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AN ALTERNATIVE PERSPECTIVE ON PROJECTIVITY OF MODULES

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Dedicated to the memory of our dear colleague and friend Francisco Raggi.

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Abstract. We approach the analysis of the extent of the projectivity of modules from a fresh perspective as we introduce the notion of relative subprojectivity. A module M is said to be N-*subprojective* if for every epimorphism $g : B \to N$ and homomorphism $f : M \to N$, there exists a homomorphism $h : M \to B$ such that $gh = f$. For a module M, the *subprojectivity domain of* M is defined to be the collection of all modules N such that M is N-subprojective. We consider, for every ring R, the subprojective profile of R, namely, the class of all subprojectivity domains for R modules. We show that the subprojective profile of R is a semi-lattice, and consider when this structure has coatoms or a smallest element. Modules whose subprojectivity domain is as smallest as possible will be called *subprojectively poor* (*sp-poor*) or *projectively indigent* (*p-indigent*), and those with co-atomic subprojectivy domain are said to be *maximally subprojective*. While we do not know if *sp*-poor modules and maximally subprojective modules exist over every ring, their existence is determined for various families. For example, we determine that artinian serial rings have *sp*-poor modules and attain the existence of maximally subprojective modules over the integers and for arbitrary V-rings. This work is a natural continuation to recent papers that have embraced the systematic study of the injective, projective and subinjective profiles of rings.

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1. Introduction and Preliminaries. The purpose of this paper is to initiate the study of an alternative perspective on the analysis of the projectivity of a module, as we introduce the notions of relative subprojectivity and assign to every module its subprojectivity domain. A module is projective if and only if its subprojectivity domain consists of all modules. Therefore, at this extreme, there is no difference in the role played by the projectivity and subprojectivity domains. Interesting things arise, however, when we focus on the subprojectivity domain of modules which are not projective. It is easy to see that every module is subprojective relative to all projective modules, and one can show (Proposition 2.8) that projective modules are the only

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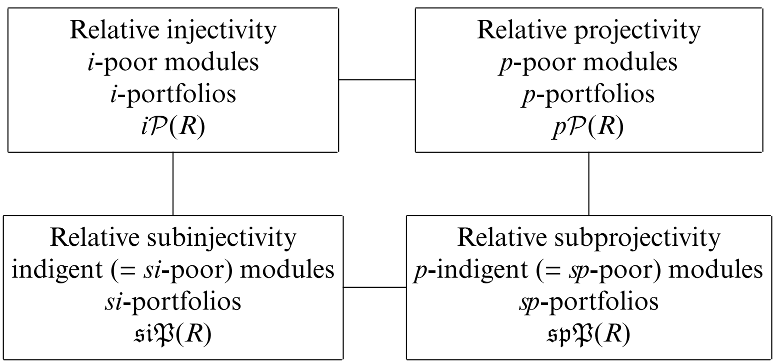
ones sharing the distinction of being in every single subprojectivity domain. It is thus tempting to ponder the existence of modules whose subprojectivity domain consists precisely of only projective modules. We refer to these modules as *sp-poor* or, to keep in line with [**5**], we sometimes use the expression *p-indigent*.

This paper is inspired by similar ideas and notions studied in several papers. On the one hand, relative injectivity, injectivity domains and the notion of a *poor* module (modules with the smallest possible injectivity domain) have been studied in [**1**, **10**, **15**]. Dually, relative projectivity, projectivity domains and the notion of a *p-poor* module have been studied in [**13**, **15**]. On the other hand, in [**15**] the authors name a class of modules an *i*-portfolio (resp. *p*-portfolio) if it coincides with the injectivity (resp. projectivity) domain of some module. Then they proceed to define the injective profile (resp. projective profile) of a ring R, an ordered structure consisting of all the *i*-portfolios (resp. *p*-portfolios) in Mod-R. In this paper, we study these concepts in the context of subinjectivity and subprojectivity domains, thus obtaining the ordered invariants $\mathfrak{si}\mathfrak{P}(R)$ and $\mathfrak{sp}\mathfrak{P}(R)$, the subinjective and subprojective profile of R respectively. We study some of its properties, such as the existence of coatoms and their relations with the lattice of torsion theories in Mod-R.

One of the first things that comes to the surface in this type of study is the potential existence of modules which are least injective or projective possible with respect to whichever measuring approach one may be using. Injectively and projectively poor modules have been studied in [**1**, **10**, **13**, **15**]. Aydoğdu and López-Permouth in [**5**] modify in a subtle yet significant way the notion of relative injectivity to obtain *relative subinjectivity*. They also study subinjectivity analogs of poor modules, calling them *indigent*. Here, we study the projective analog of relative subinjectivity and indigent modules. In order to emphasize the analogy between poor and indigent modules, we also call indigent modules *subinjectively poor*, or *si*-poor for short. In this same spirit, we call the subprojective analog of indigent modules either *p*-indigent or *sp*-poor.

We depict the different analogies between the different ways of measuring the injectivity and projectivity of a module in the following diagram.



Before tackling the rest of the paper, we finish the section with a review of some of the needed background material. Throughout, R will denote an associative ring with identity and modules will be unital right R-modules, unless otherwise explicitely stated. As usual, we denote by Mod-R the category of right R-modules. If M is an R-module, then rad(M), Soc(M) and pr.dim(M) will respectively denote the Jacobson radical, socle and projective dimension of M. The Jacobson radical of a ring R will be denoted

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by $J(R)$. A ring R is called a *right V-ring* if every simple R-module is injective; a *right hereditary ring* if submodules of projective modules are projective or, equivalently, if quotients of injective modules are injective; a *right perfect ring* if every module has a projective cover; a *semi-primary ring* if $J(R)$ is nilpotent and $R/J(R)$ is a semi-simple artinian ring; and a *right coherent ring* if every finitely generated right ideal is finitely presented or, equivalently, if products of flat left R-modules are flat.

In [**15**], torsion theory is used as a tool in the study of relative injectivity and projectivity. Such notions are also employed here, so, for easy reference, we recall them now. A torsion theory $\mathbb{T}$ is a pair of classes of modules $(\mathcal{T}, \mathcal{F})$ such that (i) $\mathrm{Hom}(M, N) = 0$ for every $M \in \mathcal{T}$, $N \in \mathcal{F}$; (ii) if $\mathrm{Hom}(A, N) = 0$ for all $N \in \mathcal{F}$, then $A \in \mathcal{T}$; and (iii) if $\mathrm{Hom}(M, B) = 0$ for all $M \in \mathcal{T}$, then $B \in \mathcal{F}$. In this situation, $\mathcal{T}$ and $\mathcal{F}$ are called the torsion class and torsion-free class of $\mathbb{T}$ respectively. A class of modules $\mathcal{T}$ is the torsion class of some torsion theory if and only if it is closed under quotients, extensions and arbitrary direct sums. Similarly, a class of modules $\mathcal{F}$ is the torsion-free class of some torsion theory if and only if it is closed under submodules, extensions and arbitrary direct products. If $\mathcal{C}$ is a class of modules, then $\mathbb{T}_{\mathcal{C}} := (\mathcal{T}_{\mathcal{C}}, \mathcal{F}_{\mathcal{C}})$, where $\mathcal{F}_{\mathcal{C}} = \{N \in \text{Mod-}R : \mathrm{Hom}(C, N) = 0 \text{ for every } C \in \mathcal{C}\}$ and $\mathcal{T}_{\mathcal{C}} = \{M \in \text{Mod-}R : \mathrm{Hom}(M, N) = 0 \text{ for every } N \in \mathcal{F}_{\mathcal{C}}\}$ is said to be the torsion theory generated by $\mathcal{C}$. Similarly, $\mathbb{T}^{\mathcal{C}} = (\mathcal{T}^{\mathcal{C}}, \mathcal{F}^{\mathcal{C}})$ where $\mathcal{T}^{\mathcal{C}} = \{M \in \text{Mod-}R : \mathrm{Hom}(M, C) = 0 \text{ for every } C \in \mathcal{C}\}$ and $\mathcal{F}^{\mathcal{C}} = \{N \in \text{Mod-}R : \mathrm{Hom}(M, N) = 0 \text{ for every } M \in \mathcal{T}^{\mathcal{C}}\}$ is said to be the torsion theory cogenerated by $\mathcal{C}$, [**17**, Chapter VI]. The torsion theory $\mathbb{T}_{\mathcal{C}}$ (resp. $\mathbb{T}^{\mathcal{C}}$) can also be characterized as the smallest torsion theory such that every object in $\mathcal{C}$ is torsion (resp. torsion-free). If $M \in \text{Mod-}R$, we write $\mathbb{T}_M$ and $\mathbb{T}^M$ for $\mathbb{T}_{\{M\}}$ and $\mathbb{T}^{\{M\}}$ respectively.

