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TYPE PRESERVATION IN LOCALLY FINITE VARIETIES WITH THE CEP

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ABSTRACT. Assume that **A** is a finite algebra contained in a variety that has the congruence extension property and that **B** is a subalgebra of **A**. If $\alpha \prec \beta$ in Con **A** and $\alpha|_B \neq \beta|_B$, then we show that $\alpha|_B \prec \beta|_B$ and that there is a close connection between the type labellings of the quotients $\langle \alpha, \beta \rangle$ and $\langle \alpha|_B, \beta|_B \rangle$.

1. **Introduction.** An algebra **A** has the *congruence extension property,* or CEP, if for every subalgebra $\mathbf{B} \leq \mathbf{A}$ and every congruence $\alpha \in \operatorname{Con}\mathbf{B}$ there is a congruence $\alpha^* \in \operatorname{Con}\mathbf{A}$ whose restriction to **B** is α. That is,

$$\alpha^*|_B \stackrel{\text{def}}{=} \alpha^* \cap (B \times B) = \alpha.$$

A class of algebras, such as a variety, whose members all have this property, is said to possess the CEP.

For an algebra **A** to have the CEP there must be a delicate interplay between the subalgebras and the congruences of **A**. The property that all algebras in a variety have the CEP is quite rare. For example, a variety of groups has the CEP if and only if it consists of abelian groups. A variety of rings has the CEP if and only if it is generated by finitely many finite fields and some zero rings. A variety of monoids has the CEP if and only if it is contained in a variety generated by a finite cyclic group and the two element semilattice.

In [8], E. Kiss provided deep insight into the structure of congruence modular varieties with the CEP. He proved that modular varieties with the CEP must satisfy the following commutator conditions:

$$\text{C2}: [\alpha, \beta] = \alpha \cdot \beta \cdot [1, 1],$$

$$\text{R}: \text{ If } \mathbf{B} \leq \mathbf{A} \text{ and } \alpha, \beta \in \operatorname{Con}\mathbf{A}, \text{ then } [\alpha, \beta]|_B = [\alpha|_B, \beta|_B].$$

Then, he showed that any modular variety satisfying C2 and R has the CEP iff all ultraproducts of subdirectly irreducible members do. His work includes the result that if **A** is finite and $\mathbf{A} \times \mathbf{A}$ has the CEP, then $\mathcal{V} = \mathcal{V}(\mathbf{A})$ satisfies C2 and R. If $\mathcal{V}$ satisfies C2 and R and $\boldsymbol{H}(\mathbf{A})$ has the CEP, then $\mathcal{V}$ has the CEP.

For modular varieties, the property C2 is a very restrictive condition on commutator arithmetic. For example, when a modular variety satisfies C2,

$$[\alpha, \alpha] = 0 \longrightarrow \alpha \cdot [1, 1] = 0 \longrightarrow [\alpha, 1] = 0,$$

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which means that abelian congruences are central. The fact that abelian congruences are central for algebras in non-modular varieties with the CEP was established by E. Kiss and myself using the TC commutator. We then observed that for modular varieties C2 implies the stronger condition that solvable congruences are central, so we asked if solvable congruences are central for algebras in non-modular varieties with the CEP. R. McKenzie answered this question affirmatively in [9] for varieties that are locally finite. His proof uses tame congruence theory.

In this paper we will prove that a version of the condition R holds for all locally finite varieties with the CEP. If $\mathbf{A}$ lies in a variety with the CEP and $\mathbf{B} \leq \mathbf{A}$ and $\alpha \prec \beta$ in $\operatorname{Con}\mathbf{A}$, then we will see that $\alpha|_B = \beta|_B$ or $\alpha|_B \prec \beta|_B$. In the latter case, we will show that β is abelian over α iff $\beta|_B$ is abelian over $\alpha|_B$. We refine this with information about the relationship between $\operatorname{typ}(\alpha,\beta)$ and $\operatorname{typ}(\alpha|_B,\beta|_B)$. In Section 3, we show how this result simplifies the task of finding the type-set of a locally finite variety with the CEP. In Section 4, we consider locally finite varieties with enough injectives which satisfy a nontrivial, idempotent Mal'cev condition. We use our CEP results to prove that a variety of this kind is congruence modular if and only if the finite, injective members constitute the class $\mathbf{P}_{\text{fin}}(\mathcal{K})$ where $\mathcal{K}$ is the class of subdirectly irreducible, injective algebras in the variety. Tame congruence theory is used throughout and we refer the reader to [5] for the definitions, notation and results of the theory.

2. **Restricting prime quotients.**

The fact stated in the second sentence of the last paragraph is contained in Lemma 2.3. In order to prove it we need a preliminary result. We use the notation that if $\gamma \in \operatorname{Con}\mathbf{A}$ and $S \subseteq A$, then $S^\gamma = \bigcup_{a\in S} a/\gamma$.

LEMMA 2.1. *$\boldsymbol{H}(\mathbf{A})$ has the* CEP *iff for every congruence $\alpha \in \operatorname{Con}\mathbf{A}$, every subalgebra $\mathbf{B} \leq \mathbf{A}$ satisfying $B = B^\alpha$ and every $\delta \in \operatorname{Con}\mathbf{B}$ we have*

$$\alpha|_B < \delta \longrightarrow \big(\alpha + Cg^{\mathbf{A}}(\delta)\big)|_B = \delta. \tag{1}$$

PROOF. Suppose that $\boldsymbol{H}(\mathbf{A})$ has the CEP, $\alpha \in \operatorname{Con}\mathbf{A}$, $\mathbf{B} \leq \mathbf{A}$ and $\delta \in \operatorname{Con}\mathbf{B}$. Then (1) must hold, for it is precisely the condition necessary for the congruence $\delta/(\alpha|_B)$ to extend from $\mathbf{B}/(\alpha|_B)$ to $\mathbf{A}/\alpha$. This does not depend on the condition that $B = B^\alpha$. On the other hand, every subalgebra of $\mathbf{A}/\alpha$ is of the form $\mathbf{B}/(\alpha|_B)$ for some $\mathbf{B} \leq \mathbf{A}$ satisfying $B = B^\alpha$ and every congruence on such a subalgebra is of the form $\delta/(\alpha|_B)$. Therefore, to verify that $\boldsymbol{H}(\mathbf{A})$ has the CEP it suffices to check that (1) holds only for those choices of α, $\mathbf{B}$ and δ for which $B = B^\alpha$. ∎

In [4] it is proved that if $\mathbf{A}$ is a group or ring, then $\boldsymbol{H}(\mathbf{A})$ has the CEP iff $\mathbf{A}$ does. Later, in [7], it is proved that if $\mathbf{A}$ lies in a congruence permutable variety, then $\boldsymbol{H}(\mathbf{A})$ has the CEP iff $\mathbf{A}$ does. Using Lemma 2.1 we can extend these results a little further.

COROLLARY 2.2. *Assume that $\mathbf{A}$ has 3-permuting congruences. $\boldsymbol{H}(\mathbf{A})$ has the* CEP *iff $\mathbf{A}$ does.*

PROOF. We need to show that if **A** has 3-permuting congruences and the CEP then it is impossible to find a congruence $\alpha \in \operatorname{Con} \mathbf{A}$, a subalgebra $\mathbf{B} \leq \mathbf{A}$ satisfying $B = B^{\alpha}$ and a $\delta \in \operatorname{Con} \mathbf{B}$ such that

$$\alpha|_B < \delta < \big(\alpha + Cg^{\mathbf{A}}(\delta)\big)|_B.$$

Assume instead that there is such an α, **B** and δ. Choose $(x,y) \in \big(\alpha + Cg^{\mathbf{A}}(\delta)\big)|_B - \delta$. Since **A** has 3-permuting congruences,

$$\alpha + Cg^{\mathbf{A}}(\delta) = \alpha \circ Cg^{\mathbf{A}}(\delta) \circ \alpha,$$

and so there exist $u, v \in A$ such that $(x,u),(v,y) \in \alpha$ and $(u,v) \in Cg^{\mathbf{A}}(\delta)$. This, along with the facts that $B = B^{\alpha}$ and $x, y \in B$, shows that $(x,u),(v,y) \in \alpha|_B$ and $(u,v) \in Cg^{\mathbf{A}}(\delta)|_B = \delta$. This last equality follows from the fact that **A** has the CEP. We conclude that $(x,y) \in \alpha|_B + \delta = \delta$, which is a contradiction. ∎

We mention that [4] contains an example of a 5-element groupoid **A** which has 4-permuting congruences and the CEP although $\boldsymbol{H}(\mathbf{A})$ does not have the CEP. Therefore the number 3, as it appears in Corollary 2.2, cannot be improved to 4.

Now we return to the project at hand.

LEMMA 2.3. *Assume that* $\mathbf{B} \leq \mathbf{A}$ *and that* $\boldsymbol{H}(\mathbf{A})$ *has the* CEP. *If* $\alpha \prec \beta$ *in* Con **A** *then* $\alpha|_B = \beta|_B$ *or* $\alpha|_B \prec \beta|_B$.

Now assume that $\mathbf{B} \leq \mathbf{A}$ and that **A** has the CEP. If $\gamma \prec \delta$ in Con **B** then there exist $\gamma^* \prec \delta^*$ in Con **A** such that $\gamma^*|_B = \gamma$ and $\delta^*|_B = \delta$.

PROOF. For the first statement suppose that $\alpha|_B < \delta \leq \beta|_B$. Then $Cg^{\mathbf{A}}(\delta) \leq \beta$ but $Cg^{\mathbf{A}}(\delta) \not\leq \alpha$. Since $\alpha \prec \beta$ we must have $\alpha + Cg^{\mathbf{A}}(\delta) = \beta$. Now, the proof of Lemma 2.1 shows that 2.1 (1) holds even if $B \neq B^{\alpha}$, so $\delta = \big(\alpha + Cg^{\mathbf{A}}(\delta)\big)|_B = \beta|_B$. Thus, $\alpha|_B \prec \beta|_B$.

For the second statement let $\delta^* = Cg^{\mathbf{A}}(\delta)$. This congruence is the least congruence on **A** whose restriction to **B** is δ. Further, every other congruence in $I[Cg^{\mathbf{A}}(\gamma), Cg^{\mathbf{A}}(\delta)]$ restricts to γ. Choose $(x,y) \in \delta - \gamma$. Now let γ^* be any congruence in $I[Cg^{\mathbf{A}}(\gamma), Cg^{\mathbf{A}}(\delta)]$ which is maximal with respect to not containing (x,y). Clearly, $\gamma^* \prec \delta^*$. ∎

Another preliminary result referred to in the introduction is the following theorem.

