J. Austral. Math. Soc. (*Series A*) **52** (1992), 1–10

ON CERTAIN METHODS OF SOLVING A CLASS OF INTEGRAL EQUATIONS OF FREDHOLM TYPE

H. M. SRIVASTAVA and R. K. RAINA

(Received 16 January 1990; revised 20 June 1990)

Communicated by W. Moran

Abstract

The authors begin by presenting a brief survey of the various useful methods of solving certain integral equations of Fredholm type. In particular, they apply the reduction techniques with a view to inverting a class of generalized hypergeometric integral transforms. This is observed to lead to an interesting generalization of the work of E. R. Love [9]. The Mellin transform technique for solving a general Fredholm type integral equation with the familiar H-function in the kernel is also considered.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*) Primary 45 B 05, 44 A 20; Secondary 33 A 35.
Keywords and phrases: Fredholm type integral equations, reduction techniques, integral transforms, Mellin transform techniques, Legendre functions, hypergeometric functions, fractional calculus, H-functions, Mellin-Barnes contour integrals, Riemann-Liouville fractional integrals, Weyl fractional integrals, Stieltjes transforms, Mellin inversion theorem.

1. Introduction and definitions

Buschman [1] and Erdélyi [2] solved certain integral equations involving the Legendre functions. Integral equations containing hypergeometric functions in their kernels were considered, among others, by Higgins [6], Love ([8], [9]), Prabhakar [11], and Srivastava and Buschman [15]. (See also Love *et al.* [10].) In particular, Love [9] solved a Fredholm type integral equation:

$$(1.1)\qquad \int_0^\infty \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} t^{-b} F\left(a, b; c; -\frac{x}{t}\right) f(t)\,dt = g(x) \qquad (x > 0)$$

by certain procedures using *fractional calculus*. A similar type of integral equation has also been considered by Prabhakar and Kashyap [12]. Our purpose in this paper is to present, in some detail, a systematic discussion of the various methods of solvability of certain interesting cases of the integral equation:

(1.2)
$$\int_0^\infty t^{-\lambda} H_{P,Q}^{M,N}\left[A(x/t)^m \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q}\end{matrix}\right] f(t)\,dt = g(x) \qquad (0<x<\infty),$$

where $H_{P,Q}^{M,N}[z|\ldots]$ denotes the familiar H-function of Fox [3, p. 408] which is defined as a contour integral of Mellin-Barnes type. (For full definition and other related details of this function, see [4] and [16]; see also Equations (2.2) and (2.3) below.)

The symbolic form $(a_j, \alpha_j)_{1,P}$, used in (1.2) and elsewhere in this paper, abbreviates the set of parameters

$$(a_1,\alpha_1),\ldots,(a_P,\alpha_P), \quad P\in\mathbb{N}=\{1,2,3,\ldots\},$$

the set being empty when $P = 0$. Also, the Mellin-Barnes contour integral representing the H-function in (1.2) converges absolutely and defines an analytic function for

$$|\arg(A)| < \frac{1}{2}\pi\Omega, \tag{1.3}$$

where

$$\Omega = \sum_{j=1}^{M}\beta_j - \sum_{j=M+1}^{Q}\beta_j + \sum_{j=1}^{N}\alpha_j - \sum_{j=N+1}^{P}\alpha_j > 0. \tag{1.4}$$

We denote by $\mathscr{A}$ the space of all functions f which are well defined on $\mathbb{R}=[0,\infty)$ with the additional constraints that

(i) $f\in\mathscr{C}^\infty(\mathbb{R})$,
(ii) $\lim_{r\to\infty}\{x^k f^{(r)}(x)\}=0$ $(\forall k, r\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$, and
(iii) $f(x)=O(1)$, $x\to 0$.

Furthermore, we suppose that $\mathscr{A}$ corresponds to the space of *good functions* (see Lighthill [7, p. 15]) defined on the whole real line $(-\infty,\infty)$.

The *Riemann-Liouville fractional integral* (of order μ) is defined by

$$\mathscr{D}^{-\mu}\{f(z)\} = {}_0\mathscr{D}_z^{-\mu}\{f(z)\} = \frac{1}{\Gamma(\mu)}\int_0^z (z-w)^{\mu-1} f(w)\,dw \tag{1.5}$$
$$(\mathrm{Re}(\mu)>0;\ f\in\mathscr{A}),$$

where $\mathscr{D}^{\mu}\{f(z)\}=\phi(z)$ is understood to mean that ϕ is a locally integrable solution of $f(z)=\mathscr{D}^{-\mu}\{\varphi(z)\}$, implying that $\mathscr{D}^{\mu}$ is the inverse of the fractional integral operator $\mathscr{D}^{-\mu}$. (Whenever necessary, we shall simply write $\mathscr{D}_z^{-\mu}$ for ${}_0\mathscr{D}_z^{-\mu}$ for the Riemann-Liouville fractional integral operator defined by Equation (1.5) above.)