Recall that a module M is said to be quasi-projective if it is projective relative to itself. Over a right perfect ring R, every quasi-projective module M satisfies the following conditions:

- (D1) For every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2 \ll M$.
- (D2) If $A \leq M$ is such that M/A is isomorphic to a direct summand of M, then A is a direct summand of M.
- (D3) If M_1 and M_2 are direct summands of M with $M_1 + M_2 = M$, then $M_1 \cap M_2$ is a direct summand of M.

Modules satisfying (D1) are called lifting, see [**9**]. Modules satisfying (D1) and (D2) are called discrete, while modules satisfying (D1) and (D3) are called quasi-discrete. Every discrete module is quasi-discrete, as it is the case that (D2) $\Rightarrow$ (D3), [**16**, Lemma 4.6]. It is not the case that every projective module is lifting, as, for example, $\mathbb{Z}$ is not a lifting $\mathbb{Z}$-module. However, if R is right perfect, then every projective module is discrete, cf. [**16**, Theorem 4.41]. Every quasi-discrete module decomposes as a direct sum of modules whose every submodule is superfluous, see [**16**, Theorem 4.15].

For additional concepts and results not mentioned here, we refer the reader to [**3**, **4**, **14**].

2. Subprojectivity and the subprojectivity domain of a module

Definition 2.1. Given modules M and N, M is said to be *N-subprojective* if for every epimorphism $g : B \to N$ and for every homomorphism $f : M \to N$, then there exists a homomorphism $h : M \to B$ such that $gh = f$. The *subprojectivity*

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domain, or *domain of subprojectivity*, of a module M is defined to be the collection

$$\underline{\mathfrak{Pr}}^{-1}(M) := \{ N \in \text{Mod-}R \, : \, M \text{ is } N\text{-subprojective} \}.$$

The domain of subprojectivity of a module is a measure of projectivity of that module. Just as with projectivity domains, a module M is projective precisely when $\underline{\mathfrak{Pr}}^{-1}(M)$ is as large as possible (i.e. equal to Mod-R.)

Before we proceed, we need to introduce two additional notions.

Definition 2.2. Let $\mathcal{C} \subseteq$ Mod-R. We say that $\mathcal{C}$ is a *subprojective-portfolio*, or *sp-portfolio* for short, if there exists $M \in$ Mod-R such that $\mathcal{C} = \underline{\mathfrak{Pr}}^{-1}(M)$. The class $\mathfrak{sp}\mathfrak{P}(R) := \{\mathcal{C} \subseteq \text{Mod-}R : \mathcal{C} \text{ is an } \mathfrak{sp}\text{-portfolio}\}$ will be named the *subprojective profile*, or sp-*profile*, of R.

Our first lemma says that, in order for M to be N-subprojective, one only needs to lift maps to projective modules that cover N, to free modules that cover N or even to a single projective module that covers N.

Lemma 2.3. *Let $M, N \in$ Mod-R. Then the following conditions are equivalent.*
(1) *M is N-subprojective.*
(2) *For every $f : M \to N$ and every epimorphism $g : P \to N$ with P projective, there exists $h : M \to P$ such that $gh = f$.*
(3) *For every $f : M \to N$ and every epimorphism $g : F \to N$ with F free, there exists $h : M \to F$ such that $gh = f$.*
(4) *For every $f : M \to N$ there exists an epimorphism $g : P \to N$ with P projective and a morphism $h : M \to P$ such that $gh = f$.*

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are clear. To show $(4) \Rightarrow (1)$, assume (4) and let $f : M \to N$ be a morphism and $\overline{g} : B \to N$ be an epimorphism. By (4), there exist an epimorphism $g : P \to N$ and a morphism $h : M \to P$ such that $gh = f$. Since P is projective, there exists a morphism $\overline{h} : P \to B$ such that $g = \overline{g}\overline{h}$. Then $\overline{h}h : M \to B$ and $\overline{g}\overline{h}h = gh = f$. Hence, M is N-subprojective. □

Using the preceding lemma, we can show that a module M is projective if and only if it is M-subprojective, thus ruling out the possibility of a non-trivial subprojective analogue to the notion of quasi-projectivity.

Proposition 2.4. *For any module M, the following are equivalent:*
(1) *M is projective.*
(2) *$M \in \underline{\mathfrak{Pr}}^{-1}(M)$.*

Proof. The implication $(1) \Rightarrow (2)$ is clear. For $(2) \Rightarrow (1)$, put $M = N$ and $f = 1_M$, the identity morphism on M in the condition of Lemma 2.3 (condition 4), to see that M is a direct summand of a projective module. Hence, M is projective. □

Some modules can be shown easily to belong to a subprojectivity domain.

Proposition 2.5. *If* $\text{Hom}_R(M, A) = 0$*, then* $A \in \underline{\mathfrak{Pr}}^{-1}(M)$.

Proof. If $\text{Hom}_R(M, A) = 0$, then given any epimorphism $g : C \to A$ if we let $h : M \to C$ be the zero mapping, then $gh = 0$. Hence, $A \in \underline{\mathfrak{Pr}}^{-1}(M)$. □

As an easy consequence of Proposition 2.5, we have the following.

Corollary 2.6. *Let M and A be right R-modules. Then,*

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(1) *If* $\mathrm{rad}(M) = M$ *and* $\mathrm{rad}(A) = 0$, *then* M *is* A*-subprojective.*
(2) *If* M *is singular and* A *is non-singular, then* M *is* A*-subprojective.*
(3) *If* M *is semi-simple and* $\mathrm{Soc}(A) = 0$, *then* M *is* A*-subprojective.*

Proposition 2.5 is also instrumental in figuring out the next example, where we see that sometimes the conditioned study in that proposition actually characterizes certain subprojectivity domains. This subject will be picked up again later in Proposition 3.2.

EXAMPLE 2.7. We see that in the category of $\mathbb{Z}$-modules, $\underline{\mathfrak{Pr}}^{-1}(\mathbb{Q})$ consists precisely of the abelian groups granted by Proposition 2.5 for, if there is a non-zero morphism $f : \mathbb{Q} \to M$, let $\pi : F \to M$ be an epimorphism with F free. Since there are no non-zero morphisms from $\mathbb{Q}$ to F, we cannot lift f to a morphism $\mathbb{Q} \to F$, so $M \notin \underline{\mathfrak{Pr}}^{-1}(\mathbb{Q})$. Consequently, the subprojectivity domain of $\mathbb{Q}$ consists precisely of the class of reduced abelian groups. Note that a similar technique can be used to find the subprojectivity domain of any divisible abelian group. We further explore this phenomena in Section 3 of this paper.

It is a natural question to ask how small $\underline{\mathfrak{Pr}}^{-1}(M)$ can be. The next proposition shows that the domain of subprojectivity of any module must contain at least the projective modules, and the projective modules are the only ones that belong to all *sp*-portfolios.

PROPOSITION 2.8. *The intersection* $\bigcap \underline{\mathfrak{Pr}}^{-1}(M)$, *running over all* R*-modules* M, *is precisely* $\{P \in \mathrm{Mod}\text{-}R \mid P \text{ is projective}\}$.

Proof. To show the containment $\subseteq$, suppose M is a module which is subprojective relative to all R-modules. Then, in particular, $M \in \underline{\mathfrak{Pr}}^{-1}(M)$. So by Proposition 2.4, M is projective.