THEOREM 2.4 (KEARNES, KISS). *Suppose that* $\mathbf{A} \times \mathbf{A}$ *has the* CEP *and* β *is an abelian congruence on* **A**. *Then* β *is in the center of* **A**.

PROOF. Let $\mathbf{A}(\beta)$ be the subalgebra of $\mathbf{A} \times \mathbf{A}$ consisting of β-related pairs. Let $\Delta_{\beta,\beta}$ be the congruence on $\mathbf{A}(\beta)$ generated by the β-diagonal, $\{\langle (x,x),(y,y)\rangle \mid (x,y) \in \beta\}$. If $\Delta_{1,\beta}$ denotes the congruence on $\mathbf{A}^2$ generated by the β-diagonal, the CEP in $\mathbf{A}^2$ insures that $\Delta_{1,\beta}|_{\mathbf{A}(\beta)} = \Delta_{\beta,\beta}$.

Now, saying that β is abelian is equivalent to saying that the diagonal subalgebra of $\mathbf{A}(\beta)$ is a union of $\Delta_{\beta,\beta}$-classes. But since $\Delta_{1,\beta}|_{\mathbf{A}(\beta)} = \Delta_{\beta,\beta}$ and $A(\beta)$ is a union of $\Delta_{1,\beta}$-classes, it follows that the diagonal subalgebra of $\mathbf{A}^2$ is a union of $\Delta_{1,\beta}$-classes. This is equivalent to saying that β is central. ∎

Lemma 2.3 shows some correspondence between the prime quotients of an algebra belonging to a CEP variety and the prime quotients of its subalgebras. For the rest of this section, $\mathbf{A}$ is a finite algebra contained in a variety with the CEP and $0_A \prec \beta$ in $\operatorname{Con}\mathbf{A}$. $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ for which $0_B \prec \beta|_B$. To further fix notation, let $U_A \in M_{\mathbf{A}}(0,\beta)$ be a $\langle 0,\beta\rangle$-minimal set and let $e_A \in E(\mathbf{A})$ be an idempotent, unary polynomial for which $e_A(A) = U_A$. Let T_A be a $\langle 0,\beta\rangle$-trace of U_A and let $\mathbf{T}_A \stackrel{\text{def}}{=} \mathbf{A}\mathbf{I}_{T_A}$ be the normally-indexed algebra $\mathbf{A}$ induces on T_A. Define U_B, e_B, T_B and $\mathbf{T}_B$ similarly. We will investigate the relationship between the algebras $\mathbf{T}_A$ and $\mathbf{T}_B$.

LEMMA 2.5. *Assume that $f \in \operatorname{Pol}_1 \mathbf{A}$ is such that $f(T_B) \subseteq T_A$ and that $\delta \in \operatorname{Con}\mathbf{T}_B^k$ contains no pair of the form $\langle (x,x,\ldots,x),(y,y,\ldots,y)\rangle$ where $x \neq y$. Then $Cg^{\mathbf{T}_A^k}(f(\delta))$ contains no pair of the form $\langle (x,x,\ldots,x),(y,y,\ldots,y)\rangle$ where $x \neq y$.*

PROOF. Our strategy will be to show that if δ contains no non-trivial "diagonal pair," then there is a subalgebra $\mathbf{C} \leq \mathbf{A}^k$ containing T_B^k such that $Cg^{\mathbf{C}}(\delta)$ contains no non-trivial diagonal pair. Using the CEP, we'll show that $Cg^{\mathbf{A}^k}(\delta)$ contains no non-trivial diagonal pair. This is already stronger than the conclusion of the lemma since $Cg^{\mathbf{T}_A^k}(f(\delta)) \subseteq Cg^{\mathbf{A}^k}(\delta)$.

Let $\mathbf{C}$ be the subalgebra of $\mathbf{A}^k$ generated by

$$T_B^k \cup \{(x,x,\ldots,x) \mid x \in B\}.$$

The fact that $\mathbf{C}$ contains the diagonal elements of $\mathbf{A}^k$ whose components lie in $\mathbf{B}$ implies that there is an idempotent unary polynomial $e' \in E(\mathbf{C})$ which is just e_B acting coordinatewise on $\mathbf{C}$. Let $U = e'(C)$. If $\theta \stackrel{\text{def}}{=} (\beta \times \beta \times \cdots \times \beta)|_C = (\beta|_B \times \beta|_B \times \cdots \times \beta|_B)|_C$, then $N = T_B^k$ is a $\theta|_U$-class.

CLAIM. $\mathbf{C}|_N$ is polynomially equivalent to $\mathbf{T}_B^k$.

PROOF OF CLAIM. This is a standard tame congruence theory argument (see Lemma 6.14 of [5], for example). These two algebras have the same universe: T_B^k. We only have to show that they have the same polynomials. To see this, it is enough to notice that $f(\bar{x})$ is an m-ary polynomial of either algebra iff

(i) $f(N^m) \subseteq N$, and

(ii) There is an $(m+n)$-ary term t and an n-tuple of elements $\bar{u}$ chosen from $N \cup \{(x,\ldots,x) \mid x \in B\}$ such that $f(\bar{x}) = t^{\mathbf{A}^k}(\bar{x},\bar{u})|_N$.

Now we use Lemma 2.4 of [5] which proves that the restriction map

$$I[0_C,\theta] \longrightarrow \operatorname{Con}\mathbf{C}|_N = \operatorname{Con}\mathbf{T}_B^k \colon \alpha \mapsto \alpha|_N$$

is a homomorphism of $I[0_C,\theta]$ onto $\operatorname{Con}\mathbf{T}_B^k$. Since this map is onto, we have $Cg^{\mathbf{C}}(\delta)|_N = \delta$. Using the CEP in $\mathbf{A}^k$ we even get that $\lambda|_N = \delta$, where $\lambda \stackrel{\text{def}}{=} Cg^{\mathbf{A}^k}(\delta)$.

Now suppose that $\langle (x,\ldots,x),(y,\ldots,y)\rangle \in \lambda$ and $x \neq y$. Then $(x,y) \in \beta - 0_A$, since $\delta \subseteq \beta \times \cdots \times \beta$. Therefore, if $\hat{\mathbf{A}}$ is the diagonal subalgebra of $\mathbf{A}^k$, then $\lambda|_{\hat{A}} =$

$\{\langle(u,\ldots,u),(v,\ldots,v)\rangle \mid (u,v)\in\beta\}$. But, if $a,b\in T_B$ are distinct, this leads to the contradiction that

$$\langle(a,\ldots,a),(b,\ldots,b)\rangle \in \lambda|_N - 0_N = \delta - 0_N.$$

Therefore, λ contains no non-trivial diagonal pair. As we pointed out in the first paragraph, this establishes the lemma. ■

Notice that the proof of Lemma 2.5 includes the proof that the congruences of $\mathbf{T}_B^k$ "extend" to $\mathbf{A}^k$. That is, if $\delta\in\operatorname{Con}\mathbf{T}_B^k$, then $Cg^{\mathbf{A}^k}(\delta)|_{\mathbf{T}_B^k}=\delta$.

LEMMA 2.6. *The congruence β is abelian iff $\beta|_B$ is.*

PROOF. The congruence β is abelian iff for every $(n+1)$-ary term p and (x,y), $(u_i,v_i)\in\beta$ for $i<n$ we have

$$p(x,\bar u)=p(x,\bar v)\longrightarrow p(y,\bar u)=p(y,\bar v).$$

If this holds, then surely the same implication holds when the elements x,y,u_i and v_i are restricted to lie in B. This is precisely what it means for $\beta|_B$ to be abelian. (A similar argument shows that if β is strongly abelian then $\beta|_B$ is strongly abelian.)

Now assume that β is nonabelian. $\mathbf{T}_A$ has exactly two elements, 0 and 1, and this algebra has a binary *pseudo-meet* operation $p(x,y)$ which satisfies $p(0,0)=p(1,0)=p(0,1)=0$ and $p(1,1)=1$. Choose two distinct elements $a,b\in T_B$. There is a unary polynomial $f\in\operatorname{Pol}_1\mathbf{A}$ such that $f(T_B)\subseteq T_A$ and $f(a)$ and $f(b)$ are distinct, β-related elements of U_A. Since 0 and 1 are the only distinct, β-related elements of U_A we may assume that $f(a)=0$ and $f(b)=1$.

Let $S=\{\langle(a,a,b),(b,b,b)\rangle,\langle(b,a,a),(b,b,b)\rangle\}$ and let $\delta=Cg^{\mathbf{T}_B^3}(S)$. Observe that $\langle(0,0,1),(1,1,1)\rangle=\langle\big(f(a),f(a),f(b)\big),\big(f(b),f(b),f(b)\big)\rangle\in\lambda\overset{\text{def}}{=}Cg^{\mathbf{T}_A^3}\big(f(\delta)\big)$. Similarly, $\langle(1,0,0),(1,1,1)\rangle\in\lambda$. Applying the pseudo-meet operation we find that

$$(1,1,1)=p\big((1,1,1),(1,1,1)\big)\lambda p\big((0,0,1),(1,0,0)\big)=(0,0,0).$$

Lemma 2.5 applies now to show that δ must contain a non-trivial diagonal pair.

CASE 1. Assume that the minimal algebra $\mathbf{T}_B$ is of type **1**. That is, that $\mathbf{T}_B$ is polynomially equivalent to a simple G-set. In this case, each polynomial of $\mathbf{T}_B$ is unary and it follows that the non-constant polynomials of $\mathbf{T}_B^3$ are precisely the non-constant polynomials of $\mathbf{T}_B$ acting on $\mathbf{T}_B^3$ coordinatewise. Suppose that $x_0,\ldots,x_n$ is a sequence of elements for which each $(x_i,x_{i+1})=\big(p(u),p(v)\big)$ or $\big(p(v),p(u)\big)$ where $(u,v)\in S$ and p is some unary polynomial of $\mathbf{T}_B$ acting coordinatewise. Further, assume that $x_0=(y,y,y)\neq(z,z,z)=x_n$ and that n is minimal for any sequence with the properties we have required. Since, for $(u,v)\in S$, $p(u)\neq p(v)$ implies that exactly one of $\{p(u),p(v)\}$ is a diagonal element, it follows that $n=2$. Arguing by symmetry, we may assume that

$$x_0=p\big((b,b,b)\big),\ x_1=p\big((a,a,b)\big)=q(u),\ x_2=q\big((b,b,b)\big)\neq x_0$$

where $u = (a,a,b)$ or (b,a,a) and p,q are unary polynomials of $\mathbf{T}_B$ acting diagonally. However, both choices for u lead to a contradiction. If $u = (a,a,b)$ then

$$p^{\mathbf{T}_B^3}\big((a,a,b)\big) = q^{\mathbf{T}_B^3}\big((a,a,b)\big), \text{ so}$$
$$p^{\mathbf{T}_B}(b) = q^{\mathbf{T}_B}(b), \text{ and}$$
$$x_0 = p^{\mathbf{T}_B^3}\big((b,b,b)\big) = q^{\mathbf{T}_B^3}\big((b,b,b)\big) = x_2.$$

Similarly, if $u = (b,a,a)$,

$$p^{\mathbf{T}_B^3}\big((a,a,b)\big) = q^{\mathbf{T}_B^3}\big((b,a,a)\big), \text{ so}$$
$$p^{\mathbf{T}_B}(b) = q^{\mathbf{T}_B}(a) = p^{\mathbf{T}_B}(a) = q^{\mathbf{T}_B}(b), \text{ and}$$
$$x_0 = p^{\mathbf{T}_B^3}\big((b,b,b)\big) = q^{\mathbf{T}_B^3}\big((b,b,b)\big) = x_2.$$

Hence, δ collapses no distinct diagonal pairs in $\mathbf{T}_B^3$. This contradicts the result of the last paragraph, so $\mathbf{T}_B$ is not of type $\mathbf{1}$.

