subset \Omega_2$. Thus $\Omega = \Omega_1 \cup \Omega_2$.

Let $G_1, G_2, \ldots$ be the components of $\Omega_1 \cap \Omega_2$. We note $\partial G_n \cap \Omega_j \neq \emptyset$ for each $n \in \mathbb{N}$ and $j = 1, 2$. Indeed, suppose to the contrary that $\partial G_n \cap \Omega_1 = \emptyset$. Let M_j be the component of Ω_j containing G_n. Then $M_1 = G_n \cup (M_1 \setminus \overline{G_n})$, where $G_n \neq \emptyset$ and where $M_1 \setminus \overline{G_n} = M_1 \setminus G_n \supset M_1 \setminus M_2 \neq \emptyset$, contradicting the fact that M_1 is connected. That $\partial G_n \cap \Omega_2 \neq \emptyset$ follows similarly.

Choose $\lambda_n \in \partial G_n \cap \Omega_1$ and $\mu_n \in \partial G_n \cap \Omega_2$. Then $\lambda_n \notin \Omega_2$ and $\mu_n \notin \Omega_1$. Select $\lambda'_n, \mu'_n \in G_n$ so that $|\lambda_n - \lambda'_n| < 2^{-n}$ and $|\mu_n - \mu'_n| < 2^{-n}$. If we construct a piecewise linear path in G_n connecting λ'_n and μ'_n, then, taking such a path with minimal number of segments, we obtain a path γ_n between λ'_n and μ'_n that does not intersect itself. Clearly it is possible to find a simply connected open set D_n so that $\gamma_n \subset D_n \subset G_n$.

Let $D = \bigcup_n D_n$ and suppose that $z \in \Omega_1 \setminus D$. We claim that there is a $\delta(z) > 0$ such that $B(z, \delta(z)) \subset \Omega_1$ and $B(z, \delta(z)) \cap \{\mu'_1, \mu'_2, \ldots\} = \emptyset$. To this end, choose $\varepsilon(z) > 0$ so that $B(z, \varepsilon(z)) \subset \Omega_1$, and let n_0 be such that $2^{-n_0} < \varepsilon(z)/2$. Now, let $\delta(z) = \min\{\varepsilon(z)/2, |z - \mu'_1|, \ldots, |z - \mu'_{n_0-1}|\}$. Then $\mu'_n \notin B(z, \delta(z))$ if $n < n_0$, and if $n \geq n_0$, then $\mu_n \notin \Omega_1$ implies that $|z - \mu'_n| \geq |z - \mu_n| - |\mu_n - \mu'_n| \geq \varepsilon(z) - 2^{-n_0} > \delta(z)$, as required. Similarly, if $z \in \Omega_2 \setminus \Omega_1$, then there is a $\delta(z) > 0$ such that $B(z, \delta(z)) \subset \Omega_2$ and $B(z, \delta(z)) \cap \{\lambda'_1, \lambda'_2, \ldots\} = \emptyset$.

We define a sequence of (possibly empty) collections of open balls recursively: for each $k \geq 1$, let $\mathcal{U}_k := \{B(z, 2^{-j}) : \delta(z) \geq 2^{-k}$ and $B(z, 2^{-k}) \not\subset \mathcal{V}_{k-1}\}$, where $\mathcal{V}_0 = \emptyset$ and $\mathcal{V}_j := \bigcup_{\ell \leq j} \bigcup_{B \in \mathcal{U}_\ell} B$ for all $j \geq 1$. If $z \in \Omega \setminus D$, then there is a least m so that $\delta(z) \geq 2^{-m}$, and so either $B(z, 2^{-m}) \subset \mathcal{V}_{m-1}$ or $B(z, 2^{-m}) \in \mathcal{U}_m$. Thus $z \in \mathcal{V}_m$ in either case. It follows that $\Omega \setminus D = \bigcup_{k=1}^{\infty} \mathcal{V}_k = \bigcup_{\ell=1}^{\infty} \bigcup_{B \in \mathcal{U}_\ell} B$, and consequently $\mathcal{U} := \{D_n\}_n \cup \bigcup_{\ell=1}^{\infty} \mathcal{U}_\ell$ is an open cover of Ω satisfying the desired conditions. □

LEMMA 3. *Let V_1, V_2 be open subsets of $\mathbb{C}$. If $T \in B(X)$ has (CR) on each V_j $(j = 1, 2)$, then T has (CR) on $V_1 \cup V_2$.*