To show the containment $\supseteq$, let P be a projective module and M be any R-module. Let $g : B \to P$ be an epimorphism and $f : M \to P$ be a homomorphism. Now, since P is projective, g splits and so there exists a homomorphism $k : P \to B$ such that $gk = 1_P$. Then $g(kf) = (gk)f = f$ and so by definition $P \in \underline{\mathfrak{Pr}}^{-1}(M)$. Since M was arbitrary, the result follows. □

Proposition 2.8 provides a lower bound on how small the domain of subprojectivity of a module can be. If a module does achieve this lower bound, then we will call it suprojectively poor.

DEFINITION 2.9. A module M is called *subprojectively poor*, *sp-poor*, or *p-indigent*, if its subprojectivity domain consists of only projective modules.

Note that it is not clear whether *sp*-poor modules over a ring R must exist. Section 4 will be devoted to this problem, but first we go deeper into our study of subprojectivity.

The following several propositions show that subprojectivity domains behave nicely with respect to direct sums.

PROPOSITION 2.10. *Let* $\{M_i\}_{i \in I}$ *be a set of* R*-modules. Then,* $\underline{\mathfrak{Pr}}^{-1}(\bigoplus_{i\in I} M_i) = \bigcap_{i \in I} \underline{\mathfrak{Pr}}^{-1}(M_i)$, *that is, the subprojectivity domain of a direct sum is the intersection of the subprojectivity domains of the summands.*

Proof. To show the containment $\subseteq$, let N be in the subprojectivity domain of $\bigoplus_{i\in I} M_i$ and fix $j \in I$. Let $g : B \to N$ be an epimorphism and $f : M \to N$ be a homomorphism. Let $p_j : \bigoplus_{i \in I} M_i \to M_j$ denote the projection map and

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$e_j : M_j \to \bigoplus_{i\in I} M_i$ denote the inclusion map. Since $N \in \underline{\mathfrak{Pr}}^{-1}(\bigoplus_{i\in I} M_i)$, there exists a homomorphism $h' : \bigoplus_{i\in I} M_i \to B$ such that $gh' = fp_j$. Letting $h := h'e_j : M_j \to B$, it is straightforward to check that $gh = f$. Hence, $N \in \underline{\mathfrak{Pr}}^{-1}(M_j)$.

To show the containment $\supseteq$, let N be in the subprojectivity domain of M_i for every $i \in I$, let $g : B \to N$ be an epimorphism, and let $f : \bigoplus_{i\in I} M_i \to N$ be a homomorphism. Since for each $j \in I$, $N \in \underline{\mathfrak{Pr}}^{-1}(M_j)$ then $\exists h_j : M_j \to B$ such that $fe_j = gh_j$. Letting $h := \bigoplus_{i\in I} h_i : \bigoplus_{i\in I} M_i \to B$, then

$$gh = \bigoplus_{i\in I} gh_i = \bigoplus_{i\in I} fe_i = f \bigoplus_{i\in I} e_i = f.$$

Hence, $N \in \underline{\mathfrak{Pr}}^{-1}(\oplus_{i\in I} M_i)$. □

Note that Proposition 2.10 tells us that $\mathfrak{sp}\mathfrak{P}(R)$ is a semi-lattice with the biggest element, namely Mod-R, the subprojectivity domain of any projective R-module. In view of Proposition 2.8, $\mathfrak{sp}\mathfrak{P}(R)$ has the smallest element if and only if R has an sp-poor module.

PROPOSITION 2.11. *If $N \in \underline{\mathfrak{Pr}}^{-1}(M)$, then every direct summand of N is in $\underline{\mathfrak{Pr}}^{-1}(M)$.*

Proof. Suppose A is a direct summand of N, and let $g : C \to A$ be an epimorphism and $f : M \to A$ be a homomorphism. Consider the epimorphism $g \oplus 1 : C \oplus N/A \to A \oplus N/A \cong N$, where $1 : N/A \to N/A$ is the identity map. Since $N \in \underline{\mathfrak{Pr}}^{-1}(M)$, there exists a homomorphism $\hat{h} : M \to C \oplus N/A$ such that $(g \oplus 1)\hat{h} = ef$, where $e : A \to N$ is the inclusion map. Therefore,

$$g(p\hat{h}) = p(g \oplus 1)\hat{h} = p(ef) = f,$$

where $p : N \to A$ denotes the projection map. Hence, $A \in \underline{\mathfrak{Pr}}^{-1}(M)$. □

PROPOSITION 2.12. *If $A_i \in \underline{\mathfrak{Pr}}^{-1}(M)$ for $i \in \{1, \cdots, m\}$, then $\bigoplus_{i=1}^m A_i \in \underline{\mathfrak{Pr}}^{-1}(M)$.*

Proof. By induction, it is sufficient to prove the proposition when $m = 2$. Let $g : C \to A \oplus B$ be an epimorphism and $f : M \to A \oplus B$ be a homomorphism. Since $A \in \underline{\mathfrak{Pr}}^{-1}(M)$, there exists a homomorphism $h_1 : M \to C$ such that $p_A gh_1 = p_A f$, where $p_A : A \oplus B \to A$ is the projection map. So $p_A(gh_1 - f) = 0$ and hence $\mathrm{Im}(gh_1 - f) \subset 0 \oplus B \cong B$. Since $B \in \underline{\mathfrak{Pr}}^{-1}(M)$, there exists a homomorhism $h_2 : M \to g^{-1}(0 \oplus B) \subset C$ such that $gh_2 = gh_1 - f$. Let $h := h_1 - h_2$. Then

$$gh = gh_1 - gh_2 = gh_1 - gh_2 - f + f = gh_2 - gh_2 + f = f.$$

Hence, $A \oplus B \in \underline{\mathfrak{Pr}}^{-1}(M)$. □

PROPOSITION 2.13. *If M is finitely generated and A_i-subprojective for every $i \in I$, then M is $\bigoplus_{i\in I} A_i$-subprojective.*

Proof. Let $f : M \to \bigoplus_{i\in I} A_i$ be a homomorphism and $g : C \to \bigoplus_{i\in I} A_i$ be an epimorphism. Let $X := \{m_1, \ldots, m_k\}$ be a set of generators for M. Then there exists a finite index set $J \subset I$ such that $f(X) \subset \bigoplus_{j\in J} A_j$. By Proposition 2.12, there exists a homomorphism $h : M \to C$ such that $gh(m_i) = f(m_i)$ for all $i \in \{1, \ldots, k\}$. Since X generates M, $gh = f$, as hoped. □

We do not know if the subprojectivity domain of a module is, in general, closed under arbitrary direct sums. However, we have some information regarding when it is closed under arbitrary direct products. If the subprojectivity domain of every module is closed under products, then, by Proposition 2.8, the class of projective modules is closed under products. By [**8**, Theorem 3.3], this means that R is a right perfect, left coherent ring. This condition is enough to ensure that subprojectivity domains are closed under products.

PROPOSITION 2.14. *Let R be a ring. The following conditions are equivalent.*
(1) *R is a right perfect, left coherent ring.*
(2) *The subprojectivity domain of any right R-module is closed under arbitrary products.*

Proof. (2) $\Rightarrow$ (1) follows from the discussion in the preceding paragraph. For (1) $\Rightarrow$ (2), let $M \in \text{Mod-}R$, and let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a set of modules in $\underline{\mathfrak{Pr}}^{-1}(M)$. Let $f = (f_\lambda)_{\lambda\in\Lambda} : M \to \prod_{\lambda\in\Lambda} N_\lambda$. For every $\lambda \in \Lambda$, be $g_\lambda : P_\lambda \to N_\lambda$ be an epimorphism with P_λ projective. By hypothesis, there exists $h_\lambda : M \to P_\lambda$ such that $f_\lambda = g_\lambda h_\lambda$. Let $h = (h_\lambda)_{\lambda\in\Lambda} : M \to \prod_{\lambda\in\Lambda} P_\lambda$, and $g : \prod_{\lambda\in\Lambda} P_\lambda \to \prod \lambda \in \Lambda N_\lambda$ be defined by $g((x_\lambda)_{\lambda\in\Lambda}) = (g_\lambda(x_\lambda))_{\lambda\in\Lambda}$. It is a routine to check that g is an epimorphism and $gh = f$. Note that, since R is right perfect and left coherent, $\prod_{\lambda\in\Lambda} P_\lambda$ is projective. By Lemma 2.3 (condition 4), M is $\prod_{\lambda\in\Lambda} N_\lambda$-subrojective. $\square$

Similarly, recall that R is said to be right perfect if submodules of projective modules are projective. If every *sp*-portfolio is closed under submodules, then R must be right hereditary. The next proposition tells us that the converse of this statement is also true.