The *Weyl fractional integral* (of order ν) is defined by

$$\tag{1.6} \begin{aligned} \mathscr{W}^{-\nu}\{f(z)\} &= {}_z\mathscr{D}_\infty^{-\nu}\{f(z)\} \\ &= \frac{1}{\Gamma(\nu)}\int_z^\infty (\zeta - z)^{\nu-1} f(\zeta)\,d\zeta \quad (\mathrm{Re}(\nu) > 0;\, f \in \mathscr{A}). \end{aligned}$$

2. Preliminary results

We first prove the following result which will be required in proving Theorem 1 below.

LEMMA 1. *Let*

(i) P, Q, M, N *be positive integers such that* $1 \le M \le Q$ *and* $0 \le N \le P$;

(ii) $\mathrm{Re}(\lambda) > \mathrm{Re}(k)$; $\mathrm{Re}[k + m(b_j/\beta_j)] > 0$ $(j = 1, \dots, M)$; $m > 0$;

(iii) $|\arg(A)| < (1/2)\pi\Omega$, *where* Ω *is given by* (1.4).

Then

$$\tag{2.1} \begin{aligned} &\mathscr{W}^{k-\lambda}\left\{ t^{-\lambda} H_{P,Q}^{M,N}\left[A(x/t)^m \,\middle|\, \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] \right\} \\ &= t^{-k} H_{P+1,Q+1}^{M,N+1}\left[A(x/t)^m \,\middle|\, \begin{matrix} (1-k, m), (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q}, (1-\lambda, m) \end{matrix} \right]. \end{aligned}$$

PROOF. The assertion (2.1) of Lemma 1 is derivable from a more general result given by Raina [13, p. 40, Equation (3.3)], which involves the H-function of two variables (cf. [16]). However, for the convenience of the interested reader, we present here a *direct* proof of Lemma 1.

Let $\omega(t)$ denote the first member of the assertion (2.1). Then, making use of (1.6) and the definition of the H-function [16, p. 3]:

$$\tag{2.2} H_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] = \frac{1}{2\pi i}\int_{\mathscr{L}} \Theta(s) z^s\, ds,$$

where $\mathscr{L}$ is a suitable contour of Mellin-Barnes type in the complex s-plane, and

$$\tag{2.3} \Theta(s) = \frac{\prod_{j=1}^{M}\Gamma(b_j - \beta_j s)\prod_{j=1}^{N}\Gamma(1 - a_j + \alpha_j s)}{\prod_{j=M+1}^{Q}\Gamma(1 - b_j + \beta_j s)\prod_{j=N+1}^{P}\Gamma(a_j - \alpha_j s)},$$

https://doi.org/10.1017/S1446788700032821 Published online by Cambridge University Press

we find that

$$(2.4)\quad \omega(t) = \frac{1}{\Gamma(\lambda-k)} \int_t^\infty (\zeta - t)^{\lambda-k-1} \left(\frac{1}{2\pi i} \int_{\mathscr{L}} \Theta(s) A^s x^{ms} \zeta^{-\lambda-ms}\, ds \right) d\zeta.$$

Assuming the inversion of the order of integration in (2.4) to be permissible by absolute (and uniform) convergence of the integrals involved above, we have

$$(2.5)\quad \omega(t) = \frac{1}{2\pi i \Gamma(\lambda-k)} \int_{\mathscr{L}} \Theta(s) A^s x^{ms} \left(\int_t^\infty (\zeta - t)^{\lambda-k-1} \zeta^{-\lambda-ms}\, d\zeta \right) ds.$$

The inner integral in (2.5) can be evaluated under hypothesis (ii) of Lemma 1, and we obtain

$$(2.6)\qquad \omega(t) = \frac{t^{-k}}{2\pi i} \int_{\mathscr{L}} \Theta(s) \frac{\Gamma(k+ms)}{\Gamma(\lambda+ms)} A^s \left(\frac{x}{t}\right)^{ms} ds,$$

which yields the second member of (2.1) by means of definitions (2.2) and (2.3).