Proof. Let $\Omega \subset V_1 \cup V_2$ be an open set. To show that T_Ω has closed range, let Ω_1, Ω_2 and $\mathcal{U}$ be as in the previous lemma, and let $f \in \overline{\operatorname{ran} T_\Omega}$. Since T has (CR) on each Ω_j, $\mathfrak{X}_T(\mathbb{C} \setminus D)$ is closed for each $D \in \mathcal{U}$ and there are $g_j \in H(\Omega_j, X)$ such that $f|_{\Omega_j} = T_{\Omega_j} g_j$ for $j = 1, 2$. Define $M := \bigcap_{D \in \mathcal{U}} \mathfrak{X}_T(\mathbb{C} \setminus D)$. We have $T_{\Omega_1 \cap \Omega_2}(g_1 - g_2) = 0$, and so $(g_1 - g_2)(\Omega_1 \cap \Omega_2) \subset M$ by Lemma 1 (ii). Thus $\widetilde{g_1}|_{\Omega_1 \cap \Omega_2} = \widetilde{g_2}|_{\Omega_1 \cap \Omega_2}$ and we can define $h \in H(\Omega, X/M)$ by $h(z) = \widetilde{g_j}(z)$ for $z \in \Omega_j$. We have $\tilde{f} = \widetilde{T}_\Omega h$ and, again by Gleason's theorem, there exists $g \in H(\Omega, X)$ such that $h = \tilde{g}$. Then $f - T_\Omega g \in H(\Omega, M)$ and so Lemma 1 (iii) and Leiterer's theorem together imply that $f - T_\Omega g = T_\Omega k$ for some $k \in H(\Omega, M)$. Hence $f = T_\Omega(g + k) \in \operatorname{ran} T_\Omega$. □

THEOREM 4. *Let $T \in B(X)$. Then there is a largest open set $\rho_{CR}(T)$ on which T has (CR).*

Proof. Let $\mathcal{W}$ be the family of all open subsets $V \subset \mathbb{C}$ such that T has (CR) on V. We show that T has (CR) on the union $W = \bigcup \mathcal{W}$, which is obviously the largest open set on which T has (CR).

Let $\Omega \subset W$ be a nonempty open subset. We show that T_Ω has closed range. For each $z \in \Omega$ choose $0 < \delta(z) < \operatorname{dist}(z, \partial\Omega)$ so that T has (CR) on $B(z, \delta(z))$. As in the proof of Lemma 2, define $\mathcal{U}_k := \{B(z, 2^{-k}) : \delta(z) \geq 2^{-k}$ and $B(z, 2^{-k}) \not\subset V_{k-1}\}$, where $V_j := \bigcup_{\ell \leq j} \bigcup_{B \in \mathcal{U}_\ell} B$ and $V_0 = \emptyset$. Then, again as in Lemma 2, $\Omega = \bigcup_{m \geq 1} V_m$, and so $\mathcal{U}' := \bigcup_{m=1}^{\infty} \mathcal{U}_m$ is a collection of open balls covering Ω such that T has (CR) on each ball $D \in \mathcal{U}'$ and also such that $D \neq D'$ in $\mathcal{U}'$ implies $D \setminus D' \neq \emptyset$. Let $\mathcal{U} = (D_k)_{k \in \mathbb{N}}$ be a countable subcover of $\mathcal{U}'$ and define $\Omega_n = \bigcup_{k \leq n} D_k$. By Lemma 3, T has (CR) on each Ω_n.

Let $M = \bigcap_n \mathfrak{X}_T(\mathbb{C} \setminus D_n)$. By Lemma 1, M is a closed subspace of X, $TM \subset M$ and $(T - z)M = M$ for all $z \in \Omega$. Denote by $\widetilde{T} : X/M \to X/M$ the operator induced by T and by $\pi : X \to X/M$ the canonical projection.

Let $f \in \overline{\operatorname{ran} T_\Omega}$. Then for each n there exists $g_n \in H(\Omega_n, X)$ such that $f|_{\Omega_n} = T_{\Omega_n} g_n$. If $n \geq 2$, then $T_{\Omega_{n-1}}(g_n|_{\Omega_{n-1}} - g_{n-1}) = 0$ and so, by Lemma 1 (ii), $g_n|_{\Omega_{n-1}} - g_{n-1} : \Omega_{n-1} \to M$, i.e., $\widetilde{g}_n|_{\Omega_{n-1}} = \widetilde{g}_{n-1}$ in $H(\Omega_{n-1}, X/M)$.

