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AN EXTENSION OF THE GENERALISED SCHUR INEQUALITY

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The well-known Schur inequality relates the sum of the squares of the absolute values of the eigenvalues of A to the elements of A. This was recently generalised to powers between one and two. Here we show that the inequality holds for powers between zero and two.

Let A be an $n \times n$ matrix, real or complex, with eigenvalues $\lambda_1, \dots, \lambda_n$. The Schur inequality

$$\sum_{i=1}^{n} |\lambda_i|^2 \leqslant \sum_{i,j=1}^{n} |a_{ij}|^2 \tag{1}$$

is well-known [**2**, p.133].

Petri and Ikramov [**3**] generalised the Schur inequality to

$$\sum_{i=1}^{n} |\lambda_i|^p \leqslant \sum_{i,j=1}^{n} |a_{ij}|^p \tag{2}$$

where $1 \leqslant p \leqslant 2$.

Ikramov [**1**] proved, for any $n \times n$ matrix A with singular values $s_1, s_2, \dots, s_n$, the following result:

$$\sum_{i=1}^{n} s_i^p \leqslant \sum_{i,j=1}^{n} |a_{ij}|^p , \tag{3}$$

where $1 \leqslant p \leqslant 2$.

Now (2) is a simple consequence of (3) by the well-known Weyl inequality:

We assume that the singular values of A constitute a non-increasing sequence

$$s_1 \geqslant s_2 \geqslant \dots \geqslant s_n$$

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and that the eigenvalues of A are numbered in accordance with their magnitudes

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|.$$

Then for $1 \leqslant k \leqslant n$ and $0 < p < \infty$

$$\sum_{i=1}^{k} |\lambda_i|^p \leqslant \sum_{i=1}^{k} s_i^p. \tag{4}$$

We now have

THEOREM 1. *For any $n \times n$ matrix A with singular values $s_1, \ldots, s_n$, inequality (3) is valid for $0 < p \leqslant 2$. For $p \geqslant 2$, the reverse inequality holds.*

PROOF: The proof follows closely that of Theorem 1 of [1]. Let $\tau_1, \ldots, \tau_n$ denote the ℓ_2 norms of the row vectors $(a_{i1}, a_{i2}, \ldots, a_{in})$ numbered so that they form the non-increasing sequence

$$\tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_n.$$

It is well-known [3] that the sequence $\tau_1^2, \ldots, \tau_n^2$ is majorised by $s_1^2, \ldots, s_n^2$, that is,

$$\sum_{i=1}^{k} \tau_i^2 \leqslant \sum_{i=1}^{k} s_i^2, \qquad 1 \leqslant k \leqslant n$$

$$\sum_{i=1}^{n} \tau_i^2 = \sum_{i=1}^{n} s_i^2.$$

So if f is a concave function, we have

$$\sum_{i=1}^{n} f(s_i^2) \leqslant \sum_{i=1}^{n} f(\tau_i^2) \tag{5}$$

and the reverse inequality holds in (5) if f is a convex function.

In particular, the function $f(x) = x^{p/2}$ is concave for $x > 0$ if $0 < p \leqslant 2$ and convex for $p \geqslant 2$. Therefore

$$\sum_{i=1}^{n} s_i^p \leqslant \sum_{i=1}^{n} \tau_i^p \tag{6}$$

holds for $0 < p \leqslant 2$ and the reverse inequality holds for $p \geqslant 2$.

On the other hand, for ℓ_p norms of any row vector, we have

$$\left(\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2} \leqslant \left(\sum_{j=1}^{n} |a_{ij}|^p\right)^{1/p} \tag{7}$$

for $0 < p \leqslant 2$ and the reverse inequality for $p \geqslant 2$.

Taking the pth power of both sides in (7) and adding the inequalities for $i = 1, \dots, n$, we obtain

$$\sum_{i=1}^{n} \tau_i^p \leqslant \sum_{i,j=1}^{n} |a_{ij}|^p \tag{8}$$

for $0 < p \leqslant 2$, and the reverse inequality for $p \geqslant 2$.

The assertion of the theorem now follows from (6) and (8) and their reversals. □

Theorem 2. *Let A be an $n \times n$ matrix, real or complex, with eigenvalues $\lambda_1, \dots, \lambda_n$. Then (2) is valid for $0 < p \leqslant 2$.*

Proof: This is a simple consequence of Theorem 1 and Weyl's inequalities (4). □

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