CASE 2. Assume that $\mathbf{T}_B$ is of type $\mathbf{2}$. In this case, $\mathbf{T}_B$ is polynomially equivalent to a 1-dimensional vector space with b for the origin. Now the δ-equivalence class containing (b,b,b) is a subspace of $\mathbf{T}_B^3$ containing (a,a,b) and (b,a,a). Indeed, it is precisely the 2-dimensional subspace of triples (x,y,z) for which $x+z = y$ in $\mathbf{T}_B$. If $\langle (x,x,x),(y,y,y)\rangle \in \delta$, then $(z,z,z) = (x-y,x-y,x-y)$ is in this 2-dimensional subspace. However, the only diagonal triple (z,z,z) in this subspace is (b,b,b), so $x-y = b$ or $x = y$. This contradicts our earlier result that δ contains a non-trivial pair of diagonal elements, so $\mathbf{T}_B$ is not of type $\mathbf{2}$.

We have ruled out the possibility that $\mathrm{typ}(0_B, \beta|_B)$ is $\mathbf{1}$ or $\mathbf{2}$. As these are the only abelian types, we conclude that $\beta|_B$ is nonabelian. ∎

LEMMA 2.7. $\mathrm{typ}(0_A,\beta) = \mathbf{1}$ *if and only if* $\mathrm{typ}(0_B,\beta|_B) = \mathbf{1}$; $\mathrm{typ}(0_A,\beta) = \mathbf{2}$ *if and only if* $\mathrm{typ}(0_B,\beta|_B) = \mathbf{2}$.

PROOF. We have already shown that $\mathrm{typ}(0_A,\beta) \in \{\mathbf{1},\mathbf{2}\}$ if and only if $\mathrm{typ}(0_B,\beta|_B) \in \{\mathbf{1},\mathbf{2}\}$. Also, the first paragraph of the proof of Lemma 2.6 explains why the assumption that $\mathrm{typ}(0_A,\beta) = \mathbf{1}$, i.e., β is strongly abelian, implies that $\mathrm{typ}(0_B,\beta|_B) = \mathbf{1}$. We need to rule out the possibility that $\mathrm{typ}(0_A,\beta) = \mathbf{2}$ and $\mathrm{typ}(0_B,\beta|_B) = \mathbf{1}$.

Assume that the type of $\mathbf{T}_B$ is $\mathbf{1}$. Choose distinct elements $a,b \in T_B$ and $f \in \mathrm{Pol}_1\,\mathbf{A}$ such that $f(T_B) \subseteq T_A$ and $f(a)$ and $f(b)$ are distinct. Let $\delta = Cg^{\mathbf{T}_B^3}(S)$ where $S = \{\langle (a,a,b),(a,a,a)\rangle, \langle (a,b,a),(a,a,a)\rangle, \langle (b,a,a),(a,a,a)\rangle\}$. Arguing as in Case 1 of the proof of Lemma 2.6 one can show that δ contains no non-trivial diagonal pairs. Lemma 2.5 applies to show that $\lambda = Cg^{\mathbf{T}_A^3}\big(f(\delta)\big)$ contains no non-trivial diagonal pairs. However, setting $0 = f(a)$ and $1 = f(b)$, we find that λ does contain the pairs $\langle (0,0,1),(0,0,0)\rangle = \langle \big(f(a),f(a),f(b)\big), \big(f(a),f(a),f(a)\big)\rangle$, $\langle (0,1,0),(0,0,0)\rangle$ and $\langle (1,0,0),(0,0,0)\rangle$. If $\mathbf{T}_A$ is of type $\mathbf{2}$, then it is polynomially equivalent to a vector space with origin at 0. $\mathbf{T}_A$ has a polynomial operation $p(x,y,z) = x+y+z$. Applying p coordinatewise in $\mathbf{T}_A^3$ yields

$$(0,0,0) = p\big((0,0,0),(0,0,0),(0,0,0)\big)\lambda p\big((0,0,1),(0,1,0),(1,0,0)\big) = (1,1,1)$$

contradicting the result that λ contains no diagonal pairs. Hence, the type of $\mathbf{T}_A$ is not **2**. ∎

LEMMA 2.8. *If* typ$(0_A, \beta) \in \{\mathbf{3}, \mathbf{4}\}$, *then* typ$(0_B, \beta|_B) \in \{\mathbf{3}, \mathbf{4}\}$.

PROOF. If typ$(0_A, \beta) \in \{\mathbf{3}, \mathbf{4}\}$, then by Lemma 2.6 we must have typ$(0_B, \beta|_B) \in \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$. Assume that typ$(0_A, \beta) \in \{\mathbf{3}, \mathbf{4}\}$ and typ$(0_B, \beta|_B) = \mathbf{5}$. T_B contains exactly two elements, say $T_B = \{a, b\}$. $\mathbf{T}_B$ has a pseudo-meet operation, p, and we may assume that $p(a,a) = p(a,b) = p(b,a) = a$ and $p(b,b) = b$. There is an $f \in \mathrm{Pol}_1\,\mathbf{A}$ such that $f(T_B) \subseteq T_A$ and $f(a) = 0, f(b) = 1$ where $T_A = \{0, 1\}$.

Let $\delta = Cg^{\mathbf{T}_B^2}(\langle (a,b),(a,a)\rangle, \langle (b,a),(a,a)\rangle)$. $\mathbf{T}_B^2$ is polynomially equivalent to the square of the 2-element semilattice on $\{a, b\}$ which has $a \leq b$. δ is the Rees congruence associated with the ideal $I = \{(a,a),(a,b),(b,a)\}$. The δ-class of any element outside of I contains exactly one element, so $\langle (a,a),(b,b)\rangle \notin \delta$. Thus, δ contains no non-trivial diagonal pairs. Using Lemma 2.5 we get that $\lambda = Cg^{\mathbf{T}_A^2}\big(f(\delta)\big)$ contains no non-trivial diagonal pairs. It does contain the pairs $\langle (0,1),(0,0)\rangle = \langle \big(f(a),f(b)\big), \big(f(a),f(a)\big)\rangle$, and $\langle (1,0),(0,0)\rangle$. Further, $\mathbf{T}_A$ has a binary polynomial operation g which satisfies $g(0,0) = 0$ and $g(0,1) = g(1,0) = g(1,1) = 1$. Using g coordinatewise in $\mathbf{T}_A^2$ we get

$$(0,0) = g\big((0,0),(0,0)\big)\lambda\, g\big((0,1),(1,0)\big) = (1,1)$$

which is a contradiction. This proves the lemma. ∎

EXAMPLE 1. In this example, we will show that even if typ$(0_A, \beta) = \mathbf{3}$ it is possible to have typ$(0_B, \beta|_B) = \mathbf{4}$. We will take $\mathbf{A}$ to be a finite, hereditarily simple algebra that generates a congruence distributive variety. Since $\mathcal{V} = \mathcal{V}(\mathbf{A})$ is congruence distributive, it satisfies C2 and R. $\boldsymbol{H}(\mathbf{A})$ has the CEP, so $\mathcal{V}$ does.

Our algebra $\mathbf{A}$ is a 3-element lattice with an additional binary operation: $\mathbf{A} = \langle \{0, a, 1\}; \vee, \wedge, r\rangle$. The lattice ordering of A is $0 < a < 1$. We define the operation r by $r(a,x) = 1$ for all $x \in A$ and $r(x,y) = y$ when $x \neq a$. It is routine to check that $\mathbf{A}$ is simple and, since it has only 3 elements, every subalgebra is simple. $\mathbf{A}$ has a lattice reduct so it generates a congruence distributive variety. Let $\beta = 1_A$. The polynomial $e(x) = x \vee a \in \mathrm{Pol}_1\,\mathbf{A}$ is an idempotent polynomial for which $e(A) = N = \{a, 1\} \in M_{\mathbf{A}}(0_A, \beta)$. $\mathbf{A}|_N$ is closed under the polynomial operations $\vee$, $\wedge$ and $r(x,a)$, which generate all the boolean operations on N. Hence, typ$(0_A, \beta) = \mathbf{3}$. Now, $\mathbf{A}$ has a subalgebra $\mathbf{B} = \langle \{0, 1\}; \vee, \wedge, r\rangle$ on which r is a trivial projection operation. $\mathbf{B}$ is just the 2-element lattice, so it is simple of type **4** and $\beta|_B = 1_A|_B = 1_B$. Thus, typ$(0_B, \beta|_B) = \mathbf{4}$.

EXAMPLE 2. In this example we prove that it is possible to have typ$(0_A, \beta) = \mathbf{4}$ and typ$(0_B, \beta|_B) = \mathbf{3}$.

This time our algebra $\mathbf{A}$ is a 3-element lattice with an additional unary operation: $\mathbf{A} = \langle \{0, a, 1\}; \vee, \wedge, f\rangle$. As in the last example, the lattice ordering of A is $0 < a < 1$. We define f by $f(0) = 1, f(a) = a$ and $f(1) = 0$. It is easy to see that $\mathbf{A}$ is (hereditarily) simple and has a lattice reduct, so $\mathcal{V}(\mathbf{A})$ has the CEP. $\mathbf{B}$ is the subalgebra with universe $\{0, 1\}$. We let $\beta = 1_A$ and so $\beta|_B = 1_B$. $\mathbf{B}$ is a 2-element boolean algebra which means that

$\operatorname{typ}(0_B, \beta|_B) = \mathbf{3}$. $e(x) = x \vee a \in \operatorname{Pol}_1 \mathbf{A}$ is an idempotent polynomial for which $e(A) = N = \{a, 1\} \in M_{\mathbf{A}}(0_A, \beta)$. N is closed under $\vee$ and $\wedge$, so $\mathbf{A}|_N$ is of type **3** or **4**. To prove that $\mathbf{A}|_N$ is of type **4** we must show that there is no $h \in \operatorname{Pol}_1 \mathbf{A}$ such that $h(a) = 1$ and $h(1) = a$. This is equivalent to proving that $(1, a) \notin \operatorname{Sg}^{\mathbf{A}^2}\big(\{(0,0), (a,a), (1,1), (a,1)\}\big)$. We leave this chore to the reader.