Finally, the H-functions occurring in (2.1) exist (and are analytic) under hypotheses (i) and (iii) of Lemma 1, and the Weyl fractional integral on the left-hand side of (2.1) converges absolutely under hypothesis (ii). Thus the assertion (2.1) of Lemma 1 holds true as stated already.

Next we apply Lemma 1 to prove an integral relation, involving the H-function.

THEOREM 1. *Under the sufficient conditions* (i), (ii), *and* (iii) *of Lemma* 1,

$$(2.7)\quad \begin{aligned} &\int_0^\infty t^{-k} H_{P+1,Q+1}^{M,N+1} \left[A(x/t)^m \middle| \begin{matrix} (1-k, m), (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q}, (1-\lambda, m) \end{matrix} \right] f(t)\, dt \\ &= \int_0^\infty t^{-\lambda} H_{P,Q}^{M,N} \left[A(x/t)^m \middle| \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] \mathscr{D}^{k-\lambda}\{f(t)\}\, dt, \end{aligned}$$

provided further that $f \in \mathscr{A}$ *and* $x > 0$.

PROOF. Let I denote the first member of the assertion (2.7) of Theorem 1. Then, by Lemma 1 and the definition (1.6), we have

$$(2.8)\quad \begin{aligned} I &= \int_0^\infty f(t) \left(\int_t^\infty \frac{(u-t)^{\lambda-k-1}}{\Gamma(\lambda-k)} u^{-\lambda} H_{P,Q}^{M,N} \left[A(x/u)^m \middle| \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] du \right) dt \\ &= \int_0^\infty u^{-\lambda} H_{P,Q}^{M,N} \left[A(x/u)^m \middle| \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] \left(\int_0^u \frac{(u-t)^{\lambda-k-1}}{\Gamma(\lambda-k)} f(t)\, dt \right) du. \end{aligned}$$

The change in the order of integration is assumed to be permissible just as in the proof of Lemma 1.

Now, by appealing to definition (1.5), (2.8) gives

$$(2.9)\quad I = \int_0^\infty u^{-\lambda} H_{P,Q}^{M,N}\left[A(x/u)^m \Big| \begin{matrix}(a_j, \alpha_j)_{1,P}\\ (b_j, \beta_j)_{1,Q}\end{matrix}\right] \mathscr{D}^{k-\lambda}\{f(u)\}\, du \quad (f \in \mathscr{A}),$$

which is precisely the right-hand member of (2.7). This completes the proof of Theorem 1.

3. Solution of a hypergeometric form of the integral equation (1.2)

We now search for methods by which we can find the solution of a certain hypergeometric form of the integral equation (1.2). A formidable method of inverting (1.2) is to use the reduction technique by means of which a given integral equation may be reduced to some simpler (and easily invertible) integral transform with the aid of results derived in the preceding sections.

To this end, we notice immediately that Theorem 1 yields a relationship expressing a certain generalized hypergeometric transform in terms of a generalized Stieltjes transform (see, for example, [14, p. 119]). Indeed, we set $\alpha_j = 1 (j = 1, \dots, P)$ and $\beta_j = 1 (j = 1, \dots, Q)$, and modify the parameters of the H-functions in such a way that use can be made of the relationship (cf., for example, [16, p. 18]):

(3.1)

$$H_{P,Q+1}^{1,P}\left[z \Big| \begin{matrix}(1-a_j, 1)_{1,P}\\ (0,1), (1-b_j, 1)_{1,Q}\end{matrix}\right] = \frac{\prod_{j=1}^{P}\Gamma(a_j)}{\prod_{j=1}^{Q}\Gamma(b_j)}\, {}_PF_Q\left[\begin{matrix}a_1, \dots, a_P;\\ b_1, \dots, b_Q;\end{matrix}\ -z\right]$$

on the left-hand side of (2.7), and of the *simpler* relationship [16, p. 18]:

$$(3.2)\qquad H_{1,1}^{1,1}\left[z \Big| \begin{matrix}(1-\lambda, 1)\\ (0,1)\end{matrix}\right] = \frac{\Gamma(\lambda)}{(1+z)^{\lambda}} = \Gamma(\lambda)\ {}_1F_0[\lambda; -; -z]$$

on the right-hand side of (2.7). Thus the following result emerges from Theorem 1.

