CONTINUITY AND DIFFERENTIABILITY PROPERTIES OF THE NEMITSKII OPERATOR IN HÖLDER SPACES

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Introduction. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the usual norm denoted by $|\cdot|$. In what follows Ω will denote an open bounded subset of $\mathbb{R}^n$, and $\bar{\Omega}$ its closure.

For $\alpha \in (0, 1]$, $C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$ is the space of all functions $u:\bar{\Omega}\to\mathbb{R}$ such that:

$$h_\alpha(u):=\sup\{|u(x)-u(y)|/|x-y|^\alpha; x,y\in\bar{\Omega}, x\neq y\}<\infty.$$

$C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$ is called the *Hölder space with exponent* α and is a Banach space when endowed with the norm:

$$\|u\|_{0,\alpha}=\|u\|_\infty+h_\alpha(u),$$

where $\|u\|_\infty$ is, as usual, defined by:

$$\|u\|_\infty=\sup\{|u(x)|; x\in\bar{\Omega}\}.$$

Let moreover $f=f(x, t)$ be a real valued function defined on $\bar{\Omega}\times\mathbb{R}$.

The aim of this paper is to find conditions on f ensuring some continuity and differentiability properties of the so called Nemitskii operator induced by f; i.e. the operator F defined by

$$F(u)(x)=f(x, u(x)) \qquad (x\in\bar{\Omega})$$

for real valued functions u defined on $\bar{\Omega}$.

More precisely we show that:

(a) if f satisfies the assumption

(H) $\qquad f\in C^{0,1}(\bar{\Omega}\times\bar{I}, \mathbb{R})$ for any bounded interval $I\subset\mathbb{R}$,

then F maps $C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$ into itself;

(b) if $f=f(x, t)$ is differentiable with respect to the real variable t and its derivative $f'_t(x, t)$ satisfies (H), then F maps $C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$ continuously into itself;

(c) finally, if f is twice differentiable with respect to t and the second derivative f''_t satisfies (H), then F is continuously differentiable.

The same results can be obtained if f is a real valued function defined on $\bar{\Omega}\times\mathbb{R}^m (m\geqq 1)$; the corresponding statements are given in §3.

Continuity properties of the Nemitskii operator operator in Sobolev spaces rather than in Hölder spaces are proved by Valent in [**3**]; he shows (Theorem 2) that if Ω has the cone property, if $f\in C^m(\bar{\Omega}\times\mathbb{R})$ and $mp>n$, then F maps $W^{m,p}(\Omega)$ continuously into itself.

We end this note with an application of the results above in the degree-theoretical approach to non linear elliptic boundary value problems of the kind:

$$\begin{cases} f(x, u, Du, D^2u) = 0 & (\text{in } \Omega) \\ u = 0 & (\text{on } \partial\Omega). \end{cases}$$

1. Continuity. Let Ω and f be as in the Introduction. In this section we state conditions on f ensuring that the corresponding Nemitskii operator maps $C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$ into itself and is continuous.

THEOREM 1.1. *If f satisfies* (H), *then F maps $C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$ into itself.*

Proof. Let $u \in C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$ and $M = \|u\|_{0,\alpha}$; then $|u(x)| \leqq M \; \forall x \in \bar{\Omega}$. Let moreover $\bar{I} = [-M, M]$ and $k = k(I)$ be the Lipschitz constant of f relative to I. Then

$$|f(x, u(x)) - f(y, u(y))|/|x-y|^{\alpha} \leqq k\{(|x-y| + |u(x) - u(y)|)/|x-y|^{\alpha}\} \qquad (x, y \in \bar{\Omega}).$$

If d denotes $(\operatorname{diam} \Omega)^{1-\alpha}$, one gets

$$h_{\alpha}(F(u)) \leqq k\{d + h_{\alpha}(u)\}. \tag{1.1}$$

Moreover, for any (x, t) in $\bar{\Omega} \times \bar{I}$,

$$|f(x, t)| \leqq c + k\{|x - x_0| + |t|\},$$

where x_0 is an arbitrary point in $\bar{\Omega}$ and $c = |f(x_0, 0)|$.