Define $h : \Omega \to X/M$ by $h|_{\Omega_n} = \widetilde{g}_n$. Then h is well-defined and analytic on Ω. Also, $\tilde{f} = \widetilde{T}_\Omega h$ in $H(\Omega, X/M)$. By Gleason's theorem, there exists $g \in H(\Omega, X)$ such that $\widetilde{g} = h$ and therefore, $\pi(f - T_\Omega g) = 0$. Exactness implies that $f - T_\Omega g \in H(\Omega, M)$, and so it again follows from Lemma 1 (iii) and Leiterer's theorem that there is a $k \in H(\Omega, M)$ such that $f - T_\Omega g = T_\Omega k$, i.e., $f = T_\Omega(g + k) \in \operatorname{ran} T_\Omega$. □

THEOREM 5. *Let $T \in B(X)$ and let $V \subset \mathbb{C}$ be an open set. Suppose that the set $\{z \in V : \operatorname{ran}(T - z)$ is not closed$\}$ is countable and that, for all $z \in V$, there is an $r_0 > 0$ for which $\mathfrak{X}_T(\mathbb{C} \setminus B(z, r))$ is closed for all $r \in (0, r_0)$. Then T has (CR) on V.*

Proof. Since the conditions of the theorem are inherited by every open subset U of V, it suffices to show that T_V has closed range in $H(V, X)$. Moreover, because the set $\{z \in \mathbb{C} : \operatorname{ran}(T - z)$ is closed and $T - z$ is not Kato$\}$ is countable by [**10**, II.12 Theorem 13], it follows that $E := V \cap \sigma_K(T)$ is countable; let $E = \{\lambda_n : n = 1, 2, \ldots\}$ be an enumeration of E (possibly finite). Note that, while E need not be discrete, the set $V \setminus E = V \cap \rho_K(T)$ is open.

We construct a sequence (B_j) of mutually disjoint open discs such that $E \subset \bigcup_j B_j$, $\overline{B_j} \subset V$ and $\mathfrak{X}_T(\mathbb{C} \setminus B_j)$ is closed for each j. Indeed, choose $r_1 > 0$ such that $\overline{B(\lambda_1, r_1)} \subset V$, $\mathfrak{X}_T(\mathbb{C} \setminus B(\lambda_1, r_1))$ is closed, and $|\lambda_j - \lambda_1| \neq r_1 \quad (j \geq 2)$. Set $B_1 = B(\lambda_1, r_1)$. Let k be the smallest index such that $\lambda_k \notin B_1$ and find $r_2 > 0$ such that $B_2 := B(\lambda_k, r_2)$ satisfies $\overline{B_2} \subset V \setminus B_1$, the space $\mathfrak{X}_T(\mathbb{C} \setminus B_2)$ is closed and $|\lambda_j - \lambda_k| \neq r_2 \quad (j > k)$. If we continue in this way, we obtain the required sequence of open discs $\mathcal{U}_E = (B_j)_j$ covering E.

For each $z_0 \in V \setminus E$ we next find a simply connected open set W_{z_0} such that $z_0 \in W_{z_0} \subset V \setminus E$ and $W_{z_0} \setminus B_n \neq \emptyset$ for each $B_n \in \mathcal{U}_E$. If $z_0 \notin \bigcup_n B_n$, choose $r > 0$ such that $B(z_0, r) \subset V \setminus E$ and set $W_{z_0} = B(z_0, r)$. Suppose then that $z_0 \in \bigcup_n B_n \setminus E$. Since the sets B_n are mutually disjoint, there is only one j with $z_0 \in B_j$, and since the set

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E is countable, there is a θ, $0 \le \theta < 2\pi$ such that $\{z_0 + te^{i\theta} : t \ge 0\} \cap E = \emptyset$. Let $t_0 = \min\{t \ge 0 : z_0 + te^{i\theta} \notin B_j\}$. Since the set $S := \{z_0 + te^{i\theta} : 0 \le t \le t_0\}$ is compact and the set $E \cup \partial V$ is closed, there is an $\varepsilon > 0$ such that the set $W_{z_0} := \{z \in \mathbb{C} : \text{dist}\,\{z, S\} < \varepsilon\}$ is disjoint with $E \cup \partial V$. Clearly W_{z_0} is an open simply connected set such that $z_0 \in W_{z_0} \subset V \setminus E \subset \rho_K(T)$. If G is the component of $\rho_K(T)$ containing W_{z_0}, then $\mathfrak{X}_T(\mathbb{C} \setminus W_{z_0}) = R^\infty(T - \lambda)$ for every $\lambda \in G$. In particular, $\mathfrak{X}_T(\mathbb{C} \setminus W_{z_0})$ is closed and $W_{z_0} \cap W_{z_1} = \emptyset$ if $z_0, z_1 \in V \setminus E$ are such that $\mathfrak{X}_T(\mathbb{C} \setminus W_{z_0}) \neq \mathfrak{X}_T(\mathbb{C} \setminus W_{z_1})$. By construction, $W_z \setminus B_k \neq \emptyset$ and $B_k \setminus W_z \neq \emptyset$ whenever $z \in V \setminus E$ and $B_k \in \mathcal{U}_E$. Thus, if $\mathcal{U}_K = \{W_z : z \in V \setminus E\}$ and $\mathcal{U} = \mathcal{U}_K \cup \mathcal{U}_E$, then $\mathcal{U}$ is an open cover of V satisfying the hypotheses of Lemma 1.