COROLLARY 1. *Let a, b, and c be complex parameters such that*

$$\mathrm{Re}(c) > \mathrm{Re}(b) > 0.$$

Also let $m \in \mathbb{N}$ and $f \in \mathscr{A}$.

Then, for all $x > 0$,

$$\int_0^\infty \frac{\Gamma(b)}{t^b}\,{}_{m+1}F_m[a, \Delta(m;b); \Delta(m;c); -(x/t)^m] f(t)\,dt \tag{3.3}$$
$$= \Gamma(c)\int_0^\infty \frac{t^{am-c}\mathscr{D}^{b-c}\{f(t)\}}{(x^m+t^m)^a}\,dt,$$

where

$$\Delta(m;\lambda) = \left\{\frac{\lambda}{m}, \frac{\lambda+1}{m}, \cdots, \frac{\lambda+m-1}{m}\right\} (m \in \mathbb{N}). \tag{3.4}$$

We are now in a position to explicitly obtain the solution of the integral equation (3.3) which is a particular (hypergeometric) case of the integral equation (1.2). Our final result is contained in

THEOREM 2. *Let* a, b, *and* c *be complex parameters such that*

$$\mathrm{Re}(a) > 1 \text{ and } \mathrm{Re}(c) > \mathrm{Re}(b) > 0.$$

Suppose also that $m \in \mathbb{N}$ *and* $f \in \mathscr{A}$.

Then, for all $x > 0$ *and* $g \in \mathscr{A}$, *the integral equation*

$$\int_0^\infty \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} t^{-b}\,{}_{m+1}F_m[a, \Delta(m;b); \Delta(m;c); -(x/t)^m] f(t)\,dt = g(x), \tag{3.5}$$

possesses the solution given by

$$f(x) = m\mathscr{D}_x^{c-b}\left\{x^{m(1-a)+c-1}\mathscr{D}_{x^m}^{1-a}\left\{(1+x)\lim_{n\to\infty} L_{n,x^m}[g(x)]\right\}\right\}, \tag{3.6}$$

where

$$L_{n,x}[g(x)] = \frac{(-x)^{n-1}}{n!(n-2)!}\frac{d^{2n-1}}{dx^{2n-1}}\{x^n g(x)\} \quad (n = 2, 3, 4, \ldots). \tag{3.7}$$

PROOF. In view of Corollary 1, we may write (3.5) in the form:

$$\Gamma(a)\int_0^\infty \frac{t^{am-c}}{(x^m+t^m)^a}\,\mathscr{D}^{b-c}\{f(t)\}\,dt = g(x). \tag{3.8}$$

A slight change in the variable on the left side (or using the linear functional relationship of Srivastava [14, p. 119]) gives

$$\frac{\Gamma(a)}{m}\int_0^\infty \frac{t^{a-1+(1-c)/m}}{(x^m+t)^a}(\mathscr{D}^{b-c}f)(t^{1/m})\,dt = g(x), \tag{3.9}$$

where, for convenience, we write [cf. Equation (1.5)]

$$(\mathscr{D}^{b-c}f)(t^{1/m}) = \mathscr{D}^{b-c}_{t^{1/m}}\left\{f(t^{1/m})\right\}. \tag{3.10}$$

Now apply the following result of Love [9, p. 281]:

$$\int_0^\infty \frac{\mathscr{D}^{\rho-1}\{\varphi(t)\}}{x+t}\,dt = \int_0^\infty \frac{\Gamma(\rho)\varphi(t)}{(x+t)^\rho}\,dt \qquad (x>0;\ \mathrm{Re}(\rho)>1), \tag{3.11}$$

where $\mathscr{D}^{\rho-1}\{\varphi(t)\}$ is assumed to exist, and

$$(1+t)^{p+q-1}t^{-q}\mathscr{D}^{\rho-1}\{\varphi(t))\}$$

is assumed to be locally integrable on the interval $[0,\infty)$ for $p\geq 0$, $q\geq 0$, and $\mathrm{Re}(\rho)>1$. We thus find from (3.9) that

$$\frac{1}{m}\int_0^\infty (x+t)^{-1}\mathscr{D}^{a-1}_t\left\{t^{a-1+(1-c)/m}\mathscr{D}^{b-c}_{t^{1/m}}\{f(t^{1/m})\}\right\}dt = g(x^{1/m}). \tag{3.12}$$

By appealing appropriately to Theorem 9 of Widder [18, p. 345] concerning the inversion of the Stieltjes transform, we get

$$\mathscr{D}^{a-1}_t\left\{t^{a-1+(1-c)/m}\mathscr{D}^{b-c}_{t^{1/m}}\left\{f(t^{1/m})\right\}\right\} = m(1+t^{1/m})\lim_{n\to\infty} L_{n,x}[g(x^{1/m})], \tag{3.13}$$

where $g(x^{1/m})$ as a function of x is operated upon by $L_{n,x}$ and then x is replaced by t. The solution (3.6) now follows from (3.13).