PROPOSITION 2.15. *Let R be a ring. The following conditions are equivalent.*
(1) *R is right hereditary.*
(2) *The subprojectivity domain of any right R-module is closed under submodules*

Proof. (2) $\Rightarrow$ (1). If the subprojectivity domain of any right R-module is closed under submodules, then, by Proposition 2.8, the class of projective modules is closed under submodules. Then R is right hereditary. For (1) $\Rightarrow$ (2), let M be a right R-module, $K \in \underline{\mathfrak{Pr}}^{-1}(M)$ and $N \leq K$. Let $f : M \to N$. We can consider f as a morphism from M to K with image in N. Let $g : P \to K$ be an epimorphism with P projective. Then there exists $h : M \to P$ such that $gh = f$. Now, let $P' = g^{-1}(N) \leq P$. Since R is right hereditary, P' is projective. Note that $h(M) \leq P'$, and $g(P') = N$. By Proposition 2.3 (condition 4), $N \in \underline{\mathfrak{Pr}}^{-1}(M)$. $\square$

In general, the subprojectivity domain of a module is not closed with respect to quotients. Consider for example the $\mathbb{Z}$-modules, $M = \mathbb{Z}/(2)$, $A = \mathbb{Z}$ and $B = 2\mathbb{Z}$. Since A and B are projective, by Proposition 2.8, $A, B \in \underline{\mathfrak{Pr}}^{-1}(M)$. But $M = A/B$ is not projective, so by Proposition 2.4, $A/B \notin \underline{\mathfrak{Pr}}^{-1}(M)$. Using similar arguments, note that $\underline{\mathfrak{Pr}}^{-1}(M)$ is closed under quotients if and only if M is projective.

3. Subprojectivity domains and torsion-free classes.

Hereditary pretorsion classes are an important tool in the study of the injective and projective profiles of a ring R, see [**15**]. For this reason, it seems reasonable to see if torsion-theoretic notions or techniques may help in the study of $\mathfrak{sp}\mathfrak{P}(R)$. Our next result tells us that it is torsion-free classes that play a role in the study of this semi-lattice.

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Proposition 2.5 tells us that, for every module M, the torsion-free class generated by M, $\mathcal{F}_M$ is contained in $\underline{\mathfrak{Pr}}^{-1}(M)$. In Example 2.7 we found that in the category of $\mathbb{Z}$-modules one actually has that $\mathcal{F}_{\mathbb{Q}} = \underline{\mathfrak{Pr}}^{-1}(\mathbb{Q})$. Our next goal is to characterize those subprojective portfolios for which this phenomenon happens. First, we give a definition.

Definition 3.1. Let $\mathcal{C} \in \mathfrak{sp}\mathfrak{P}(R)$. We say that $\mathcal{C}$ is a *basic* $\mathcal{sp}$-portfolio if there exists $M \in \text{Mod-}R$ such that $\mathcal{C} = \underline{\mathfrak{Pr}}^{-1}(M) = \{N \in \text{Mod-}R : \text{Hom}(M, N) = 0\}$.

As a quick example, note that Mod-R is always a basic $\mathcal{sp}$-portfolio as Mod-$R = \underline{\mathfrak{Pr}}^{-1}(0)$. Moreover, if R is a cogenerator for Mod-R (e.g. a QF-ring) then the only basic subportfolio is Mod-R.

It is clear that if $\mathcal{C}$ is a basic $\mathcal{sp}$-portfolio and M is a module as in Definition 3.1, then $\text{Hom}(M, P) = 0$ for every projective module P. The following proposition tells us that this condition is indeed sufficient for $\underline{\mathfrak{Pr}}^{-1}(M)$ to be basic.

Proposition 3.2. *Let $\mathcal{C} \subseteq$ Mod-R be an $\mathcal{sp}$-portfolio. The following conditions are equivalent.*

(1) *$\mathcal{C}$ is basic.*

(2) *There exists a module M such that $\mathcal{C} = \underline{\mathfrak{Pr}}^{-1}(M)$ and* $\text{Hom}(M, R) = 0$.

Proof. (1) $\Rightarrow$ (2) is clear. We show (2) $\Rightarrow$ (1). Let N be a module such that $\text{Hom}(M, N) \neq 0$, and let $f : M \to N$ be a non-zero morphism. Let $p : F \to N$ be an epimorphism with F free. By (2), $\text{Hom}(M, F) = 0$, so f cannot be lifted to a morphism $M \to F$, so $N \notin \underline{\mathfrak{Pr}}^{-1}(M)$. Hence, $\underline{\mathfrak{Pr}}^{-1}(M)$ is basic. □

Note that if $\mathcal{C} = \underline{\mathfrak{Pr}}^{-1}(M)$ is basic, then it does not follow that $\text{Hom}(M, R) = 0$. For example, by proposition, if $M \in \text{Mod-}R$ then M and $M \oplus R$ have the same subprojectivity domain, while it is always the case that $\text{Hom}(M \oplus R, R) \neq 0$.

As a consequence of Proposition 3.2, we can list the subprojectivity domain of some classes of modules.

(1) $\underline{\mathfrak{Pr}}^{-1}(\mathbb{Z}_{p^n}) = \{M \in \text{Mod-}\mathbb{Z} : M$ does not have elements of order $p\}$.

(2) $\underline{\mathfrak{Pr}}^{-1}(\bigoplus_{p \text{ prime}} \mathbb{Z}_p) = \{M \in \text{Mod-}\mathbb{Z} : \text{Soc}(M) = 0\} = \{M \in \text{Mod-}\mathbb{Z} : t(M) = 0\}$.

(3) $\underline{\mathfrak{Pr}}^{-1}(\mathbb{Z}_{p^\infty}) = \{M \in \text{Mod-}\mathbb{Z} : \mathbb{Z}_{p^\infty}$ is not isomorphic to a submodule of $M\}$.

As we have said, for every module M, the class $\{N \in \text{Mod-}R : \text{Hom}(M, N) = 0\}$ is a torsion-free class, that is, it is closed under submodules, extensions and arbitrary direct products. Then we have the following consequence of Proposition 3.2.

Corollary 3.3. *Let M be a module such that* $\text{Hom}(M, R) = 0$. *Then, $\underline{\mathfrak{Pr}}^{-1}(M)$ is closed under arbitrary products, submodules and extensions.*

The module $\mathbb{Z}_{p^\infty}$ exhibits an interesting behaviour. It is not projective, but its subprojectivity domain is in some sense large. We formalize this in the following definition.

Definition 3.4. Let M be an R-module. We say that M is *maximally subprojective if* $\underline{\mathfrak{Pr}}^{-1}(M)$ is a coatom in $\mathfrak{sp}\mathfrak{P}(R)$.

Proposition 3.5. *Let M be a module such that $\underline{\mathfrak{Pr}}^{-1}(M) = \{K \in$ Mod-R : K does not have a direct summand isomorphic to $M\}$. Then M is maximally subprojective.*

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Proof. Assume N is a module with $\underline{\mathfrak{Pr}}^{-1}(M) \subsetneq \underline{\mathfrak{Pr}}^{-1}(N)$. If $N \cong M \oplus K$, then $\underline{\mathfrak{Pr}}^{-1}(M) \subsetneq \underline{\mathfrak{Pr}}^{-1}(N) = \underline{\mathfrak{Pr}}^{-1}(M) \cap \underline{\mathfrak{Pr}}^{-1}(K) \subseteq \underline{\mathfrak{Pr}}^{-1}(M)$, a contradiction. Hence, N does not have direct summands isomorphic to M. By our assumptions, $N \in \underline{\mathfrak{Pr}}^{-1}(M) \subsetneq \underline{\mathfrak{Pr}}^{-1}(N)$, so N is N-subprojective, that is, N is projective. Hence, M is maximally subprojective. □

Corollary 3.6.