Now we begin a sequence of results which culminate in the proof that if β is abelian, then T_B and T_A are polynomially isomorphic in $\mathbf{A}$ and $\mathbf{T}_B$ and $\mathbf{T}_A$ are weakly isomorphic. (Subsets $X, Y \subseteq A$ are *polynomially isomorphic in* $\mathbf{A}$ if there are polynomials $f, g \in \operatorname{Pol}_1 \mathbf{A}$ such that $f(X) = Y$, $g(Y) = X$, $gf|_X = \mathrm{id}_X$ and $fg|_Y = \mathrm{id}_Y$. The algebra $\mathbf{C}$ is said to be *weakly isomorphic* to $\mathbf{D}$ if $\mathbf{C}$ is isomorphic to an algebra $\mathbf{D}'$ whose universe is D and for which $\operatorname{Clo}\mathbf{D}' = \operatorname{Clo}\mathbf{D}$.)

LEMMA 2.9. *If $h \in \operatorname{Pol}_1 \mathbf{A}$, then*

$$h|_{T_B}\colon T_B \longrightarrow A$$

is either 1-1 or constant. In particular, $|T_B| \leq |T_A|$.

PROOF. If $|T_B| = 2$ the result is trivial so assume that $|T_B| > 2$; necessarily $\mathbf{T}_B$ is abelian. Also, assume that h is neither 1-1 nor constant on T_B. There must be distinct elements $a, b, c \in T_B$ for which $h(a) \neq h(b) = h(c)$. We will show that this leads to a contradiction.

Choose $g \in \operatorname{Pol}_1 \mathbf{A}$ such that $g\big(h(a)\big) \neq g\big(h(b)\big)$ and $g\big(h(T_B)\big) \subseteq T_A$ and set $f(x) = g\big(h(x)\big)$. Let $S = \{\langle (a,b,c), (a,a,a)\rangle, \langle (a,b,b), (b,b,b)\rangle\}$ and $\delta = Cg^{\mathbf{T}_B^3}(S)$. If $0 = f(a)$ and $1 = f(b) = f(c)$, then clearly $\lambda = Cg^{\mathbf{T}_A^3}\big(f(\delta)\big)$ contains $\langle (0,1,1), (0,0,0)\rangle$ and $\langle (0,1,1), (1,1,1)\rangle$, so λ contains the non-trivial diagonal pair $\langle (0,0,0), (1,1,1)\rangle$. Lemma 2.5 demands that δ contain a non-trivial diagonal pair.

CLAIM 1. If $\mathbf{T}_B$ has type **1**, then let $x_0, \ldots, x_n$ be a sequence of elements from T_B^3 where each $(x_i, x_{i+1}) = \big(p(u), p(v)\big)$ or $\big(p(v), p(u)\big)$ with $(u, v) \in S$ and $p \in \operatorname{Pol}_1 \mathbf{T}_B$ acts coordinatewise. We assume that $x_0 = (y,y,y) \neq (z,z,z) = x_n$ and that n is minimal for any sequence with these properties. As in the proof of Lemma 2.6, $n = 2$. Now, let η_i denote the kernel of the projection onto the i^{th} factor of $\mathbf{T}_B^3$. Observe that

$$\langle (y,y,y), (z,z,z)\rangle \notin Cg^{\mathbf{T}_B^3}\big(\langle (a,b,c), (a,a,a)\rangle\big) \leq \eta_0$$

and

$$\langle (y,y,y), (z,z,z)\rangle \notin Cg^{\mathbf{T}_B^3}\big(\langle (a,b,b), (b,b,b)\rangle\big) \leq \eta_2.$$

Thus, the only way that x_0, x_1, x_2 could be a sequence satisfying all of our conditions is if

$$x_0 = p\big((a,a,a)\big) \neq x_1 = p\big((a,b,c)\big) = q\big((a,b,b)\big) \neq x_2 = q\big((b,b,b)\big) \neq x_0$$

is such a sequence. Here p and q are unary polynomials of $\mathbf{T}_B$ acting coordinatewise. Since $p\big((a,a,a)\big) \neq p\big((a,b,c)\big)$ it follows that p is non-constant. However, $p\big((a,b,c)\big) =$

$q\big((a,b,b)\big)$, so $p(b) = q(b) = p(c)$. Hence, p is not a permutation. This is impossible, since $\mathbf{T}_B$ is a minimal algebra. Therefore, $\mathbf{T}_B$ is not of type **1**.

CLAIM 2. Assume that the type of $\mathbf{T}_B$ is **2**. $\mathbf{T}_B$ is polynomially equivalent to a 1-dimensional vector space with origin at a. The δ-class of (a,a,a) is a subspace of $\mathbf{T}_B^3$ spanned by (a,b,c) and $(b,b,b)-(a,b,b) = (b,a,a)$. There is a unary polynomial, $k(x)$, of $\mathbf{T}_B$ such that $k(a) = a$ and $k(b) = c \neq b$. No element of T_B other than a is fixed by k, since no unary polynomial of a 1-dimensional vector space other than the polynomial $p(x) = x$ fixes more than one element and $k(b) \neq b$. The set V of $(x,y,z) \in T_B^3$ satisfying $z = k(y)$ is a subspace of $\mathbf{T}_B^3$ containing (a,b,c) and (b,a,a). If $\langle (y,y,y),(z,z,z)\rangle \in \delta$, then $(w,w,w) = (y-z, y-z, y-z)$ is δ-related to (a,a,a), so $(w,w,w) \in V$. But then $w = k(w)$ and $a = w = y-z$, which forces $y = z$. This shows that δ contains no non-trivial diagonal pair, so $\mathbf{T}_B$ is not of type **2**

We've reached the contradiction that $\mathbf{T}_B$ is abelian, but not of type **1** or **2**. Therefore, h is 1-1 or constant, as we claimed.

For the second statement of the lemma, assume that $a', b' \in T_B$ are distinct. There is a unary polynomial $f' \in \mathrm{Pol}_1\,\mathbf{A}$ such that $f'(T_B) \subseteq T_A$ and $f'(a') \neq f'(b')$. Since $f'|_{T_B}$ is not constant it must be 1-1, so $|T_B| \leq |T_A|$. ∎

LEMMA 2.10. *If $T_A \subseteq B$, then T_A and T_B are polynomially isomorphic in* $\mathbf{A}$. *In fact, there is an $f \in \mathrm{Pol}_1\,\mathbf{B}$ such that $f : T_A \overset{\mathbf{A}}{\simeq} T_B$.*

PROOF. We need to show that if $T_A \subseteq B$, then there is an $f \in \mathrm{Pol}_1\,\mathbf{B}$ and a $g \in \mathrm{Pol}_1\,\mathbf{A}$ such that $f(T_A) = T_B$, $g(T_B) = T_A$, $gf|_{T_A} = \mathrm{id}_{T_A}$ and $fg|_{T_B} = \mathrm{id}_{T_B}$.

Choose distinct elements $0, 1 \in T_A$. Since $(0,1) \in \beta|_B - 0_B$ there is an $f \in \mathrm{Pol}_1\,\mathbf{B}$ such that $f(B) = U_B, f(\{0,1\}) \subseteq T_B$ and $f(0) \neq f(1)$. This claim relies on Theorem 2.8 of [5]. T_A is a $\langle 0_A, \beta\rangle$-trace and f is a polynomial of $\mathbf{A}$ that is non-constant on T_A, so f is 1-1 on T_A. Now, $T_A \subseteq B$, so $f(T_A) \subseteq U_B$. Every element of $f(T_A)$ is β-related to $f(0)$ and the set of all elements β-related to $f(0)$ in U_B is just T_B. Hence, $f(T_A) \subseteq T_B$ and $f\colon T_A \longrightarrow T_B$ is 1-1. But Lemma 2.9 proves that $|T_B| \leq |T_A|$, so $f\colon T_A \longrightarrow T_B$ is onto, also.

The polynomial image of a $\langle 0_A, \beta\rangle$-trace is either a singleton or another $\langle 0_A, \beta\rangle$-trace, so the argument of the last paragraph proves that T_B is a $\langle 0_A, \beta\rangle$-trace. All $\langle 0_A, \beta\rangle$-traces are polynomially isomorphic in $\mathbf{A}$, so there is an $h \in \mathrm{Pol}_1\,\mathbf{A}$ such that $h\colon T_B \longrightarrow T_A$ is a bijection. There is a suitable choice of k such that $g = h(fh)^k \in \mathrm{Pol}_1\,\mathbf{A}$ is an inverse to f. This f and g witness the polynomial isomorphism between T_A and T_B. ∎

The proof of our next result uses a slight modification of the arguments in Lemma 4.4 and Lemma 4.6 of [9].

LEMMA 2.11. *Assume that β is abelian. Then* $\mathbf{B}$ *contains a $\langle 0_A, \beta\rangle$-trace.*

PROOF. First, we reduce this lemma to a special case. If this lemma is false, then there is a finite algebra $\mathbf{A}$ which generates a CEP variety and has an abelian congruence $\beta \succ 0_A$ which restricts non-trivially to some subalgebra $\mathbf{B}$ that contains no $\langle 0_A, \beta\rangle$-trace. By extending $\mathbf{B}$ if necessary, we may assume that $\mathbf{B}$ is maximal among subalgebras of $\mathbf{A}$ which contain no $\langle 0_A, \beta\rangle$-trace. Further, expanding $\mathbf{A}$ by adding constant operations to denote the elements of **B** will not injure our assumptions, so we do this. Now observe that if $\mathbf{A}'$ is a subalgebra of **A** which properly contains **B**, then **B** contains no $\langle 0_{A'}, \beta|_{A'}\rangle$-trace, either. The argument for this is as follows. $\mathbf{A}'$ must contain a $\langle 0_A, \beta\rangle$-trace by the maximality of **B**; so, by Lemma 2.10, every $\langle 0_{A'}, \beta|_{A'}\rangle$-trace is a $\langle 0_A, \beta\rangle$-trace and none of these are contained in **B**. It is also true that $0_{A'} \prec \beta|_{A'}$, as Lemma 2.3 shows, and that $\beta|_{A'}$ is abelian. This shows that, replacing **A** by $\mathbf{A}'$ if necessary, we may assume that the only subuniverses of **A** are A and B. For the rest of the proof we assume this (and also that every element of **B** is a constant term).