For $m=1$, Theorem 2 is seen to correspond to a result given by Love [9, p. 284] under less stringent conditions.

4. Use of other methods

The solution of the integral equation (3.5) can be derived by other methods also. In fact, a solution of the main integral equation (1.2) involving the H-function in the kernel is obtainable by resorting to the application of Mellin transforms (see Gupta and Mittal [5]). Thus, on replacing f by $\mathscr{D}^{\lambda-k}f$ in (2.7), and applying (2.1), we have

$$g(x) = \int_0^\infty t^{-k}H^{M,N+1}_{P+1,Q+1}\left[A(x/t)^m \middle| \begin{matrix}(1-k,m),(a_j,\alpha_j)_{1,P}\\ (b_j,\beta_j)_{1,Q},(1-\lambda,m)\end{matrix}\right]\mathscr{D}^{\lambda-k}\{f(t)\}\,dt. \tag{4.1}$$

Multiplying both sides of (4.1) by x^{s-1}, and integrating with respect to x

https://doi.org/10.1017/S1446788700032821 Published online by Cambridge University Press

from 0 to ∞, we have

(4.2)

$$\varphi(s) = \int_0^\infty x^{s-1} g(x)dx$$
$$= \int_0^\infty t^{-k} \mathscr{D}^{\lambda-k}\{f(t)\}$$
$$\cdot \left(\int_0^\infty x^{s-1} H_{P+1,Q+1}^{M,N+1} \left[A(x/t)^m \middle| \begin{matrix} (1-k, m), (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q}, (1-\lambda, m) \end{matrix} \right] dx \right) dt,$$

where we have assumed the absolute (and uniform) convergence of the integrals involved, with a view to justifying the inversion of the order of integration.

Now evaluate the inner integral in (4.2) by a simple change of variables, followed by an appeal to the familiar results (cf., for example, [4] and [16, p. 15]):

$$\int_0^\infty x^{s-1} H_{P,Q}^{M,N} \left[zx \middle| \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] dx = z^{-s} \Theta(-s), \tag{4.3}$$

where, for convergence, $\Omega > 0$, $|\arg(z)| < (1/2)\pi\Omega$, and

$$-\min\{\operatorname{Re}(b_j/\beta_j)\} < \operatorname{Re}(s) < \min\{\operatorname{Re}[(1-a_l)/\alpha_l]\} \qquad (j = 1, \dots, M;\, l = 1, \dots, N), \tag{4.4}$$

$\Theta(s)$ and Ω being given by (2.3) and (1.4), respectively. We thus find from (4.2) that

$$\varphi(s) = \frac{\Gamma(k-s)}{m\Gamma(\lambda-s)} A^{-s/m} \Theta(-s/m) \int_0^\infty t^{s-k} \mathscr{D}^{\lambda-k}\{f(t)\}\, dt, \tag{4.5}$$

where $\varphi(s)$ is given by (4.2).

Inverting (4.5) by applying the Mellin inversion theorem [17, p. 46], we have

$$\mathscr{D}^{\lambda-k}\{f(t)\} = \frac{m}{2\pi i} \lim_{\gamma\to\infty} \int_{\sigma-i\gamma}^{\sigma+i\gamma} \frac{\Gamma(\lambda-s)}{\Gamma(k-s)} [\Theta(-s/m)]^{-1} t^{k-s-1} \varphi(s)\, ds. \tag{4.6}$$

Operating upon both sides by $\mathscr{D}^{k-\lambda}$, (4.6) gives us

$$f(t) = \frac{m}{2\pi i} \mathscr{D}^{k-\lambda} \left\{ \lim_{\gamma\to\infty} \int_{\sigma-i\gamma}^{\sigma+i\gamma} \frac{\Gamma(\lambda-s)}{\Gamma(k-s)} [\Theta(-s/m)]^{-1} t^{k-s-1} \varphi(s)\, ds \right\}, \tag{4.7}$$

which finally yields

$$f(x) = \frac{mx^{\lambda-1}}{2\pi i} \lim_{\gamma\to\infty} \int_{\sigma-i\gamma}^{\sigma+i\gamma} [\Theta(-s/m)]^{-1} x^{-s} \varphi(s)\, ds \tag{4.8}$$

as the solution of the integral equation (1.2), provided that the involved interchange is permissible. The above details may be summarized in