(1) *$\mathbb{Z}_{p^\infty}$ is a maximally subprojective $\mathbb{Z}$-module.*
(2) *For any ring R, if S is a simple injective non-projective module over any ring R, then S is maximally subprojective.*

Proof. Condition (1) is clear from the examples after Proposition 3.2. For condition (2), if S is a simple injective non-projective module, then $\mathrm{Hom}(S, R) = 0$, for otherwise S would be a summand of a projective module. Then $\underline{\mathfrak{Pr}}^{-1}(S)$ is basic. Finally, note that those modules M for which $\mathrm{Hom}(S, M) = 0$ are precisely the modules that do not contain a direct summand isomorphic to S. Then S is maximally subprojective. □

Corollary 3.7. *Let R be a non-semi-simple right V-ring. Then R has maximally subprojective modules.*

Our next result tells us that every *sp*-portfolio is basic if and only if every *sp*-portfolio is a torsion-free class and describes precisely the rings for which these conditions hold. To state it, we need the following known result.

Proposition 3.8. *[(see e.g. [**18**])] Let R be a ring. The following conditions are equivalent.*

(1) *R is a right hereditary, right perfect, left coherent ring.*
(2) *R is a semi-primary, right hereditary, left coherent ring.*

We will use this result freely throughout the paper, most notably in Propositions 3.9, 4.7, 4.8 and 4.19.

Proposition 3.9. *Let R be a ring. The following are equivalent.*

(1) *R is a semi-primary, right hereditary, left coherent ring.*
(2) *Every sp-porfolio is basic.*
(3) *Every sp-portfolio is a torsion-free class.*

Proof. (2) $\Rightarrow$ (3) is clear. (3) $\Rightarrow$ (1) follows because torsion-free classes are closed under arbitrary intersections, so the class of projective modules is a torsion-free class, cf. Proposition 2.8. Finally, for (1) $\Rightarrow$ (2), let M be a right R-module. Since R is right hereditary, right perfect, left coherent, the class of projective modules is closed under direct products and submodules. Let $N = \bigcap\{\mathrm{Ker}(f) : f : M \to P,$ and P is projective$\}$. M/N is projective and N is the smallest submodule of M that yields a projective quotient. Now, $M \cong M/N \oplus K$. If $\mathrm{Hom}(K, R) \neq 0$ then K has a projective quotient and we can find a submodule of M properly contained in N that yields a projective quotient of M, a contradiction. Hence, $\mathrm{Hom}(K, R) = 0$, so $\underline{\mathfrak{Pr}}^{-1}(K)$ is basic, and $\underline{\mathfrak{Pr}}^{-1}(M) = \underline{\mathfrak{Pr}}^{-1}(M/N) \cap \underline{\mathfrak{Pr}}^{-1}(K) = \underline{\mathfrak{Pr}}^{-1}(K)$. □

Note that if we ignore (2) in Proposition 3.9, the equivalence (1) $\Leftrightarrow$ (3) can be easily obtained from Propositions 2.14 and 2.15.

The use of torsion-theoretic techniques can also be applied to study the notion of subinjectivity as defined in [**5**].

DEFINITION 3.10. Let $\mathcal{I} \subseteq \text{Mod-}R$. We say that $\mathcal{I}$ is a *subinjective portfolio*, or *si-portfolio* for short, if there exists $M \in \text{Mod-}R$ such that $\mathcal{I} = \underline{\mathfrak{In}}^{-1}(M)$. The class $\mathfrak{si}\mathfrak{P}(R) := \{\mathcal{I} \subseteq \text{Mod-}R : I \text{ is an si-portfolio}\}$ will be called the *subinjective profile*, or *si-profile* of R. By [**5**, Proposition 2.4 (1)], $\mathfrak{si}\mathfrak{P}(R)$ is a semi-lattice with the biggest element.

Analog to Proposition 2.5, we have the following result that tells us that for every module M the torsion class cogenerated by M $\mathcal{T}^M$ is contained in $\underline{\mathfrak{In}}^{-1}(M)$.

PROPOSITION 3.11. *Let* $M, N \in \text{Mod-}R$. *If* $\text{Hom}(N, M) = 0$, *then* $N \in \underline{\mathfrak{In}}^{-1}(M)$.

Motivated by Proposition 3.11, we have the following definition.

DEFINITION 3.12. We say that an *si*-portfolio $\mathcal{I}$ is *basic* if there exists a module M for which $\mathcal{I} = \underline{\mathfrak{In}}^{-1}(M) = \{N \in \text{Mod-}R : \text{Hom}(N, M) = 0\}$.

PROPOSITION 3.13. *Let* $\mathcal{I} \subseteq \text{Mod-}R$ *be an si-portfolio. The following conditions are equivalent.*

(1) $\mathcal{I}$ *is basic.*
(2) *There exists* $M \in \text{Mod-}R$ *such that* $\mathcal{I} = \underline{\mathfrak{In}}^{-1}(M)$ *and* $\text{Hom}(E, M) = 0$ *for every injective module* E.

If R *is right noetherian, this happens if and only if* $\text{Hom}(\mathcal{E}, M) = 0$, *where* $\mathcal{E} = \bigoplus\{E : E \text{ is an indecomposable injective}\}$.

Proof. (1) $\Rightarrow$ (2) is clear from [**5**, Proposition 2.3]. Now assume condition (2), and let $N \in \text{Mod-}R$ be such that $\text{Hom}(N, M) \neq 0$. Then there exists a non-zero $f : N \to M$ that cannot be extended to a morphism $\overline{f} : E(N) \to M$. Hence, $N \notin \underline{\mathfrak{In}}^{-1}(M)$. The last assertion is clear. □

It is again worth noting that if $\mathcal{I} = \underline{\mathfrak{In}}^{-1}(M)$ is basic, it does not follow that $\text{Hom}(E, M) = 0$ for every injective module E. For example, if E_0 is any injective module, it follows from [**5**, Proposition 2.4 (2)] that M and $M \oplus E_0$ have the same subinjectivity domain, while it is always the case that $\text{Hom}(E_0, E_0 \oplus M) \neq 0$.

EXAMPLE 3.14. The subinjectivity domain of $\mathbb{Z}$ is $\underline{\mathfrak{In}}^{-1}(\mathbb{Z}) = \{N \in \mathbb{Z}\text{-Mod} : \text{Hom}(N, \mathbb{Z}) = 0\} = \{N \in \mathbb{Z}\text{-Mod} : \mathbb{Z} \text{ is not isomorphic to a direct summand of } N\}$.

Clearly, $\mathbb{Z}$ is not an injective (= divisible) $\mathbb{Z}$-module. However, it is proved in [**15**] that $\mathfrak{In}^{-1}(\mathbb{Z})$ is a coatom in the injective profile of $\mathbb{Z}$. Interestingly enough, $\underline{\mathfrak{In}}^{-1}(\mathbb{Z})$ is also a coatom in the subinjective profile of $\mathbb{Z}$.

DEFINITION 3.15. Let $M \in \text{Mod-}R$. We say that M is *maximally subinjective* if $\underline{\mathfrak{In}}^{-1}(M)$ is a coatom in the subinjective profile of R.