As in Lemma 1, let $M = \bigcap_{D \in \mathcal{U}} \mathfrak{X}_T(\mathbb{C} \setminus D)$ and let $\widetilde{T} : X/M \to X/M$ be the operator induced by T. By Lemma 1 (iii), we have $(T - z)M = M$ for all $z \in V$. We show that $\widetilde{T} - z$ is bounded below for each $z \in V \setminus E$, i.e., if $z \in V \setminus E$ and $(x_n)_n \subset X$ is such that $(\widetilde{T} - z)\tilde{x}_n \to 0$ in X/M, then $\tilde{x}_n \to 0$ in X/M.

Fix $z \in V \setminus E$ and let $x \in \ker(T - z)$. Then $\ker(T - z) \subset R^\infty(T - z) = \mathfrak{X}_T(\mathbb{C} \setminus W_z)$, and so there exists $g \in H(W_z, X)$ so that $(T - \omega)g(\omega) = x$ for all $\omega \in W_z$. If $h = (T - z)g$, then $h \in \ker T_{W_z}$ and, since $W_z \in \mathcal{U}$, it follows from Lemma 1 (ii) that $h : W_z \to M$. In particular, $x = h(z) \in M$. Thus $\ker(T - z) \subset M$.

A sequence $(x_n)_n \subset X$ satisfies $(\widetilde{T} - z)\tilde{x}_n \to 0$ only if there exists $(y_n)_n \subset M$ so that $(T - z)x_n - y_n \to 0$ in X. Since $(T - z)M = M$, there exists $(w_n)_n \subset M$ so that $(T - z)w_n = y_n$ and therefore, $(T - z)(x_n - w_n) \to 0$. Since ran$\,(T - z)$ is closed, it follows that dist$\,(x_n - w_n, \ker(T - z)) \to 0$. But $\ker(T - z) \subset M$, and so dist$\,(x_n, M) \to 0$, i.e., $\tilde{x}_n \to 0$ in X/M as required. Hence $\widetilde{T} - z$ is bounded below for each $z \in V \setminus E$. In particular, $V \setminus E \subset \rho_K(\widetilde{T})$.

We wish to show that $\widetilde{T}_V$ is injective with closed range. Suppose then that $(f_n)_n$ is a sequence in $H(V, X/M)$ such that $\widetilde{T}_V f_n \to 0$. In order to show that $f_n \to 0$ in $H(V, X/M)$, it suffices to show that $p_F(f_n) = \sup_{z\in F} \|f_n(z)\| \to 0$ for every closed rectangle $F \subset V$. Suppose that a, b, c, d are real numbers such that the rectangle $F = [a, b] \times [c, d] \subset V$. Choose $\delta > 0$ so that $[a - \delta, b + \delta] \times [c - \delta, d + \delta] \subset V$. Since E is countable, the projections $P_1 = \{\text{Re}\lambda : \lambda \in E\}$ and $P_2 = \{\text{Im}\lambda : \lambda \in E\}$ are countable and we may choose $a', b' \in \mathbb{R} \setminus P_1$ and $c', d' \in \mathbb{R} \setminus P_2$ so that $a - \delta < a' < a < b < b' < b + \delta$ and $c - \delta < c' < c < d < d' < d + \delta$. Define Γ to be the positively oriented boundary of the rectangle $[a', b'] \times [c', d'] \subset V$. Then $\Gamma \subset V \setminus E$ surrounds F in the sense of Cauchy's theorem. By continuity of the minimum modulus function $z \mapsto \gamma(\widetilde{T} - z)$ on $V \setminus E$, there is a constant $c > 0$ so that $\sup_{z\in\Gamma} \|f_n(z)\| \le c \sup_{z\in\Gamma} \|(\widetilde{T} - z)f_n(z)\|$ for all n. Thus for each $\lambda \in F$ the maximum principle implies that

$$\|f_n(\lambda)\| \le \sup_{z\in\Gamma} \|f_n(z)\| \le C\, p_\Gamma(\widetilde{T}_V f_n)$$

where $C = c\,|\Gamma|/(2\pi\, \text{dist}\,(\Gamma, F))$. Thus $p_F(f_n) \to 0$ as $n \to \infty$ as required. Since $(T - z)M = M$ for all $z \in V$ by part (iii) of Lemma 1, Leiterer's theorem implies that $T_V H(V, M) = H(V, M)$, and T_V therefore has closed range in $H(V, X)$ by [**9**, Prop. 2.1]; the theorem is established. $\square$