Since $\beta|_B > 0_B$, there is a pair $(b,c) \in \beta|_B - 0_B$. Theorem 2.8 (5) of [5] implies that there is a chain of elements $b = x_0, \ldots, x_n = c$ such that $(x_i, x_{i+1}) \in N_i$ where N_i is a $\langle 0_A, \beta\rangle$-trace. If $N_0 \subseteq B$ we are done, so assume that there is an element $a \in N_0 \cap (A - B)$. Our assumptions from the last paragraph imply that a generates **A**. Since $(b,c) \in \beta - 0_A$, there is an $f \in \operatorname{Pol}_1 \mathbf{A}$ such that $f(A) \in M_{\mathbf{A}}(0_A, \beta)$ and $f(b) \neq f(c)$. The element a generates **A** so we can find a binary term $t(x,y)$ such that $f(x) = t^{\mathbf{A}}(x,a)$. Let $g(x)$ be the unary term $t(x,b)$. Theorem 2.4 proves that β is central; so, for $u, v \in A$, $t(u,a) = t(v,a)$ holds iff $t(u,b) = t(v,b)$ holds. That is, f and g have the same kernel. Hence $|g(A)| = |f(A)|$ and $g(b) \neq g(c)$, implying that $g(A) = U \in M_{\mathbf{A}}(0_A, \beta)$. Now choose an idempotent polynomial $e \in E(\mathbf{A})$ such that $e(A) = U$. We can write $e(x) = s^{\mathbf{A}}(x,a)$ for some term $s(x,y)$ chosen so that **A** satisfies the equation $s\big(s(x,y),y\big) \approx s(x,y)$. In fact, the β,1-term condition implies that **A** satisfies $s\big(s(x,y),z\big) \approx s(x,z)$ whenever $(y,z) \in \beta$. This implies that $e'(x) = s(x,b)$ is an idempotent, unary term and that $ee' = e$ and $e'e = e'$. If $W = e'(A)$, then $e'(U) = W$ and $e(W) = U$. Thus, $e'g(b)$ and $e'g(c)$ are distinct elements of $W \cap B$ and $W \in M_{\mathbf{A}}(0_A, \beta)$. Let N be the $\langle 0_A, \beta\rangle$-trace of W that contains $e'g(b)$ and $e'g(c)$. We will proceed to show that $N \subseteq B$. The conclusions of this paragraph that we will need to remember is that e' is an idempotent unary term, $e'(A) = W \in M_{\mathbf{A}}(0_A, \beta)$, N is a trace of W and $0 = e'g(b)$, $1 = e'g(c) \in N \cap B$ are distinct.

CASE 1. Assume that $\operatorname{typ}(0_A, \beta) = \mathbf{1}$. This means that $\mathbf{A}|_N$ is polynomially equivalent to a simple G-set.

Suppose that $N \not\subseteq B$. Choose $a' \in N - B$; **A** is generated by a'. Of course, 0, 1 and a' are distinct elements of N. Since $\mathbf{A}|_N$ is polynomially equivalent to a simple G-set with at least 3 elements, there is a $h \in \operatorname{Pol}_1 \mathbf{A}$ such that $h(N) = N$ and $h(0) = a'$. Choose a term $q(x,y)$ such that $h(x) = q^{\mathbf{A}}(x,a')$. N is closed under $e'q^{\mathbf{A}}(x,y)$ and this operation depends on its first variable. But the type of $\langle 0_A, \beta\rangle$ is **1**, so $e'q^{\mathbf{A}}$ restricted to N does not depend on its second variable. Hence, $a' = e'q(0,a') = e'q(0,0) \in B$, which is a contradiction. We conclude that if the type of $\langle 0_A, \beta\rangle$ is **1**, then $N \subseteq B$.

CASE 2. Assume that $\operatorname{typ}(0_A, \beta) = \mathbf{2}$. This means that $\mathbf{A}|_N$ is polynomially equivalent to a 1-dimensional vector space. We will argue that the polynomial operations of $\mathbf{A}|_N$ are the restrictions of terms.

There is a term operation $d(x,y,z,w)$ such that $d^{\mathbf{A}}(x,y,z,a)$ is a pseudo-Mal'cev operation of $\mathbf{A}|_W$. Therefore, if C is the body of W, $d^{\mathbf{A}}(x,u,0,a)$ and $d^{\mathbf{A}}(u,x,0,a)$ are permutations of W whenever $u \in C$. Hence, the term $r(x,y) = e'd(x,y,0,b)$ has the property that the polynomials $r^{\mathbf{A}}(x,u)$ and $r^{\mathbf{A}}(u,x)$ are permutations of W whenever $u \in C$. A polynomial which is a permutation cannot map a trace into the tail, so these permutations leave the body (and the tail) of W invariant. In particular, C is closed under r.

The term r is a quasigroup operation on C so, by the technique of Lemma 4.6 of [5], we can construct from r a term $p(x,y,z)$ such that $\mathbf{A}|_C$ is closed under p and satisfies

$$p(x,y,y) = x = p(y,y,x).$$

$N \subseteq C$ is closed under p, too, since p is idempotent and N is a $\beta|_C$-class. The only vector space polynomial satisfying the equations just listed is $p(x,y,z) = x - y + z$. Therefore, the abelian group operations of $\mathbf{A}|_N$ are the restriction of terms: $0 \in B$, $x + y = p(x,0,y)$ and $-x = p(0,x,0)$.

Now we need to show that the unary polynomials which are the scalar multiplications of $\mathbf{A}|_N$ are induced by unary terms. Suppose that $k_0(x),\ldots,k_n(x)$ are polynomials of $\mathbf{A}$ whose restrictions to N are the distinct scalar multiplications. Let $t_i(x,y)$ be terms such that $k_i(x) = t_i^{\mathbf{A}}(x,a)$. Let $m_i(x)$ be the term $e't_i(x,b) - e't_i(0,b)$. N is closed under the operations $m_i^{\mathbf{A}}(x)$ and clearly $m_i^{\mathbf{A}}(0) = 0$. Further, the kernel of $m_i^{\mathbf{A}}(x)$ restricted to N is the same as the kernel of $e't_i^{\mathbf{A}}(x,b)$ which is the same as the kernel of $e't_i^{\mathbf{A}}(x,a)$; that is, it is 0_N. Therefore, by showing that the m_i are distinct we will show that they induce all the scalar multiplications.

Assume that

$$e't_i^{\mathbf{A}}(x,b) - e't_i^{\mathbf{A}}(0,b) = e't_j^{\mathbf{A}}(x,b) - e't_j^{\mathbf{A}}(0,b).$$

Rewrite this as

$$e't_i^{\mathbf{A}}(x,b) - e't_j^{\mathbf{A}}(x,b) = e't_i^{\mathbf{A}}(0,b) - e't_j^{\mathbf{A}}(0,b).$$

Now, using the β,1-term condition we get

$$e't_i^{\mathbf{A}}(x,a) - e't_j^{\mathbf{A}}(x,a) = e't_i^{\mathbf{A}}(0,a) - e't_j^{\mathbf{A}}(0,a),$$

from which it follows that $e't_i^{\mathbf{A}}(x,a) = e't_j^{\mathbf{A}}(x,a)$, or $k_i(x) = k_j(x)$. This implies $i = j$.

We have succeeded in showing that the algebra $\mathbf{A}|_N$ is polynomially equivalent to a 1-dimensional vector space whose operations are the restriction of term operations of $\mathbf{A}$. $N \cap B$ is closed under these operations, so it is a subspace. The distinct elements 0 and 1 lie in $N \cap B$, so $N = N \cap B$. Hence, if $\mathrm{typ}(0_A,\beta) = \mathbf{2}$, then $N \subseteq B$. This establishes the lemma. ∎

LEMMA 2.12. *If β is abelian, then T_A and T_B are polynomially isomorphic in* $\mathbf{A}$ *and* $\mathbf{T}_A$ *and* $\mathbf{T}_B$ *are weakly isomorphic.*

PROOF. If β is abelian, then Lemma 2.11 proves that B contains a $\langle 0_A, \beta \rangle$-trace, N. N is polynomially isomorphic in $\mathbf{A}$ to T_A since any two $\langle 0_A, \beta \rangle$-traces are polynomially isomorphic in $\mathbf{A}$ (this follows from Corollary 5.2 (2) of [5]). Further, T_B is polynomially isomorphic in $\mathbf{A}$ to N by Lemma 2.10. Since the notion of polynomial isomorphism of subsets of A is an equivalence relation, $T_A \overset{\mathbf{A}}{\simeq} T_B$.

To show that $\mathbf{T}_A$ and $\mathbf{T}_B$ are weakly isomorphic, we must show that $\mathbf{T}_A$ is isomorphic to an algebra $\mathbf{C}$ where the universe of $\mathbf{C}$ is T_B and where Clo $\mathbf{C}$ = Clo $\mathbf{T}_B$. The polynomial isomorphism of the last paragraph provides a way of transferring the structure of $\mathbf{T}_A$ onto the set T_B. That is, if $f : T_A \overset{\mathbf{A}}{\simeq} T_B$ has polynomial inverse g, then we will let the operations of $\mathbf{C}$ be the operations of $\mathbf{A}|_{T_B}$ indexed as follows: for every k-ary operation symbol t in the type of $\mathbf{T}_A$ we let $t^{\mathbf{C}}(\bar{x}) = ft^{\mathbf{T}_A}\big(g(\bar{x})\big) \in \operatorname{Pol}_k \mathbf{A}|_{T_B}$. For this indexing, f is obviously an isomorphism from $\mathbf{T}_A$ to $\mathbf{C}$. Now Clo $\mathbf{C}$ is just the clone of polynomial operations of $\mathbf{A}$ under which T_B is closed. Clo $\mathbf{T}_B$ is the clone of polynomial operations of $\mathbf{B}$ under which T_B is closed. This means that Clo $\mathbf{T}_B \subseteq$ Clo $\mathbf{C}$. We will be finished if we can show that every polynomial of $\mathbf{A}$ under which T_B is closed is equal to a polynomial of $\mathbf{T}_B$.