PROPOSITION 3.16. *Let* M *be an* R*-module such that* $\underline{\mathfrak{In}}^{-1}(M) = \{N \in \text{Mod-}R : M \text{ is not isomorphic to a direct summand of } N\}$. *Then* M *is maximally subinjective.*

Proof. Let K be a module such that $\underline{\mathfrak{In}}^{-1}(M) \subsetneq \underline{\mathfrak{In}}^{-1}(K)$. Note that if $K \cong L \oplus M$, then $\underline{\mathfrak{In}}^{-1}(K) = \underline{\mathfrak{In}}^{-1}(L) \cap \underline{\mathfrak{In}}^{-1}(M) \subseteq \underline{\mathfrak{In}}^{-1}(M)$, a contradiction. Hence, M is not isomorphic to a direct summand of K, so $K \in \underline{\mathfrak{In}}^{-1}(M) \subsetneq \underline{\mathfrak{In}}^{-1}(K)$, that is, K is K-subinjective. Therefore, K is injective and M is maximally subinjective. □

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Again, note that Mod-$R = \underline{\mathfrak{In}}^{-1}(0)$ is always a basic *si*-portfolio. If Mod-R has an injective generator (e.g. a right self-injective ring), then Mod-R is the only basic *si*-portfolio. The next proposition tackles the extreme opposite case, that is, when every *si*-portfolio is basic.

PROPOSITION 3.17. *Let R be a ring. The following conditions are equivalent.*

(1) *R is a right hereditary, right noetherian ring.*
(2) *Every si-portfolio is basic.*
(3) *Every si-portfolio is a torsion class.*

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (1) is clear. For (1) $\Rightarrow$ (2), assume R is right hereditary, right noetherian and let $M \in$ Mod-R. Then $M = d(M) \oplus r(M)$, where $d(M)$ is a divisible part of M, and $r(M)$ is its reduced part. Since $r(M)$ does not have injective submodules, $\mathrm{Hom}(E, r(M)) = 0$ for every injective module E. Then, $\underline{\mathfrak{In}}^{-1}(M)$ is basic and $\underline{\mathfrak{In}}^{-1}(M) = \underline{\mathfrak{In}}^{-1}(d(M)) \cap \underline{\mathfrak{In}}^{-1}(r(M)) = \underline{\mathfrak{In}}^{-1}(r(M))$. □

Now, assume R is a right hereditary, right noetherian ring. Then the class $\mathfrak{E}$ of injective modules is a torsion class. Let $\mathfrak{F}$ denote its corresponding torsion-free class. Then $M \in \mathfrak{F}$ if and only if $\mathrm{Hom}(E, M) = 0$ for every $E \in \mathfrak{E}$, that is, if and only if $\underline{\mathfrak{In}}^{-1}(M) = \{N \in \text{Mod-}R : \mathrm{Hom}(N, M) = 0\}$. Then a module $K \in \mathfrak{F}$ is indigent if and only if $\{N \in \text{Mod-}R : \mathrm{Hom}(N, K) = 0\} = \mathfrak{E}$ if and only if K is a cogenerator of the torsion theory $(\mathfrak{E}, \mathfrak{F})$. With these observations, we have proved the following result.

PROPOSITION 3.18. *Let R be a right hereditary right noetherian ring, and let* $\mathbb{T} = (\mathfrak{E}, \mathfrak{F})$ *be the torsion theory where* $\mathfrak{E}$ *is the class of injective modules. Then module M is si-poor if and only if* $M/d(M)$ *is a cogenerator of* $\mathbb{T}$*. In particular, R has an si-poor module if and only if* $\mathbb{T}$ *can be cogenerated by a single module.*

As an application of Proposition 3.18, we show that *si*-poor modules exist over a hereditary finite dimensional algebra over an algebraically closed field $k = \overline{k}$. Recall that, over such an algebra A, in mod-A we have the Auslander–Reiten translate τ, where τM is the k-dual of the transpose of M. The Auslander–Reiten translate has several interesting properties, a number of which can be found in [**4**, Chapter IV].

COROLLARY 3.19. *Let A be a hereditary finite dimensional algebra over an algebraically closed field k, and let* $\mathcal{E}$ *be the direct sum of indecomposable injectives. Then* $\tau\mathcal{E}$ *is an si-poor module, where* τ *denotes the Auslander–Reiten translate.*

Proof. Since $\mathcal{E}$ is injective, $\mathrm{Ext}_A^1(\mathcal{E}, \mathcal{E}) = 0$. Since A is hereditary, $\mathrm{pr.dim}(\mathcal{E}) \leq 1$. The number of non-isomorphic indecomposable summands of $\mathcal{E}$ equals the rank of the Grothendieck group $K_0(A)$. Then $\mathcal{E}$ is a tilting A-module, see [**4**, Corollary VI.4.4]. It follows that $(\mathrm{Gen}(\mathcal{E}), \mathrm{Cogen}(\tau\mathcal{E}))$ is a torsion theory [**4**, Theorem VI.2.5]. Note that $\mathrm{Gen}(\mathcal{E})$ is the class of injective modules. Hence, $\tau\mathcal{E}$ is an indigent module. □

EXAMPLE 3.20. Let A be the path algebra of the quiver

$$1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4\,.$$

The indecomposable projectives are $P(1) = K \to K \leftarrow 0 \to 0$, $P(2) = 0 \to K \leftarrow 0 \to 0$, $P(3) = 0 \to K \leftarrow K \to K$, and $P(4) = 0 \to 0 \leftarrow 0 \to K$; and the indecomposable injectives $I(1) = K \to 0 \leftarrow 0 \to 0$, $I(2) = K \to K \leftarrow K \to 0$,

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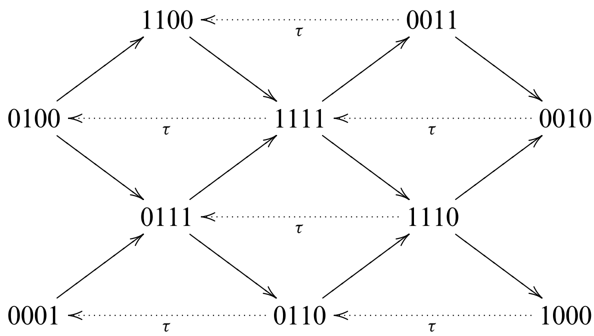
$I(3) = 0 \to 0 \leftarrow K \to 0$ and $P(4) = 0 \to 0 \leftarrow K \to K$. The Auslander-Reiten quiver of A, $\Gamma(A)$, is



where we represent each module by its dimension vector. Now the sum of indecomposable injectives is $E = 1000 \oplus 1110 \oplus 0010 \oplus 0011$, so $\tau E = 0110 \oplus 0111 \oplus 1111 \oplus 1100$ is *si*-poor.

4. Bounds in the subprojectivity domain of a module. In this section, we investigate both upper and lower bounds that one may impose on the subprojectivity domain of a module. Recall that a module is said to be *sp*-poor, or *p*-indigent, if its subprojectivity domain consists precisely of projective modules. We will show next that this condition may be softened with equivalent results.

PROPOSITION 4.1. *Consider the following conditions on module M.*

(1) $\underline{\mathfrak{Pr}}^{-1}(M) = \{P \in \text{Mod-}R : P \textit{ is projective}\}$.
(2) $\underline{\mathfrak{Pr}}^{-1}(M) \subseteq \{P \in \text{Mod-}R : P \textit{ is quasi-projective}\}$.
(3) $\underline{\mathfrak{Pr}}^{-1}(M) \subseteq \{P \in \text{Mod-}R : P \textit{ is discrete}\}$.
(4) $\underline{\mathfrak{Pr}}^{-1}(M) \subseteq \{P \in \text{Mod-}R : P \textit{ is quasi-discrete}\}$.

Then, conditions (1) and (2) are equivalent, and the four conditions are equivalent if R is a right perfect ring.