For $T \in B(X)$ denote by $K(T)$ the analytic core of T, i.e., the set of all $x_0 \in X$ such that there exists a sequence $(x_n)_n \subset X$ such that $Tx_n = x_{n-1}$ $(n \ge 1)$ and $\sup \|x_n\|^{1/n} < \infty$. Clearly $K(T) = \bigcup_n \mathfrak{X}_T(\mathbb{C} \setminus D(0, 1/n))$. This set has been shown to play a significant role in the Fredholm theory of Banach space operators; see, for example [**1**].

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COROLLARY 6. *Let $T \in B(X)$ and let $V \subset \mathbb{C}$ be an open set. Suppose that $K(T-z)$ is closed for each $z \in V$ and that the set $\{z \in V : \operatorname{ran}(T-z)$ is not closed$\}$ is countable. Then T has (CR) on V.*

Proof. Let $z \in V$ and $K(T-z)$ be closed. Clearly $(T-z)K(T-z) = K(T-z)$ and, by the Banach open mapping theorem, there is an $\varepsilon > 0$ such that $K(T-z) = \mathfrak{X}_T(\mathbb{C} \setminus B(z, \varepsilon))$. In fact, $\varepsilon = \gamma((T-z)|_{K(T-z)})^{-1}$. Clearly $\mathfrak{X}_T(\mathbb{C} \setminus W) = K(T-z)$ for each open set W with $z \in W \subset B(z, \varepsilon)$. By Theorem 5, T has (CR) on V. □

A generalized Kato decomposition for $T \in B(X)$ is a pair of subspaces $X_1, X_2 \in \operatorname{Lat}(T)$ such that $X = X_1 \oplus X_2$, $T|_{X_1}$ is Kato and $T|_{X_2}$ is quasinilpotent. The operator T is said to be of Kato-type if $T|_{X_2}$ is nilpotent. It is well known that semi-Fredholm operators are of Kato-type, see e.g. **[1]**, **[10]**.

If $\rho_{gk}(T)$ denotes the set of $\lambda \in \mathbb{C}$ such that $T - \lambda$ has a generalized Kato decomposition, then $\rho_{gk}(T)$ is open and $\rho_{gk}(T) \cap \sigma_K(T)$ accumulates only on $\partial\rho_{gk}(T)$. Indeed, suppose that $0 \in \rho_{gk}(T)$ and that $X_1, X_2 \in \operatorname{Lat}(T)$ such that $X = X_1 \oplus X_2$, $T|_{X_1}$ is Kato and $T|_{X_2}$ is quasinilpotent. If $\varepsilon > 0$ is such that $B(0, \varepsilon) \subset \rho_K(T|_{X_1})$, then for $0 < |z| < \varepsilon$, $(T-z)X_2 = X_2$. Thus $\operatorname{ran}(T-z) = (T-z)X_1 \oplus X_2$ is closed and $N^\infty(T-z) = N^\infty(T|_{X_1} - z) \subset R^\infty(T|_{X_1} - z)$.

Moreover, if T has generalized Kato decomposition (X_1, X_2) as above, then by the remarks preceding Lemma 1, $R^\infty(T|_{X_1}) \subseteq K(T)$. On the other hand, if $x \in K(T)$, write $x = u_0 + v_0$ with $u_0 \in X_1$ and $v_0 \in X_2$. We show that $v_0 = 0$.

Suppose to the contrary that $v_0 \neq 0$. Then, by definition, there are sequences $(u_n) \subset X_1$ and $(v_n) \subset X_2$ such that $Tu_n = u_{n-1}$ and $Tv_n = v_{n-1}$ for all n and $C := \sup \|u_n + v_n\|^{1/n} < \infty$. Let $P \in B(X)$ be the projection with $\ker P = X_1$ and $\operatorname{ran} P = X_2$. We have $\|v_n\|^{1/n} = \|P(u_n + v_n)\|^{1/n} \leq \|P\|^{1/n} \cdot C$. Thus

$$\lim_{n\to\infty} \|T^n|_{X_2}\|^{1/n} \geq \limsup_{n\to\infty} \left(\frac{\|v_0\|}{\|v_n\|}\right)^{1/n} = \frac{1}{\liminf_{n\to\infty} \|v_n\|^{1/n}} \geq 1/C > 0,$$

a contradiction to the assumption that $T|_{X_2}$ is quasinilpotent. Hence $v_0 = 0$ and $K(T) \subseteq X_1$. Therefore

$$K(T) = K(T|_{X_1}) = R^\infty(T|_{X_1});$$

in particular, $K(T)$ is closed.