Therefore, for all $x \in \bar{\Omega}$,

$$|f(x, u(x))| \leqq c + k(c_1 + \|u\|_{\infty}), \tag{1.2}$$

where c_1 is the radius of a ball centered at x_0 and containing $\bar{\Omega}$.

Finally, taking into account (1.1) and (1.2), we get

$$\|F(u)\|_{0,\alpha} \leqq c + k(c_2 + \|u\|_{0,\alpha}),$$

where $c_2 = d + c_1$.

THEOREM 1.2. *Let f'_t denote the partial derivative of f with respect to the real variable t and assume that f'_t satisfies (H). Then*:

(i) *the Nemitskii operator G induced by f'_t maps $C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$ into itself*;
(ii) *the Nemitskii operator F induced by f is locally Lipschitzian and hence continuous.*

Proof. (i) is a consequence of Theorem 1.1. (ii) Fix $u \in C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$, let $N = \|u\|_{0,\alpha} + 1$, $\bar{J} = [-N, N]$ and let k be the Lipschitz constant of f'_t corresponding to $\bar{J}$. Then, arguing as in the proof of Theorem 1.1, we get

$$\|G(u + \xi v)\|_{0,\alpha} \leqq c + k(c_2 + \|u + \xi v\|) \tag{1.3}$$

whenever $\xi \in [0, 1]$ and $v \in C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$ is such that $\|v\|_{0,\alpha} \leqq 1$. Now write

$$f(x, u(x)+v(x)) - f(x, u(x)) = \int_0^1 f_t'(x, u(x)+\xi v(x))v(x)\,d\xi$$
$$= \int_0^1 G(u+\xi v)(x)v(x)\,d\xi,$$

whence

$$\|F(u+v) - F(u)\|_\infty \leqq \int_0^1 \|G(u+\xi v)v\|_\infty\,d\xi. \tag{1.5}$$

From (1.4) we also get

$$|f(x, u(x)+v(x)) - f(x, u(x)) - f(y, u(y)+v(y)) + f(y, u(y))|/|x-y|^\alpha$$
$$\leqq \int_0^1 |G(u+\xi v)(x)v(x) - G(u+\xi v)(y)v(y)|/|x-y|^\alpha\,d\xi, \tag{1.6}$$

which shows that

$$h_\alpha(F(u+v) - F(u)) \leqq \int_0^1 h_\alpha(G(u+\xi v)v)\,\mathrm{d}\xi. \tag{1.7}$$

Therefore, from (1.5) and (1.7),

$$\|F(u+v) - F(u)\|_{0,\alpha} \leqq \int_0^1 \|G(u+\xi v)v\|_{0,\alpha}\,d\xi.$$

One checks easily that $\|wv\|_{0,\alpha} \leqq m\,\|w\|_{0,\alpha}\,\|v\|_{0,\alpha}$ for some $m \geqq 0$ and all $w, v \in C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$; therefore we have

$$\|F(u+v) - F(u)\|_{0,\alpha} \leqq m\,\|v\|_{0,\alpha} \int_0^1 \|G(u+\xi v)\|_{0,\alpha}\,d\xi$$

whence, using (1.3), we finally get, if $\|v\|_{0,\alpha} \leqq 1$,

$$\|F(u+v) - F(u)\|_{0,\alpha} \leqq L\,\|v\|_{0,\alpha},$$

where $L = m[c + k(c_2 + N)]$. This proves that F is Lipschitz continuous around u.