Proof. (1) ⇒ (2) is clear. Assume condition (2). Let $K \in \underline{\mathfrak{Pr}}^{-1}(M)$, and let P be a projective module that covers K. By Proposition 2.12, $K \oplus P \in \underline{\mathfrak{Pr}}^{-1}(M)$. Then $K \oplus P$ is quasi-projective, so K is P-projective and hence projective. If R is right perfect, then every quasi-projective module is discrete (cf. [**16**, Theorem 4.41]), so we have (1) ⇒ (2) ⇒ (3) ⇒ (4). To show (4) ⇒ (1), assume condition (4) and let $K \in \underline{\mathfrak{Pr}}^{-1}(M)$. Then K is quasi-discrete, so by [**16**, Theorem 4.15] there exists a decomposition $K = \bigoplus_{i \in I} K_i$, where each K_i is a quasi-discrete hollow module. We show that each K_i is projective. Indeed, let P be the projective cover of K_i. Since R is perfect, P is quasi-discrete, so $P = \bigoplus_{j \in J} P_j$, where each P_j is a projective hollow module. By Proposition 2.11, each P_j is in $\underline{\mathfrak{Pr}}^{-1}(M)$. Then $P_j \oplus K_i \in \underline{\mathfrak{Pr}}^{-1}(M)$, so $P_j \oplus K_i$ is quasi-discrete. Then by [**16**, Theorem 4.48] K_i is P_j-projective. Since R is right perfect, projectivity domains are closed under arbitrary direct sums [**3**, Exercise 7.16], which implies that K_i is P-projective. Then K_i is projective. □

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Note that if R is not right perfect, then the implication (2) $\Rightarrow$ (3) does not hold, as, by [**16**, Theorem 4.41], a ring is right perfect if and only if every quasi-projective right module is discrete.

DEFINITION 4.2. A ring R is called right manageable if there exist a set S of non-projective right R-modules such that for every non-projective R-module M, there exists $A \in S$ such that A is isomorphic to a direct summand of M. For convenience, we refer to the set S as the manageable set associated with R.

Recall that a ring R is said to be Σ-cyclic if every right R-module is a direct sum of cyclic modules. See [**11**, Chapter 25].

EXAMPLE 4.3. If R is a right Σ-cyclic ring, then R is right manageable. In particular, an artinian serial ring is both right and left manageable.

PROPOSITION 4.4. *Every manageable ring R has an sp-poor module.*

Proof. Let S be a manageable set of modules associated with R. Let $X = \bigoplus_{A \in S} A$. We claim that X is sp-poor. To see this, let $B \in \underline{\mathfrak{Pr}}^{-1}(X)$. If B is not projective, then there exists $C \in S$ such that C is isomorphic to a direct summand of B. By Proposition 2.11, $C \in \underline{\mathfrak{Pr}}^{-1}(X)$. By Proposition 2.10, $C \in \underline{\mathfrak{Pr}}^{-1}(C)$ and by Proposition 2.4, C is projective, a contradiction. Then B is projective and X is sp-poor. □

If R is an artinian chain ring then Proposition 4.4 implies that the direct sum of non-projective cyclic right R-modules is sp-poor. The next proposition tells us that, in fact, for such a ring every non-projective right R-module is sp-poor. This is an interesting discovery giving us a glance into the phenomenon of a ring R having no subprojective middle class. It should be noted that artinian chain rings also fail to have a subinjective middle class [**5**]. The study of rings without a projective or injective middle class has been undertaken in [**1**, **10**, **13**, **15**].

PROPOSITION 4.5. *If R is an artinian chain ring, then every non-projective module is sp-poor.*

Proof. Since R is an artinian chain ring, every R-module is a direct sum of cyclic uniserial modules. Consequently, it suffices to consider cyclic modules by Propositions 2.10 and 2.11. Because R is an artinian chain ring, the ideals of R are zero or the powers $J(R)^n$ of $J(R)$, the Jacobson radical of R. Moreover, if $p \in J(R)$ but $p \notin J(R)^2$, then $J(R)^n = p^n R$ for every $n \geq 0$. Hence, we have the finite chain for some positive integer n:

$$R \supset pR \supset p^2 R \supset \ldots \supset p^n R = 0.$$

Therefore, it is enough to show that $p^k R$ is sp-poor for every positive integer k.

Let $A = p^k R$, where $k \neq 0$ and let $g : R \to p^k R$ be the quotient map. If $k > m$, then let $f : A \to p^m R$ be the inclusion map. Assume there exists $h : A \to R$ such that $gh = f$. Since R is a chain ring, either $\operatorname{Ker} g \subset \operatorname{Im} h$ or $\operatorname{Im} h \subset \operatorname{Ker} g$. If $\operatorname{Im} h \subset \operatorname{Ker} g$, then $gh = 0$, a contradiction. Hence, $\operatorname{Ker} g \subset \operatorname{Im} h$. But since g is not monic, $\operatorname{Ker} g \neq 0$. Hence, there is a non-zero element $x \in A$ such that $0 = gh(x) = f(x)$, a contradiction. Thus, $p^m R \notin \underline{\mathfrak{Pr}}^{-1}(A)$.

If $k < m$, then consider the homomorphism $f : A \to p^m R$, where $f(p^k) = p^m$. Assume there exists $h : A \to R$ such that $gh = f$. But then $p^m = f(p^k) = gh(p^k) = g(1)p^m \in p^{(2m)} R$, a contradiction. Thus, $p^m R \notin \underline{\mathfrak{Pr}}^{-1}(A)$.

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Finally, we note that by Proposition 2.4, $p^kR \notin \underline{\mathfrak{Pr}}^{-1}(p^kR)$, as p^kR is not projective. □

EXAMPLE 4.6. If p is a prime, then $\mathbb{Z}/p^i\mathbb{Z}$ is a p-indigent $(\mathbb{Z}/p^k\mathbb{Z})$-module for every $i < k$.

Now we investigate the existence of *sp*-poor modules over a semi-primary, right hereditary, left coherent ring. We choose this class of rings because it is precisely when the projective modules form a torsion-free class $\mathfrak{P}$, see Proposition 3.8.

PROPOSITION 4.7. *Let R be a semi-primary, right hereditary, left coherent ring. Let $\mathfrak{P}$ be the torsion-free class consisting of the projective R-modules, and let $\mathfrak{T}$ be the corresponding torsion class. Then R has an* sp*-poor module if and only if there exists a module M that generates* $(\mathfrak{T}, \mathfrak{P})$.

Proof. Assume that there exists a module M that generates $(\mathfrak{T}, \mathfrak{P})$. Then, since $(\mathfrak{T}, \mathfrak{P})$ is a torsion theory, the class $\{N : \mathrm{Hom}(M, N) = 0\} = \mathfrak{P}$. Then M is *sp*-poor. Now, assume M is an *sp*-poor module. Let M' be the smallest module that yields a projective quotient, so $M \cong M/M' \oplus N$ and $\underline{\mathfrak{Pr}}^{-1}(M) = \underline{\mathfrak{Pr}}^{-1}(N)$, with $N \in \mathfrak{T}$. Now $\underline{\mathfrak{Pr}}^{-1}(N)$ is basic, and $\underline{\mathfrak{Pr}}^{-1}(N) = \mathfrak{P}$. Therefore, N is a generator of $(\mathfrak{T}, \mathfrak{P})$. □

Now we investigate the existence of an *sp*-poor $\mathbb{Z}$-module. Note that $\mathbb{Z}$ is not perfect, so we cannot apply the preceding proposition. In fact, we have the following.

PROPOSITION 4.8. *Let R be a ring which is not semi-primary, or not right hereditary, or not left coherent. If there exists an* sp*-poor module M, then* $\mathrm{Hom}(M, R) \neq 0$.

Proof. If $\mathrm{Hom}(M, R) = 0$, then $\underline{\mathfrak{Pr}}^{-1}(M)$ is a torsion-free class. But this cannot happen, as the class of projective modules is either not closed under submodules or not closed under arbitrary direct products. Then $\mathrm{Hom}(M, R) \neq 0$. □

COROLLARY 4.9. *If M is an sp-poor $\mathbb{Z}$-module, then* $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \neq 0$ *and, consequently,* $\mathrm{Hom}_{\mathbb{Z}}(M, N) \neq 0$ *for every abelian group N.*

Since $\mathbb{Z}$ is a principal ideal domain, the last corollary tells us that if M is an *sp*-poor $\mathbb{Z}$-module, then $M \cong \mathbb{Z} \oplus N$, and $\underline{\mathfrak{Pr}}^{-1}(M) = \underline{\mathfrak{Pr}}^{-1}(\mathbb{Z}) \cap \underline{\mathfrak{Pr}}^{-1}(N) = \underline{\mathfrak{Pr}}^{-1}(N)$, so N is also an *sp*-poor $\mathbb{Z}$-module. Iterating the process and taking a direct limit, we have the following result.