Thus we have established the following special case of Corollary 6, generalizing **[9**, Theorem 2.5].

COROLLARY 7. *$T \in B(X)$ has (CR) on $\rho_{gk}(T)$.*

Duality and weak–∗ closed ranges. Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and for U an open neighborhood of ∞, let $P(U, X)$ denote the Fréchet space of analytic functions $f : U \to X$ with $f(\infty) = 0$. If $T \in B(X)$, then T induces a continuous mapping T^U on $P(U, X)$ defined by $T^U f(z) = (T-z)f(z) + \lim_{|\omega|\to\infty} \omega f(\omega)$. For F closed in $\mathbb{C}_\infty$ with $\infty \in F$, let $P(F, X)$ denote the inductive limit of the spaces $P(U, X)$, $U \supset F$ open; i.e., $P(F, X)$ is the (LF)-space consisting of germs of analytic X–valued functions defined in a neighborhood of F and vanishing at infinity. If $\infty \in F$ is closed and U is open with $F \subset U$, let $i_U : P(U, X) \to P(F, X)$ be defined by $i_U f = [f]$. Then a mapping S from $P(F, X)$ to an arbitrary topological vector space E is continuous if

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and only if $S \circ i_U$ is continuous for every open neighborhood U of F. In particular, the mappings T^U induce a continuous mapping T^F on $P(F, X)$. Recall further the Grothendieck-Köthe duality principle: given $V \subset \mathbb{C}$ open, the Fréchet space $H(V, X^*)$ may be canonically identified with the strong dual of $P(\mathbb{C}_\infty \setminus V, X)$ via

$$\langle f, g\rangle = \int_\gamma \langle f(z), \widetilde{g}(z)\rangle\, dz,$$

where $f \in H(V, X^*)$, $\widetilde{g} \in P(U, X)$ is a representative of $g \in P(\mathbb{C}_\infty \setminus V, X)$ and γ is a contour surrounding $\mathbb{C} \setminus U$ in V. In this sense, we have that $T_V^* = (T^{\mathbb{C}\setminus V})^*$, [**6**, Theorem 2.5.12 and Lemma 2.5.13]. Moreover, by the duality results of Albrecht and Eschmeier, specifically, Theorem 21 and the proof of Theorem 5 of [**2**], T^* has property (β) on U if and only if $T^F P(F, X) = P(F, X)$ for every closed set $F \subseteq \mathbb{C}_\infty$ with $\mathbb{C}_\infty \setminus U \subseteq F$. In this case, for every open $V \subseteq U$, T_V^* is injective with weak–$*$ closed range in $H(V, X^*)$ by a theorem of Köthe, [**6**, Theorem 2.5.9].

Let us say that T^* has the property $(\mathrm{CR})^{\text{weak}-*}$ on U provided that ran T_V^* is weak–$*$ closed in $H(V, X^*)$ for every open $V \subseteq U$.

Proposition 8. *Let $T \in B(X)$ and $U \subset \mathbb{C}$ open and suppose that F is closed in $\mathbb{C}$ with $\mathbb{C} \setminus U \subset F$.*

(i) *If T has (CR) on U, then $\mathfrak{X}_T(F) = {}^{\perp}\mathfrak{X}_{T^*}^*(\mathbb{C} \setminus F)$, the preannihilator of $\mathfrak{X}_{T^*}^*(\mathbb{C} \setminus F) := \bigcup\{\mathfrak{X}_{T^*}^*(K) : K \text{ compact},\ K \subset \mathbb{C} \setminus F\}$.*
(ii) *If T^* has $(CR)^{\text{weak}-*}$ on U, then $\mathfrak{X}_{T^*}^*(F) = \mathfrak{X}_T(\mathbb{C} \setminus F)^{\perp}$, the annihilator of $\mathfrak{X}_T(\mathbb{C} \setminus F) := \bigcup\{\mathfrak{X}_T(K) : K \text{ compact},\ K \subset \mathbb{C} \setminus F\}$. In particular, $\mathfrak{X}_{T^*}^*(\mathbb{C} \setminus V)$ is weak–$*$ closed whenever $V \subseteq U$ is open.*

Proof. If F is closed and $\mathbb{C} \setminus U \subseteq F$, then $V := \mathbb{C} \setminus F$ is an open subset of U. Thus ran T_V is closed in case (i), and ran T_V^* is weak–$*$ closed in case (ii). The result now follows from parts (c) and (d) of [**4**, Lemma I.2.5]; alternatively, one could argue as in the proof of [**6**, Prop 2.5.14]. □

As a consequence of the Proposition 8, we obtain weak–$*$ analogs of Theorems 4 and 5.