2. Differentiability

THEOREM 2.1. *Let Ω be as before, let f be twice differentiable with respect to the real variable t, and assume that its second derivative f_t'' satisfies (H). Then:*

(i) *the Nemitskii operator G induced by f_t' is continuous;*

(ii) *the Nemitskii operator F induced by f is continuously differentiable, with derivative $F'(u)[v] = G(u)v$.*

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Proof. (i) This is a consequence of Theorem 1.2. (ii) Set

$$w(u, v, x) := f(x, u(x) + v(x)) - f(x, u(x)) - f_t'(x, u(x))v(x)$$

so that

$$w(u, v, x) = \int_0^1 [f_t'(x, u(x) + \xi v(x)) - f_t'(x, u(x))]v(x)\, d\xi$$
$$= \int_0^1 (G(u + \xi v) - G(u))(x)v(x)\, d\xi,$$

whence

$$\|F(u+v) - F(u) - G(u)v\|_\infty \leqq \int_0^1 \|(G(u+\xi v) - G(u))v\|_\infty\, d\xi.$$

Moreover,

$$|w(u, v, x) - w(u, v, y)|/|x-y|^\alpha$$
$$\leqq \int_0^1 |(G(u+\xi v) - G(u))(x)v(x) - (G(u+\xi v) - G(u))(y)v(y)|/|x-y|^\alpha\, d\xi.$$

In other words,

$$h_\alpha[F(u+v) - F(u) - G(u)v] \leqq \int_0^1 h_\alpha[(G(u+\xi v) - G(u))v]\, d\xi.$$

We conclude that

$$\|F(u+v) - F(u) - G(u)v\|_{0,\alpha} \leqq \int_0^1 \|(G(u+\xi v) - G(u))v\|_{0,\alpha}\, d\xi$$
$$\leqq m\, \|v\|_{0,\alpha} \int_0^1 \|G(u+\xi v) - G(u)\|_{0,\alpha}\, d\xi.$$

Now let $\varepsilon > 0$. By continuity of G (part (i)) there exists $\delta > 0$ such that $\|G(u+\xi v) - G(u)\|_{0,\alpha} < \varepsilon$ whenever $\|v\|_{0,\alpha} < \delta$. Therefore,

$$\|F(u+v) - F(u) - G(u)v\|_{0,\alpha} \leqq \varepsilon\, \|v\|_{0,\alpha}$$

whenever $\|v\|_{0,\alpha} < \delta$, showing that F is differentiable at u with derivative $F'(u)[v] = G(u)v$.

Finally, to show that F is continuously differentiable, let $\mathscr{L}$ denote the Banach space of all linear bounded mappings of $C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$ into itself, equipped with its usual norm $\|T\|_{\mathscr{L}} = \sup\{\|T[v]\|_{0,\alpha} : \|v\|_{0,\alpha} = 1\}$.

Since

$$\|F'(u+w)[v] - F'(u)[v]\|_{0,\alpha} = \|G(u+w)v - G(u)v\|_{0,\alpha}$$
$$\leqq m\, \|G(u+w) - G(u)\|_{0,\alpha}\, \|v\|_{0,\alpha}$$

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we have

$$\|F'(u+w)-F'(u)\|_{\mathscr{L}} \leqq m\,\|G(u+w)-G(u)\|_{0,\alpha}$$

and the conclusion follows again from the continuity of G.

3. Vector-valued functions. If Ω denotes, as before, an open bounded subset of $\mathbb{R}^n$, the same results given in Sections 1 and 2 can be stated when $f=f(x,s)=f(x,s_1,\dots,s_m)$ is a real-valued function defined on $\bar{\Omega}\times\mathbb{R}^m (m \geqq 1)$.

We let here $f'_s=(f'_{s_1},\dots,f'_{s_m})$ denote the gradient of f with respect to the variable $s\in\mathbb{R}^m$, while f''_s will denote the $m\times m$ Hessian matrix $(f''_{s_is_j})$ $(i,j=1,\dots,m)$ of f with respect to the same variable.