COROLLARY 4.10. *Let M be a p-indigent $\mathbb{Z}$-module. Then there exists a submodule $N \leq M$ such that $N \cong \mathbb{Z}^{(\mathbb{N})}$.*

Our next goal is to show that the $\mathbb{Z}$-modules $T = \prod_i(\mathbb{Z}/p_i\mathbb{Z})$ and $S = \left(\prod_i(\mathbb{Z}/p_i\mathbb{Z}) / \left(\bigoplus_i \mathbb{Z}/p_i\mathbb{Z}\right)\right)$ are not *sp*-poor, where $p_1 < p_2 < \ldots$ are the rational primes in increasing order. To do so, we will need the following result, which can be found in **[12]**. For each $i \in \mathbb{N}$, let $e_i \in \mathbb{Z}^{\mathbb{N}}$ be the standard unit vectors in $\mathbb{Z}^{\mathbb{N}}$, that is, $e_i(j) = \delta_{ij}$, the Kronecker delta.

PROPOSITION 4.11. *Every homomorphism $f : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$ is completely determined by its action on $\mathbb{Z}^{(\mathbb{N})}$. In particular, if $f(e_i) = 0$ for all i, then $f = 0$.*

PROPOSITION 4.12. $\mathrm{Hom}(T, \mathbb{Z}) = 0$.

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Proof. Let $f \in \mathrm{Hom}(T, \mathbb{Z})$. Define $P = \mathbb{Z}^{\mathbb{N}}$ and $g : P \to T$ by $[g(\alpha)](i) := \alpha(i) + p_i\mathbb{Z} \in \mathbb{Z}/p_i\mathbb{Z}$; that is

$$(\alpha_1, \alpha_2, \ldots) \mapsto (\alpha_1 + p_1\mathbb{Z}, \alpha_2 + p_2\mathbb{Z}, \ldots).$$

Then g is epic and $fg \in \mathrm{Hom}(P, \mathbb{Z})$.

Fix $k \in \mathbb{N}$. Then $g(p_k e_k) = 0$. Hence, $0 = (fg)(p_k e_k) = p_k(fg)(e_k) \in \mathbb{Z}$. Hence, $(fg)(e_k) = 0$.

So what we have shown is that $(fg)(e_i) = 0$ for all i, and so by the last statement of Proposition 4.11, $fg = 0$. Since g is epic, it follows that $f = 0$, which concludes the proof. □

PROPOSITION 4.13. $\mathrm{Hom}(S, \mathbb{Z}) = 0$.

Proof. Let $f \in \mathrm{Hom}(S, \mathbb{Z})$. Let $h : T \to S$ be the epic mapping each element to its equivalence class. Then $fh \in \mathrm{Hom}(T, \mathbb{Z})$. By Proposition 4.12, $fh = 0$. Since h is epic, it follows that $f = 0$, which concludes the proof. □

From Propositions 4.12 and 4.13, we conclude that both S and T are not *sp*-poor $\mathbb{Z}$-modules. In view of Corollary 4.10, another natural candidate for an *sp*-poor $\mathbb{Z}$-module is the Baer–Specker group $\mathbb{Z}^{\mathbb{N}}$. However, we do not know if this is the case. Also note that, by [**14**, Lemma 2.8], the group $\mathbb{Z}^{\mathbb{N}}/\mathbb{Z}^{(\mathbb{N})}$ is also not p-indigent.

Now we consider a lower bound on $\underline{\mathfrak{Pr}}^{-1}(M)$, which is inspired by [**2**], where they define a module M to be strongly soc-injective if, for every $N \in \mathrm{Mod}\text{-}R$, every morphism $f : \mathrm{Soc}(N) \to M$ can be extended to a morphism $\overline{f} : N \to M$. It is not hard to see that the requirement of M to be strongly soc-injective is equivalent to saying that $\mathrm{SSMod}\text{-}R \subseteq \underline{\mathfrak{In}}^{-1}(M)$.

DEFINITION 4.14. Let $M \in \mathrm{Mod}\text{-}R$. We say that M is strongly soc-projective if $\mathrm{SSMod}\text{-}R \subseteq \underline{\mathfrak{Pr}}^{-1}(M)$.

Of course, a strongly soc-projective module need not be projective, as shown by the $\mathbb{Z}$-module $\mathbb{Q}_{\mathbb{Z}}$. However, in the category of abelian groups, a finitely generated strongly soc projective module is projective.

PROPOSITION 4.15. *Let M be a finitely generated abelian group such that every semi-simple module is in $\underline{\mathfrak{Pr}}^{-1}(M)$. Then M is projective.*

Proof. By the Fundamental Theorem of Finitely Generated Abelian groups, $M \cong \mathbb{Z}^n \oplus \mathbb{Z}_{p_1^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p_n^{\alpha_n}}$. If $\alpha_i \neq 0$ for some i, then $\mathbb{Z}_{p_i} \notin \underline{\mathfrak{Pr}}^{-1}(M)$, a contradiction. Then, $M \cong \mathbb{Z}^n$ is projective. □

If R is a semi-perfect ring, then every simple module has a projective cover, which has to be a local module, that is, with only one maximal submodule. In this case, we have the following proposition.

PROPOSITION 4.16. *Let R be a semi-perfect ring and let M be a right R-module. Assume S is a simple module such that $S \in \underline{\mathfrak{Pr}}^{-1}(M)$. Then, either* $\mathrm{Hom}(M, S) = 0$ *or $M = P(S) \oplus K$, where $P(S)$ stands for the projective cover of S.*

Proof. Assume $\mathrm{Hom}(M, S) \neq 0$, and let $f : M \to S$ be a non-zero morphism. By hypothesis, we can lift this morphism to a morphism $\overline{f} : M \to P(S)$. Now, $\overline{f}(M)$ cannot be contained in the unique maximal ideal of $P(S)$, which is the kernel of the epimorphism $P(S) \to S$. Then $\overline{f}(M) = P(S)$ and, by the projectivity of $P(S)$, we conclude that $M = P(S) \oplus K$. □

COROLLARY 4.17. *Let R be a semi-perfect ring and let M be a finitely generated right R-module of finite uniform dimension that is subprojective with respect to every simple (equivalently, with respect to every semi-simple) module. Then M is projective.*

Proof. Iterating the process of Proposition 4.16, and using the fact that M has finite uniform dimension, we have that $M = P \oplus K$, with P projective and K has no non-zero morphisms to a simple module. If $K \neq 0$, then K has maximal submodules, a contradiction. Hence, $M = P$ □

Note that the conclusion of Corollary 4.17 may hold even if R is not semi-perfect. For example, by Proposition 4.15, it holds for the ring of integers $\mathbb{Z}$.

If we assume that R is right perfect, we can drop the finitely generated assumption in Corollary 4.17, as the existence of a maximal submodule of K is guaranteed by the conditions on R. Then we have the following.

COROLLARY 4.18. *Let R be a right perfect ring and let M be a right R-module of finite uniform dimension that is subprojective with respect to every semi-simple module. Then M is projective.*

If, moreover, we assume that our ring is semi-primary, right hereditary, left coherent, we can also remove the finite uniform dimension assumption in Corollary 4.18.

PROPOSITION 4.19. *Let R be a semi-primary, right hereditary, left coherent ring. Then right module M is projective if and only if it is strongly soc-projective.*

Proof. As we've seen, in this case every module has the smallest module that yields a projective quotient. Then we can decompose every module M as $M \cong P \oplus X$, with P projective and X a module without projective quotients. Moreover, since R is right perfect, every non-zero module has maximal submodules, cf. [7]. Then if $X \neq 0$ there exists a simple module S such that $\mathrm{Hom}(X, S) \neq 0$. Since $\underline{\mathfrak{Pr}}^{-1}(M) = \underline{\mathfrak{Pr}}^{-1}(X)$, $S \in \underline{\mathfrak{Pr}}^{-1}(X)$. This implies, by Proposition 4.16, that $P(S)$ is a direct summand of X, a contradiction. Hence, $X = 0$ and $M \cong P$ is projective. □

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