CASE 1. Assume that $\operatorname{typ}(0_A, \beta) = \mathbf{1}$. Each of the algebras $\mathbf{C}$ and $\mathbf{T}_B$ possesses all of the polynomials of a simple G-set on T_B. Let $h(x)$ be a non-constant polynomial of $\mathbf{C}$; necessarily, h is a permutation of $T_B = \{a_0, \ldots, a_{n-1}\}$. We fix an $r < n$ such that $a_r \neq h(a_0)$. Let δ be the congruence on $\mathbf{T}_B^n$ that is generated by the pairs

$$\langle (a_0, a_0, \ldots, a_0), (a_0, a_1, \ldots, a_{n-1})\rangle, \langle (a_r, a_r, \ldots, a_r), \big(h(a_0), h(a_1), \ldots, h(a_{n-1})\big)\rangle.$$

We apply Lemma 2.5; for the polynomial f in the lemma we take id_{T_B}. In order to apply this lemma we are using the fact that T_B is a $\langle 0_A, \beta\rangle$-trace. This follows from the polynomial isomorphism between T_A and T_B. Let $\lambda = Cg^{\mathbf{C}}(\delta)$. Since $h \in \operatorname{Pol}_1 \mathbf{C}$,

$$\begin{aligned}\big(h(a_0), h(a_0), \ldots, h(a_0)\big)\lambda\big(h(a_0), h(a_1), \ldots, h(a_{n-1})\big)\\ = \big(h(a_0), h(a_1), \ldots, h(a_{n-1})\big)\lambda(a_r, a_r, \ldots, a_r).\end{aligned}$$

The pair $\langle \big(h(a_0), h(a_0), \ldots, h(a_0)\big), (a_r, a_r, \ldots, a_r)\rangle \in \lambda$ is a non-trivial diagonal pair, so δ must contain a non-trivial diagonal pair. Arguing as we did in the proofs of Lemmas 2.6 and 2.9, there must be non-constant, unary polynomials $p, q \in \operatorname{Pol}_1 \mathbf{T}_B$ such that

$$\begin{aligned}x_0 &= \big(p(a_0), p(a_0), \ldots, p(a_0)\big) \neq (p(a_0), p(a_1), \ldots, p(a_{n-1})\big) = x_1\\ &= \big(qh(a_0), qh(a_1), \ldots, qh(a_{n-1})\big) \neq \big(q(a_r), q(a_r), \ldots, q(a_r)\big) = x_2 \neq x_0.\end{aligned}$$

But this says that $p(x) = qh(x)$ for all $x \in T_B$. Since q is a permutation of T_B, $q^{-1}(x) \in \operatorname{Pol}_1 \mathbf{T}_B$ and $h = q^{-1}p \in \operatorname{Pol}_1 \mathbf{T}_B$. Thus, Clo $\mathbf{T}_B$ = Clo $\mathbf{C}$.

CASE 2. Now assume that $\operatorname{typ}(0_A, \beta) = \mathbf{2}$. Each of Clo $\mathbf{T}_B$ and Clo $\mathbf{C}$ is the polynomial clone of a 1-dimensional vector space on T_B; further, Clo $\mathbf{T}_B \subseteq$ Clo $\mathbf{C}$. The polynomial clone of a 1-dimensional vector space is generated by the constant operations, the (unique) Mal'cev operation $d(x, y, z) = x - y + z$ and the non-constant, unary operations given by scalar multiplication. Clearly, both Clo $\mathbf{T}_B$ and Clo $\mathbf{C}$ contain the same constant operations. The Mal'cev operation of $\mathbf{T}_B$ is an operation of $\mathbf{C}$, so it must be the unique Mal'cev operation of $\mathbf{C}$. Further, Clo $\mathbf{C}$ contains all the scalar multiplications in Clo $\mathbf{T}_B$. We only need to observe that Clo $\mathbf{C}$ contains no other scalar multiplication operations. This is so because the number of distinct, non-constant, scalar multiplication operations of a 1-dimensional vector space depends only on the cardinality of the vector space. This number is $|T_B| - 1$ for both $\mathbf{T}_B$ and $\mathbf{C}$. Thus, $\operatorname{Clo}\mathbf{T}_B = \operatorname{Clo}\mathbf{C}$. ∎

The final result of this section is essentially a summary of the information that we have discovered about the way the type is preserved when we restrict prime quotients to subalgebras.

THEOREM 2.13. *Assume that $\mathcal{V}$ has the* CEP *and that $\mathbf{A} \in \mathcal{V}$ is a finite algebra. Assume also that $\mathbf{B} \leq \mathbf{A}$, that $0_A \prec \beta$ in* Con **A** *and that $0_B \neq \beta|_B$ in* Con **B**. *The following are true:*

(1) $0_B \prec \beta|_B$.
(2) $\operatorname{typ}(0_A, \beta) = \mathbf{1}$ *iff* $\operatorname{typ}(0_B, \beta|_B) = \mathbf{1}$.
(3) $\operatorname{typ}(0_A, \beta) = \mathbf{2}$ *iff* $\operatorname{typ}(0_B, \beta|_B) = \mathbf{2}$.
(4) If $\operatorname{typ}(0_A, \beta) \in \{\mathbf{3}, \mathbf{4}\}$, *then* $\operatorname{typ}(0_B, \beta|_B) \in \{\mathbf{3}, \mathbf{4}\}$.
(5) If β is abelian, then B contains a $\langle 0_A, \beta \rangle$-trace.

Further, if B contains a $\langle 0_A, \beta \rangle$-trace, then the following are true:

(6) Every $\langle 0_B, \beta|_B \rangle$-trace is a $\langle 0_A, \beta \rangle$-trace.
(7) If $\operatorname{typ}(0_A, \beta) \neq \mathbf{3}$, *then* $\operatorname{typ}(0_A, \beta) = \operatorname{typ}(0_B, \beta|_B)$.
(8) If $\operatorname{typ}(0_A, \beta) \neq \mathbf{3}$, *$M$ is a $\langle 0_A, \beta \rangle$-trace and N is a $\langle 0_B, \beta|_B \rangle$-trace, then $\mathbf{AI}_M$ is weakly isomorphic to $\mathbf{BI}_N$.*

PROOF. Items (1) – (6) have already been proved. Both (7) and (8) have been proved under the assumption that $\operatorname{typ}(0_A, \beta) \in \{\mathbf{1}, \mathbf{2}\}$. Since (8) implies (7), it will suffice for us to explain why (8) holds in the case that $\operatorname{typ}(0_A, \beta) \in \{\mathbf{4}, \mathbf{5}\}$.

Using (6) and the fact that all $\langle 0_A, \beta \rangle$-traces are weakly isomorphic we may suppose that $M = N$. Since we are only considering the nonabelian case, $|N| = 2$. Clearly, $\mathbf{BI}_N$ is a reduct of $\mathbf{AI}_N$. If $\operatorname{typ}(0_A, \beta) = \mathbf{5}$, then $\operatorname{Clo}\mathbf{BI}_N$ is the polynomial clone of a nonabelian algebra on a two-element set. Further, $\operatorname{Clo}\mathbf{BI}_N \subseteq \operatorname{Clo}\mathbf{AI}_N$, which is the polynomial clone of a semilattice. The description of the polynomial clones on a two-element set given in Lemma 4.8 of [5] proves that $\operatorname{Clo}\mathbf{BI}_N = \operatorname{Clo}\mathbf{AI}_N$. If $\operatorname{typ}(0_A, \beta) = \mathbf{4}$, then $\operatorname{typ}(0_B, \beta|_B) \in \{\mathbf{3}, \mathbf{4}\}$ so $\operatorname{Clo}\mathbf{BI}_N$ is the polynomial clone of a 2-element boolean algebra or a 2-element lattice. Since $\mathbf{BI}_N$ is a reduct of $\mathbf{AI}_N$, it must be that $\operatorname{Clo}\mathbf{BI}_N = \operatorname{Clo}\mathbf{AI}_N$. ∎

The important ingredient missing from Theorem 2.13 is the information on how type **5** quotients restrict. The theorem does show that if $\operatorname{typ}(0_A, \beta) = \mathbf{5}$, then $\operatorname{typ}(0_B, \beta|_B) \in \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$. However, we know of no example for which $\operatorname{typ}(0_B, \beta|_B) \neq \mathbf{5}$. Our technique based on Lemma 2.5 cannot eliminate the possibility that a type **5** quotient restricts to a type **3** or type **4** quotient.

3. **The type-set of a CEP variety.** One of the most significant and beautiful aspects of tame congruence theory is that the structural properties of a locally finite variety are strongly influenced by the type labels that occur in the congruence lattices of the algebras in the variety. Unfortunately, it is sometimes difficult to discover exactly which type labels occur for a given variety. Of course, every type label that occurs will appear in the labelled congruence lattice of a finite algebra. Indeed, it will occur as $\operatorname{typ}(0_S, \mu)$ for some

finite, subdirectly irreducible algebra **S** with monolith μ. It is easy to show that **1** appears in typ$\{\mathcal{V}\}$ iff it appears in typ$\{\mathbf{F}_{\mathcal{V}}(2)\}$. It is known that $\mathcal{V}$ satisfies certain Mal'cev conditions in 3 or 4 variables iff it omits certain types (see Chapter 9 of [5]). Therefore, for certain subsets $X \subseteq \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5}\}$ it is possible to ascertain whether typ$\{\mathcal{V}\} \subseteq X$ by examining $\mathbf{F}_{\mathcal{V}}(3)$ and $\mathbf{F}_{\mathcal{V}}(4)$. However, except for $X = \emptyset$, $\{\mathbf{1}\}$, $\{\mathbf{2}\}$, $\{\mathbf{1},\mathbf{2}\}$, $\{\mathbf{3}\}$ or $\{\mathbf{2},\mathbf{3}\}$, there is no known procedure for determining when typ$\{\mathcal{V}(\mathbf{A})\} = X$ for a given finite **A**. The problem is more complicated when no finite, generic algebra for $\mathcal{V}$ is known or when none exists. Say, for example, that one only knows a recursive procedure for computing $\mathbf{F}_{\mathcal{V}}(n)$. In this case, there is no known algorithm for determining that typ$\{\mathcal{V}\} = X$ unless $X = \emptyset$ or $\{\mathbf{3}\}$. Probably no algorithm exists for any other subsets. That there is an algorithm when $X = \{\mathbf{3}\}$ follows from Theorem 9.15 of [5]; an algorithm when $X = \emptyset$ involves only checking that $|F_{\mathcal{V}}(2)| > 1$. Now, if Y is any non-empty subset of types not containing **3** then one can find a locally finite variety whose type-set is Y. With care, one can often construct a sequence of locally finite varieties $\mathcal{V}_n$ such that typ$\{\mathcal{V}_n\} = Y \cup \{\mathbf{3}\}$ and $\mathbf{F}_{\mathcal{V}}(n) = \mathbf{F}_{\mathcal{V}_n}(n)$. When one can do so it is impossible to determine whether typ$\{\mathcal{V}\} = Y$ or $Y \cup \{\mathbf{3}\}$ with any algorithm that only examines the free algebras because no algorithm can prove that $\mathcal{V} \notin \{\mathcal{V}_n\}$.

The problem of determining the type-set of a finitely generated variety is investigated in [3] where several pathological examples are given. In this section we prove that it is usually not very difficult to determine the type-set of a locally finite variety that has the CEP.