Moreover, the symbol I will denote here a bounded interval in $\mathbb{R}^m$:

$$I=\{x=(x_1,\dots,x_m)\in\mathbb{R}^m : a_i<x_i<b_i,\ i=1,2,\dots,m\}$$

(with a_i, b_i real numbers such that $a_i<b_i$, $i=1,\dots,m$) and $\bar{I}$ will denote the closure of $\bar{I}$.

Finally, we choose for the space $C^{0,\alpha}(\bar{\Omega},\mathbb{R}^m)$ the norm:

$$\|u\|_{0,\alpha}=\sum_{i=1}^{m}\|u_i\|_{0,\alpha} \qquad (u=(u_1,u_2,\dots,u_m)).$$

THEOREM 3.1. *Let Ω be as before and let $f:\bar{\Omega}\times\mathbb{R}^m\to\mathbb{R}$ be of class $C^{0,1}(\bar{\Omega}\times\bar{I},\mathbb{R})$ for any bounded interval $I\subset\mathbb{R}^m$; then the Nemitskii operator F induced by f, defined by $F(u)(x)=f(x,u(x))$ for vector valued functions $u:\bar{\Omega}\to\mathbb{R}^m$, maps $C^{0,\alpha}(\bar{\Omega},\mathbb{R}^m)$ into $C^{0,\alpha}(\bar{\Omega},\mathbb{R})$.*

THEOREM 3.2. *With the same notations as before, assume moreover that f is differentiable with respect to the $\mathbb{R}^m$ variable and that $f'_s\in C^{0,1}(\bar{\Omega}\times\bar{I},\mathbb{R}^m)$ for any bounded interval $I\subset\mathbb{R}^m$. Then:*

(i) *the Nemitskii operator G induced by f'_s maps $C^{0,\alpha}(\bar{\Omega},\mathbb{R}^m)$ into itself;*

(ii) *the Nemitskii operator F induced by f maps $C^{0,\alpha}(\bar{\Omega},\mathbb{R}^m)$ into $C^{0,\alpha}(\bar{\Omega},\mathbb{R})$ and is locally Lipschitzian.*

THEOREM 3.3. *If f is twice differentiable with respect to the $\mathbb{R}^m$ variable and $f''_s\in C^{0,1}(\bar{\Omega}\times\bar{I},\mathbb{R}^m)$ for any bounded interval $I\subset\mathbb{R}^m$, then:*

(i) *the Nemitskii operator G induced by f'_t maps continuously $C^{0,\alpha}(\bar{\Omega},\mathbb{R}^m)$ into itself;*

(ii) *the Nemitskii operator F induced by f maps $C^{0,\alpha}(\bar{\Omega},\mathbb{R}^m)$ into $C^{0,\alpha}(\bar{\Omega},\mathbb{R})$ and is continuously differentiable with derivative*

$$(F'(u)[v]=G'(u)\cdot v \qquad (u,v\in C^{0,\alpha}(\bar{\Omega},\mathbb{R}^m)),$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^m$; explicitly,

$$\begin{aligned}(F'(u)[v])(x)&=f'_s(x,u(x))\cdot v(x)\\&=\sum_{i=1}^{m}f'_{s_i}(x,u(x))v_i(x).\end{aligned} \tag{3.1}$$

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4. An application to nonlinear elliptic problems. Let $C^{2,\alpha}(\bar{\Omega}, \mathbb{R})$ be the space of real functions defined on $\bar{\Omega}$, with derivative up to the second order in $C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$. We equip $C^{2,\alpha}(\bar{\Omega}, \mathbb{R})$ with the usual norm:

$$\|u\|_{2,\alpha} = \sum_{|k| \leq 2} \|D^k u\|_{0,\alpha},$$

where $k = (k_1, \ldots, k_n)$ is a multiindex, $|k| = k_1 + \ldots + k_n$ and

$$D^k u = \frac{\partial^{|k|} u}{\partial^{k_1} x_1 \ldots \partial^{k_n} x_n}.$$

Let moreover $f = f(x, t, p, q)$ be a real valued function defined on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} = \bar{\Omega} \times \mathbb{R}^m$ $(m = 1 + n + n^2)$, and consider the following nonlinear boundary value problem:

$$\begin{cases} f(x, u, Du, D^2 u) = 0 & (\text{in } \Omega), \\ u = 0 & (\text{on } \partial\Omega), \end{cases} \tag{4.1}$$

where Ω has smooth boundary $\partial\Omega$ and Du, $D^2 u$ are shorthand notations for the first (resp. second) order derivatives of u.