Theorem 9. *There is a largest open set V on which $T^* \in B(X^*)$ has $(CR)^{\text{weak}-*}$.*

Proof. First we establish an analog of Lemma 3. Suppose that $T^* \in B(X^*)$ has $(\mathrm{CR})^{\text{weak}-*}$ on open sets V_1 and V_2 and that Ω is an open subset of $V_1 \cup V_2$. Let $\Omega_1 \subset V_1 \cap \Omega$, $\Omega_2 \subset V_2 \cap \Omega$ be open sets and $\mathcal{U}$ an open cover of Ω as in Lemma 2. Let $M = \bigcap_{D\in\mathcal{U}} \mathfrak{X}_{T^*}^*(\mathbb{C} \setminus D)$. By Proposition 8, for each $D \in \mathcal{U}$, $\mathfrak{X}_{T^*}^*(\mathbb{C} \setminus D)$ is weak–$*$ closed and therefore M is also weak–$*$ closed. Evidently, the restriction mapping $f \mapsto f|_{\Omega_j}$ from $H(\Omega, X^*)$ to $H(\Omega_j, X^*)$ is weak-$*$ continuous and intertwines T_Ω^* and $T_{\Omega_j}^*$, $j = 1, 2$. Therefore, if $f \in \overline{\operatorname{ran} T_\Omega^*}^{\text{weak}-*}$, then $f|_{\Omega_j} \in \overline{\operatorname{ran} T_{\Omega_j}^*}^{\text{weak}-*}$, and so, by assumption, there are $g_j \in H(\Omega_j, X^*)$ such that $f|_{\Omega_j} = T_{\Omega_j}^* g_j$ for each j. As in the proof of Lemma 3, it follows from Lemma 2 that $T_{\Omega_1\cap\Omega_2}^*(g_1 - g_2) = 0$, and so $(g_1 - g_2)(\Omega_1 \cap \Omega_2) \subset M$ by Lemma 1 (ii). Thus $\widetilde{g}_1|_{\Omega_1\cap\Omega_2} = \widetilde{g}_2|_{\Omega_1\cap\Omega_2}$ in $H(\Omega_1 \cap \Omega_2, X^*/M)$, and we can define $h \in H(\Omega, X^*/M)$ by $h(z) = \widetilde{g}_j(z)$ for $z \in \Omega_j$. We have $\widetilde{f} = (T^*)\widetilde{}_\Omega h$ and, by Gleason's theorem, there exists $g \in H(\Omega, X^*)$ such that $h = \widetilde{g}$. Moreover, $f - T_\Omega^* g \in H(\Omega, M)$, and so again Lemma 1 (iii) and Leiterer's theorem imply that $f - T_\Omega^* g = T_\Omega^* k$ for some $k \in H(\Omega, M)$. Hence $f = T_\Omega^*(g + k) \in \operatorname{ran} T_\Omega^*$. Thus $T^* \in B(X^*)$ has $(\mathrm{CR})^{\text{weak}-*}$ on $V_1 \cup V_2$.

To complete the argument, we adapt the proof of Theorem 4 similarly. The routine details are left to the reader. □

Recall that ran T^* is weak–$*$ closed in X^* if and only if ran T is closed in X, [**6**, A.1.10]. Also, $\sigma_K(T^*) = \sigma_K(T)$, [**10**, II.12 Theorem 11].