THEOREM 3.1. *Assume that $\mathcal{V}$ is a locally finite variety with the* CEP. *Then,*

$$\operatorname{typ}\{\mathcal{V}\} \subseteq \operatorname{typ}\{\mathbf{F}_{\mathcal{V}}(2)\} \cup \{\mathbf{3}\}.$$

If $\mathbf{4} \notin$ typ$\{\mathbf{F}_{\mathcal{V}}(2)\}$, *then* typ$\{\mathcal{V}\} =$ typ$\{\mathbf{F}_{\mathcal{V}}(2)\}$.

PROOF. If $\mathbf{i} \in$ typ$\{\mathcal{V}\}$, then there is a finite algebra $\mathbf{A} \in \mathcal{V}$ and a congruence $\beta \in \operatorname{Con}\mathbf{A}$ for which $0_A \prec \beta$ and typ$(0_A, \beta) = \mathbf{i}$. Choose $\{0,1\} \subseteq N$ for some $\langle 0_A, \beta\rangle$-trace, N. Let **B** be the image of the homomorphism

$$f\colon \mathbf{F}_{\mathcal{V}}(x,y) \longrightarrow \mathbf{A}\colon x \longmapsto 0,\, y \longmapsto 1.$$

If $\langle 0_A, \beta\rangle$ is nonabelian, then $N = \{0,1\} \subseteq B$. If $\langle 0_A, \beta\rangle$ is abelian, then B contains some $\langle 0_A, \beta\rangle$-trace by Lemma 2.11. In either case, we conclude from Theorem 2.13 that if $\mathbf{i} \neq \mathbf{3}$ then $\mathbf{i} \in$ typ$\{\mathbf{B}\} \subseteq$ typ$\{\mathbf{F}_{\mathcal{V}}(2)\}$. Hence, typ$\{\mathcal{V}\}$ − typ$\{\mathbf{F}_{\mathcal{V}}(2)\}$ is a subset of $\{\mathbf{3}\}$. This establishes the first claim of the theorem. Now suppose that $\mathbf{i} = \mathbf{3}$ and $\mathbf{j} =$ typ$(0_B, \beta|_B)$. Then $\mathbf{j} \in \{\mathbf{3},\mathbf{4}\}$ by Lemma 2.8 and $\mathbf{j} \in$ typ$\{\mathbf{B}\} \subseteq$ typ$\{\mathbf{F}_{\mathcal{V}}(2)\}$. Hence, if $\mathbf{4} \notin$ typ$\{\mathbf{F}_{\mathcal{V}}(2)\}$ then we must have

$$\mathbf{i} = \mathbf{3} = \mathbf{j} \in \operatorname{typ}\{\mathbf{F}_{\mathcal{V}}(2)\}.$$

That is, under the assumption that $\mathbf{4} \notin$ typ$\{\mathbf{F}_{\mathcal{V}}(2)\}$ we can prove that $\mathbf{i} \in$ typ$\{\mathbf{F}_{\mathcal{V}}(2)\}$ for all $\mathbf{i} \in$ typ$\{\mathcal{V}\}$. This establishes the second claim. ■

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EXAMPLE 3. In this example we show that for any $n < \omega$ there is a locally finite variety $\mathcal{V}$ with the CEP such that $\mathbf{3} \in \operatorname{typ}\{\mathcal{V}\}$, but that $\mathbf{3} \notin \operatorname{typ}\{\boldsymbol{HSP}(\mathbf{F}_{\mathcal{V}}(n))\}$. This shows that the result of Theorem 3.1 cannot be much improved.

Consider the poset $\langle A; \leq \rangle$ where $A = \{0, 1, \ldots, n\}$ and $0 \leq 1 \leq \cdots \leq n$. We define an algebra on A as follows. Let F be the collection of all finitary operations on A which preserve the order and all the subsets of A. Define one more operation

$$f(x_0, \ldots, x_{n+1}) = \begin{cases} 0, & \text{if } x_i = i \text{ for } i \leq n \text{ and } x_{n+1} = n \\ n, & \text{if } x_i = i \text{ for } i \leq n \text{ and } x_{n+1} \neq n \\ x_{n+1}, & \text{otherwise.} \end{cases}$$

Now, let $\mathbf{A} = \langle A; F \cup \{f\} \rangle$.

The ordering $\leq$ is a lattice ordering, so the lattice operations preserve this order. Since $\langle A; \leq \rangle$ is a chain, the lattice operations preserve the subsets, too. (That is, $x \vee y \in \{x, y\}$ and $x \wedge y \in \{x, y\}$.) This shows that $\mathbf{A}$ has a lattice reduct, so it generates a congruence distributive variety. The operation f also preserves all the subsets of A, so every non-empty subset of A is a subuniverse. Any subalgebra $\mathbf{B}$ is ordered by the restriction of the ordering of $\langle A; \leq \rangle$ and every order preserving operation on B is a polynomial operation of $\mathbf{B}$. This is enough to imply that $\mathbf{A}$ is hereditarily simple. By our remarks in Example 1, $\mathcal{V} = \mathcal{V}(\mathbf{A})$ has the CEP.

By Jónsson's Lemma, the subdirectly irreducible members of $\mathcal{V}$ are contained in $\boldsymbol{HS}(\mathbf{A}) = \boldsymbol{S}(\mathbf{A})$. Since any type label that appears in $\mathcal{V}$ appears in the congruence lattice of a subdirectly irreducible, we need to discover which types appear in $\boldsymbol{S}(\mathbf{A})$.

The fact that $\mathcal{V}$ is congruence distributive implies that $\operatorname{typ}\{\mathcal{V}\} \subseteq \{\mathbf{3}, \mathbf{4}\}$. If $\mathbf{B}$ is a proper subalgebra of $\mathbf{A}$, then f is trivial on $\mathbf{B}$, so all operations of $\mathbf{B}$ respect a certain connected partial order: the restriction of $\leq$ to B. By Theorem 5.26 (1) of [5], this fact implies that $\operatorname{typ}(0_B, 1_B) \in \{\mathbf{4}, \mathbf{5}\}$, so $\operatorname{typ}(0_B, 1_B) = \mathbf{4}$. Now let $h \in \operatorname{Pol}_1 \mathbf{A}$ be the operation $f(0, 1, \ldots, n, x)$. $U = \{0, n\} = h(A) \in M_{\mathbf{A}}(0_A, 1_A)$ is closed under the lattice operations of $\mathbf{A}$ and also under h. Since $h(0) = n$ and $h(n) = 0$, $\mathbf{A}|_U$ is polynomially equivalent to a boolean algebra. Therefore, $\mathbf{A}$ is of type $\mathbf{3}$. This proves that $\operatorname{typ}\{\mathcal{V}\} = \{\mathbf{3}, \mathbf{4}\}$. To prove that $\mathbf{3} \notin \operatorname{typ}\{\boldsymbol{HSP}(\mathbf{F}_{\mathcal{V}}(n))\}$ it suffices to observe that the subdirectly irreducible algebras of $\boldsymbol{HSP}(\mathbf{F}_{\mathcal{V}}(n))$ are precisely the n-generated subdirectly irreducibles of $\mathcal{V}$. Thus, they are precisely the proper subalgebras of $\mathbf{A}$. Hence, $\operatorname{typ}\{\boldsymbol{HSP}(\mathbf{F}_{\mathcal{V}}(n))\} = \{\mathbf{4}\}$.

Example 3 also shows that Theorem 2.13 (7) cannot be improved to say that if B contains a $\langle 0_A, \beta \rangle$-trace then $\operatorname{typ}(0_A, \beta) = \operatorname{typ}(0_B, \beta|_B)$. The algebra $\mathbf{A}$ of Example 3 is simple of type $\mathbf{3}$ and $\{0, n\} \subseteq A$ is a $\langle 0_A, 1_A \rangle$-trace which is also the universe of a simple, type $\mathbf{4}$ subalgebra. Thus, if we let $\mathbf{B} = \operatorname{Sg}^{\mathbf{A}}(\{0, n\})$ we find that B contains a $\langle 0_A, \beta \rangle$-trace, $\{0, n\}$, but $\operatorname{typ}(0_A, 1_A) = \mathbf{3} \neq \mathbf{4} = \operatorname{typ}(0_B, 1_A|_B)$. This also shows that 2.13 (8) cannot be improved.

It would be worthwhile to try to prove that $\mathbf{3} \in \mathcal{V}(\mathbf{A})$ for some finite algebra $\mathbf{A}$ which generates a CEP variety if and only if $\mathbf{3} \in \operatorname{typ}\{\mathbf{A}^n\}$ for some finite n. A result in [3] proves that, if this is true, then we may choose $n = |A|^2$. Together with Theorem 3.1 such a result would lead to an algorithm for determining the type-set of any finitely generated variety with the CEP.

4. **The class of finite injectives.** An object **I** in a category $\mathcal{K}$ is *injective* over $\mathcal{K}$ if whenever there is a diagram in $\mathcal{K}$:

$$\begin{array}{ccc} \mathbf{B} & & \\ {\scriptstyle f}\uparrow & & \\ \mathbf{A} & \xrightarrow{g} & \mathbf{I} \end{array}$$

where f is a monomorphism and g is arbitrary then there is a morphism $\hat{g}\colon \mathbf{B} \longrightarrow \mathbf{I}$ for which $\hat{g} \circ f = g$. When we consider a variety as a category we choose the morphisms to be the homomorphisms.

A subalgebra **A** of **B** is a *retract* of **B** if it is the image of an idempotent endomorphism of **B**. $\mathbf{A} \in \mathcal{V}$ is an *absolute retract in* $\mathcal{V}$ if it is a retract of any of its extensions in $\mathcal{V}$.

A variety of algebras $\mathcal{V}$ is said to have the *amalgamation property* if whenever we have embeddings $f_1\colon \mathbf{A} \longrightarrow \mathbf{B}$ and $g_1\colon \mathbf{A} \longrightarrow \mathbf{C}$ we can find an algebra **D** and embeddings f_2, g_2 which complete a commutative diagram:

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{f_2} & \mathbf{D} \\ {\scriptstyle f_1}\uparrow & & {\scriptstyle g_2}\uparrow \\ \mathbf{A} & \xrightarrow[g_1]{} & \mathbf{C} \end{array}$$

$\mathcal{V}$ is *residually small* just in case it has a bound on the size of its subdirectly irreducible members.

A variety has *enough injectives* if every member can be embedded into an injective member. It is known that a variety has enough injectives if and only if it is residually small, has the congruence extension property and has the amalgamation property. This follows from a combination of results due to B. Banaschewski [2] and W. Taylor [10]. In a variety with enough injectives the injective algebras are just the absolute retracts.