One seeks $C^{2,\alpha}$ solutions of (4.1).

One way of attacking (4.1) is to use degree theory for Fredholm mappings, as suggested by K. D. Elworthy and A. J. Tromba in their paper [**2**]. To do this, one basic requirement to fulfill is that the Nemitskii operator F induced by f be a smooth (e.g. C^1) mapping between $C^{2,\alpha}(\bar{\Omega}; \mathbb{R})$ and $C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$; moreover, one needs the explicit expression of the derivative $F'(u)$ in order to check that F is a Fredholm mapping of index zero (see e.g. Berger [**1**] for the definition). To this end we prove the following result.

THEOREM 4.1. *Let $f = f(x, t, p, q)$ be as above and assume that it satisfies the assumptions of Theorem* 3.3. *Then the induced Nemitskii operator*

$$\bar{F}(u)(x) = f(x, u(x), Du(x), D^2 u(x)) \qquad (x \in \bar{\Omega})$$

maps $C^{2,\alpha}(\bar{\Omega}, \mathbb{R})$ into $C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$ and is continuously differentiable, with derivative

$$\begin{aligned}(\bar{F}'(u)[v])(x) &= f'_t(x, u(x), Du(x), D^2 u(x)) v(x) \\ &+ \sum_{i=1}^{n} f'_{p_i}(x, u(x), Du(x), D^2 u(x)) \frac{\partial v}{\partial x_i}(x) \\ &+ \sum_{i,j=1}^{n} f'_{q_{i,j}}(x, u(x), Du(x), D^2 u(x)) \frac{\partial^2 v}{\partial x_i \partial x_j}(x)\end{aligned} \tag{4.2}$$

for any $u, v \in C^{2,\alpha}(\bar{\Omega}, \mathbb{R})$.

Proof. Let j be the isometry of $C^{2,\alpha}(\bar{\Omega}, \mathbb{R})$ onto $C^{0,\alpha}(\bar{\Omega}, \mathbb{R}^m)$, defined by

$$ju = (u, Du, D^2 u),$$

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and let F be the Nemitskii operator induced by f on $C^{0,\alpha}(\bar{\Omega}, \mathbb{R}^m)$; i.e.

$$F(v)(x) = f(x, v(x)), \qquad v \in C^{0,\alpha}(\bar{\Omega}, \mathbb{R}^m).$$

We have

$$\bar{F}(u) = F(ju) \qquad (v \in C^{2,\alpha}(\bar{\Omega}, \mathbb{R}));$$

i.e. $\bar{F} = F \circ j$. Therefore, by Theorem (3.3), $\bar{F}$ maps continuously $C^{2,\alpha}(\bar{\Omega}, \mathbb{R})$ into $C^{0,\alpha}(\bar{\Omega}, \mathbb{R})$ and is continuously differentiable; moreover, by the chain rule,

$$\bar{F}'(u) = F'(ju) \circ j$$

or

$$\bar{F}(u)[v] = F'(ju)[jv] \qquad (u, v \in C^{2,\alpha}(\bar{\Omega}, \mathbb{R})).$$

Therefore, by the explicit formula (3.1),

$$\begin{aligned}(\bar{F}'(u))[v](x) &= f_s'(x, ju(x)) \cdot jv(x) \\ &= f_s'(x, u, Du(x), D^2u(x)) \cdot (v(x), Dv(x), D^2v(x)),\end{aligned}$$

which is nothing but the shorthand version of (4.2).

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