THEOREM 10. *Let $T \in B(X)$ and let $V \subset \mathbb{C}$ be an open set. Suppose that the set $\{z \in V : \operatorname{ran}(T - z)$ is not closed$\}$ is countable and that, for all $z \in V$, there is a $r_0 > 0$ for which $\mathfrak{X}_T(\mathbb{C} \setminus B(z, r))$ is weak–$*$ closed for all $r \in (0, r_0)$. Then T^* has $(CR)^{\text{weak}-*}$ on V.*

Proof. Since the conditions of the theorem are inherited by every open subset U of V, it suffices to show that T^*_V has weak–$*$ closed range. Let $E := V \cap \sigma_K(T)$ and construct a covering $\mathcal{U} = \mathcal{U}_K \cup \mathcal{U}_E$ exactly as in the proof of Theorem 5, noting that if $z_0 \in V \setminus E$ and if λ is in the component of $\rho_K(T)$ containing z_0, then $\mathfrak{X}^*_{T^*}(\mathbb{C} \setminus W_{z_0}) = R^\infty(T^* - \lambda)$ is weak–$*$ closed. Let $M = \bigcap_{D \in \mathcal{U}} \mathfrak{X}^*_{T^*}(\mathbb{C} \setminus D)$ and denote by $(T^*)\tilde{}$ the operator on X^*/M induced by T^*. Then Lemma 1 (iii) implies that $(T^* - z)M = M$ for all $z \in V$, and, as in the proof of Theorem 5, $(T^*)\tilde{} - z$ is bounded below for each $z \in V \setminus E$. The conclusion now follows from [**9**, Prop. 3.1], noting that, as indicated in the proof of Theorem 5, it suffices in [**9**, Prop. 3.1] that the exceptional set E be merely countable rather than discrete. □

COROLLARY 11. *Let $T \in B(X)$ and let $V \subset \mathbb{C}$ be an open set. Suppose that the analytic core $K(T^* - z)$ is weak–$*$ closed for each $z \in V$ and that the set $\{z \in V : \operatorname{ran}(T - z)$ is not closed$\}$ is countable. Then T^* has $(CR)^{\text{weak}-*}$ on V. In particular, T^* has $(CR)^{\text{weak}-*}$ on $\rho_{gk}(T)$.*

Proof. The first statement follows from Theorem 10 just as Corollary 6 follows from Theorem 5. If $T \in B(X)$ has generalized Kato decomposition (X_1, X_2), then $(X_2^\perp, X_1^\perp)$ is a generalized Kato decomposition for T^* consisting of weak–$*$ closed subspaces of X^*. Thus $\rho_{gk}(T) \subseteq \rho_{gk}(T^*)$. If $z \in \rho_{gk}(T)$, and (X_1, X_2) is a generalized Kato decomposition for T, then $K(T^* - z) = K((T^* - z)|_{X_2^\perp}) = R^\infty((T^* - z)|_{X_2^\perp})$; in particular, $K(T^* - z)$ is weak–$*$ closed in X^*. Since $\rho_{gk}(T) \cap \sigma_K(T)$, is discrete, the last result now follows. □

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REFERENCES

1. P. Aiena, *Fredholm and local spectral theory, with applications to multipliers* (Kluwer Academic Publ., Dordrecht, 2004).

2. E. Albrecht and J. Eschmeier, Analytic functional models and local spectral theory, *Proc. London Math. Soc.* (3) **75** (1997), 323–348.

3. E. Bishop, A duality theory for an arbitrary operator, *Pacific J. Math.* **9** (1959), 379–397.

4. J. Eschmeier, *Analytische Dualität und Tensorprodukte in der mehrdimensionalen Spektraltheorie*, Habilitationsschrift, Schriftenreihe des Mathematischen Instituts der Universität Münster, 2. Serie, Heft 42 (Münster, 1987).

5. J. Eschmeier, On the essential spectrum of Banach space operators, *Proc. Edinburgh Math. Soc.* (2) **43** (2000), 511–528.

6. K. B. Laursen and M. M. Neumann, *An introduction to local spectral theory* (Clarendon Press, Oxford, 2000).

7. T. L. Miller and V. G. Miller, An operator satisfying Dunford's condition (C) but without Bishop's property (β), *Glasgow Math. J.* **40** (1998), 427–430.

8. T. L. Miller, V. G. Miller and M. M. Neumann, Localization in the spectral theory of operators on Banach spaces, in *Proceedings of the Fourth Conference on Function Spaces at Edwardsville*, Contemp. Math. **328** (Amer. Math. Soc., Providence, RI, 2003), 247–262.

9. T. L. Miller, V. G. Miller and M. M. Neumann, The Kato-type spectrum and local spectral theory, *Czech. Math. J.* **57** (2007), 831–842.

10. V. Müller, Spectral theory of linear operators and spectral systems in banach algebras (Birkhäuser Verlag, Basel, 2003).

11. B. Nagy, On S-decomposable operators, *J. Operator Theory* **2** (1979), 277–286.

12. F.-H. Vasilescu, *Analytic functional calculus and spectral decompositions* (Editura Academiei and D. Reidel Publishing Company, Bucharest and Dordrecht, 1982).