THEOREM 3. *If* $f(t) \in \mathscr{A}$, $\mathscr{D}^{\lambda-k}\{f(t)\}$ *exists,* $m > 0$, $x > 0$, $|\arg(A)| < (1/2)\pi\Omega$, $\Omega > 0$ (Ω *being given by* (1.4)), *and* $\mathrm{Re}(\lambda) > \mathrm{Re}(k) > 0$, *then the solution of the integral equation* (4.1) *is given by* (4.8), *provided further that*

$$\max\{\mathrm{Re}[(a_l - 1)/\alpha_l]\} < -\mathrm{Re}(s/m) < \min\{\mathrm{Re}(b_j/\beta_j)\} \quad (j = 1, \ldots, M; l = 1, \ldots, N). \tag{4.9}$$

Acknowledgements

The present investigation was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353. The second author wishes to thank the University Grants Commission of India for providing financial support for this work. The authors wish to thank the referee for suggesting a number of improvements in the paper.

References

[1] R. G. Buschman, 'An inversion integral', *Proc. Amer. Math. Soc.* **13** (1962), 675–677.
[2] A. Erdélyi, 'An integral equation involving Legendre functions', *J. Soc. Indust. Appl. Math.* **12** (1964), 15–30.
[3] C. Fox, 'The G and H functions as symmetrical Fourier kernels', *Trans. Amer. Math. Soc.* **98** (1961), 395–429.
[4] K. C. Gupta and U. C. Jain, 'The H-function. II', *Proc. Nat. Acad. Sci. India Sect. A* **36** (1966), 594–609.
[5] K. C. Gupta and P. K. Mittal, 'The H-function transform', *J. Austral. Math. Soc.* **11** (1970), 142–148.
[6] T. P. Higgins, 'A hypergeometric function transform', *J. Soc. Industr. Appl. Math.* **12** (1964), 601–612.
[7] M. J. Lighthill, *Introduction to Fourier analysis and generalised functions* (Cambridge Monographs on Mechanics and Applied Mathematics, Cambridge University Press, Cambridge, London, and New York, 1958).
[8] E. R. Love, 'Some integral equations involving hypergeometric functions', *Proc. Edinburgh Math. Soc.* (2) **15** (1967), 169–198.
[9] E. R. Love, 'A hypergeometric integral equation', in *Fractional calculus and its applications*, edited by B. Ross, pp. 272–288 (Springer-Verlag, New York, 1975).
[10] E. R. Love, T. R. Prabhakar, and N. K. Kashyap, ' A confluent hypergeometric integral equation', *Glasgow Math. J.* **23** (1982), 31–40.
[11] T. R. Prabhakar, 'A class of integral equations with Gauss functions in the kernels', *Math. Nachr.* **52** (1972), 71–83.
[12] T. R. Prabhakar and N. K. Kashyap, 'A new class of hypergeometric integral equations', *Indian J. Pure Appl. Math.* **11** (1980), 92–97.
[13] R. K. Raina, 'On the Weyl fractional differentiation', *Pure Appl. Math. Sci.* **10** (1979), 37–41.

[14] H. M. Srivastava, 'Some remarks on a generalization of the Stieltjes transform', *Publ. Math. Debrecen* **23** (1976), 119–122.

[15] H. M. Srivastava and R. G. Buschman, *Convolution integral equations with special function kernels* (Halsted Press, John Wiley and Sons, New York, 1977).

[16] H. M. Srivastava, K. C. Gupta, and S. P. Goyal, *The H-functions of one and two variables with applications* (South Asian Publishers, New Delhi, 1982).

[17] E. C. Titchmarsh, *Introduction to the theory of Fourier integrals* (Second Edition, Oxford University Press, London, Clarendon Press, Oxford, 1948).

[18] D. V. Widder, *The Laplace transform* (Princeton University Press, Princeton, New Jersey, 1946).

Department of Mathematics
and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada

Department of Mathematics
College of Technology and
Agricultural Engineering
(Rajasthan Agricultural University)
Udaipur-313001, Rajasthan
India