The class of all injective algebras in a variety is known for only a few special varieties. For example, varieties of **R**-modules are fairly well-behaved, as far as varieties go; and we will see in Theorem 4.1 that the structure of the injective **R**-modules with DCC on submodules is quite nice. However, unless **R** is noetherian, not much is known about the "large" injective **R**-modules. (For the case when **R** is noetherian it is known that every injective **R**-module is isomorphic to a direct sum of indecomposable **R**-modules. In fact, this property characterizes noetherian rings.) Another well-behaved variety is the variety of boolean algebras. The injective boolean algebras are precisely the complete boolean algebras. Thus, any theorem describing the structure of arbitrary injective algebras in, say, an arbitrary modular variety must generalize the notion of a complete boolean algebra and also of the construction of direct sums for modules. Such a theorem ought to apply to varieties of modules over non-noetherian rings. We expect that such a general result would be very difficult to discover and to prove. Nevertheless, the structure of injectives satisfying DCC on congruences is easy to describe.

THEOREM 4.1 [6]. *Assume that $\mathcal{V}$ is a congruence modular variety and that $\mathbf{A} \in \mathcal{V}$ has* DCC *on congruences.* $\mathbf{A}$ *is injective if and only if* $\mathbf{A}$ *is isomorphic to a finite direct product of subdirectly irreducible, injective algebras.* ■

This result makes it fairly easy to locate all the finite injectives in a congruence modular variety. For example, it follows immediately from this theorem that the finite, injective, distributive lattices are precisely the boolean lattices. On the other hand, a variety of non-distributive lattices has no finite injectives other than the trivial lattice. (In fact, no lattice in a non-distributive variety can be injective, for assume that $\mathbf{L}$ is a lattice which is injective in a variety of lattices containing $\mathbf{K}$ where $\mathbf{K} \in \{\mathbf{M}_3, \mathbf{N}_5\}$. Let $\mathbf{3}$ denote the 3-element chain. Embed $\mathbf{3}$ into $\mathbf{L}$ with a map f. If $\mathbf{K} = \mathbf{N}_5$ then choose f so that the image of $\mathbf{K}$ does not contain the top element of $\mathbf{N}_5$. Now, we can map $\mathbf{3}$ into $\mathbf{L}$ with a map g which identifies only the two smaller elements of $\mathbf{K}$. There is no way to extend g to a mapping of $\mathbf{K}$ into $\mathbf{L}$ contradicting the injectivity of $\mathbf{L}$.)

Theorem 4.1 seemed to us to be so strong that we were led to make, in [6], the following conjecture:

CONJECTURE 1. Assume that $\mathcal{V}$ is a locally finite variety with enough injectives. $\mathcal{V}$ is congruence modular iff every finite injective in $\mathcal{V}$ factors as a direct product of subdirectly irreducible injectives.

The forward direction of the conjecture is true without the assumptions of local finiteness or of enough injectives. This follows from Theorem 4.1. However, for the reverse direction to be true we certainly need assumptions to guarantee that $\mathcal{V}$ has a lot of finite injectives around; this explains our formulation of the conjecture. A closely related conjecture is the following:

CONJECTURE 2. Assume that $\mathcal{V}$ is a locally finite variety with enough injectives. $\mathcal{V}$ is congruence distributive iff every finite injective in $\mathcal{V}$ factors *uniquely* as a direct product of subdirectly irreducible injectives.

If Conjecture 1 is true, then Conjecture 2 is true, as well. To show this, first notice that any finite algebra in a congruence distributive variety factors uniquely into a direct product of directly indecomposable algebras. Thus, the forward direction of Conjecture 2 follows from Conjecture 1. For the other direction, observe that a locally finite, non-distributive, modular variety which satisfies C2 has a non-trivial, finite, abelian algebra, $\mathbf{A}$. If the variety satisfies R then any maximal essential extension of this algebra, $\mathbf{A}'$, is a finite, injective, abelian algebra. Now, it is not hard to see that $\mathbf{A}' \times \mathbf{A}'$ is a finite injective algebra that has more than one direct decomposition into subdirectly irreducible factors, each of which must be injective.

In this section we will prove that these conjectures hold for any variety satisfying a nontrivial, idempotent Mal'cev condition.

LEMMA 4.2. *Assume that $\mathcal{V}$ is a locally finite variety with enough injectives. If* $\mathbf{i} \in \operatorname{typ}\{\mathcal{V}\} \cap \{\mathbf{1},\mathbf{2},\mathbf{5}\}$, *then there is a finite, injective, subdirectly irreducible algebra* $\mathbf{A} \in \mathcal{V}$, *with monolith* μ, *such that* $\operatorname{typ}(0,\mu) = \mathbf{i}$.

PROOF. If $\mathbf{i} \in \text{typ}\{\mathcal{V}\}$ then there is a finite algebra $\mathbf{A}'$ with $\mathbf{i} \in \text{typ}\{\mathbf{A}'\}$. If $|A'|$ is minimal for this property, then $\mathbf{A}'$ is subdirectly irreducible with monolith μ' where $\text{typ}(0, \mu') = \mathbf{i}$. Let $\mathbf{A}$ be a maximal essential extension of $\mathbf{A}'$; $\mathbf{A}$ exists since $\mathcal{V}$ is residually small (see [10]). Essential extensions of subdirectly irreducible algebras are subdirectly irreducible, so $\mathbf{A}$ is subdirectly irreducible with monolith μ. Since $\mathcal{V}$ has the CEP, $\mu|_{A'} = \mu'$. By Theorem 2.13 (2), (3) and (4), and the fact that $\mathbf{i} \in \{\mathbf{1}, \mathbf{2}, \mathbf{5}\}$, it follows that $\text{typ}(0_A, \mu) = \text{typ}(0_{A'}, \mu') = \mathbf{i}$. Now, in a variety with enough injectives an algebra is injective iff it is an absolute retract iff it has no proper essential extensions. Hence, $\mathbf{A}$ is injective. We will be done if we show that $\mathbf{A}$ is finite. This fact follows from Theorems 3 and 4 of [1] which prove that a locally finite, residually small variety with the CEP has only finitely many subdirectly irreducible algebras and all are of finite cardinality. ∎

Lemma 4.2 fails without the hypothesis that $\mathbf{i} \in \{\mathbf{1}, \mathbf{2}, \mathbf{5}\}$. It can be shown by the criteria of Corollary 3.16 of [6] that the finite algebras described in Examples 1 and 2 generate varieties with enough injectives. In each example the algebra $\mathbf{A}$ is the only nontrivial, injective, subdirectly irreducible member of $\mathcal{V} = \mathcal{V}(\mathbf{A})$. Hence, in Example 1, $\mathbf{4} \in \text{typ}\{\mathcal{V}\}$ but $\mathbf{4}$ is not the type of the monolith of any injective, subdirectly irreducible algebra in $\mathcal{V}$. Similarly, in Example 2, $\mathbf{3} \in \text{typ}\{\mathcal{V}\}$ but $\mathbf{3}$ is not the type of the monolith of any injective, subdirectly irreducible algebra in $\mathcal{V}$.

THEOREM 4.3. *Assume that $\mathcal{V}$ is a locally finite variety with enough injectives. If every finite injective algebra factors into a direct product of subdirectly irreducible algebras, then* $\mathbf{5} \notin \text{typ}\{\mathcal{V}\}$.

PROOF. Assume that $\mathbf{5} \in \text{typ}\{\mathcal{V}\}$. Let $\mathbf{A} \in \mathcal{V}$ be a finite, injective, subdirectly irreducible algebra with monolith μ where $\text{typ}(0, \mu) = \mathbf{5}$. Choose $U \in M_{\mathbf{A}}(0, \mu)$ and let $e \in E(\mathbf{A})$ be an idempotent polynomial for which $e(A) = U$ and let $N = \{0, 1\}$ be the unique $\langle 0, \mu \rangle$-trace of U. If Δ_{A^2} denotes the diagonal of $\mathbf{A}^2$, let $\mathbf{B} = \text{Sg}^{\mathbf{A}^2}(\{(0,1)\} \cup \Delta_{A^2})$ and let $\mathbf{C} = \text{Sg}^{\mathbf{A}^2}(N^2 \cup \Delta_{A^2}) = \text{Sg}^{\mathbf{A}^2}(\{(1,0)\} \cup B)$.

CLAIM 1. $\mathbf{B} \neq \mathbf{C}$.

PROOF OF CLAIM 1. We need to prove that $(1,0) \notin B$. If, instead, $(1,0) \in B$, then for some n there is an $(n+1)$-ary term, t, and n elements $(a_i, a_i) \in \Delta_{A^2}$ such that $t^{\mathbf{A}^2}\big((0,1), \overline{(a_i, a_i)}\big) = (1,0)$. U is closed under $f(x) = et^{\mathbf{A}}(x, \overline{a_i})$ and $f(0) = 1, f(1) = 0$. This contradicts the fact that $\text{typ}(0, \mu) = \mathbf{5}$. Hence, $(1,0) \in C - B$.

The algebra $\mathbf{C}$ is examined in Chapter 5 of [5] and it is proved in Theorem 5.27 of that book that Con $\mathbf{C}$ has an interval sublattice as pictured in Figure 1.

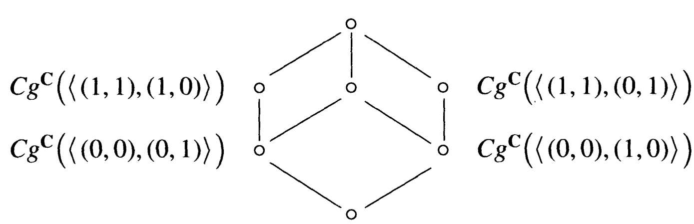

FIGURE 1

CLAIM 2. $Cg^{\mathbf{C}}\big(\langle (0,0),(1,0)\rangle\big)|_B = 0_B$.

PROOF OF CLAIM 2. Assume instead that $\langle (a,b),(c,d)\rangle \in Cg^{\mathbf{C}}\big(\langle (0,0),(1,0)\rangle\big)|_B - 0_B$. Since $\langle (a,b),(c,d)\rangle \in Cg^{\mathbf{C}}(\langle (0,0),(1,0)\rangle)$, we must have $(a,c) \in \mu - 0_A$ and $b = d$. Also, since $(x,y) \in \mu$ for all $(x,y) \in B$, we must have $a \,\mu\, b \,\mu\, c$. Choose $g \in \mathrm{Pol}_1\, \mathbf{A}$ such that $g(A) = U$, $g(a/\mu) \subseteq \{0,1\}$ and $g(a) \neq g(c)$. Say, $g(a) = 1$ and $g(c) = 0$. Then $g(b)$ is either 0 or 1; we will show that both choices lead to a contradiction.

If $g(b) = 0$, let $\hat{g} \in \mathrm{Pol}_1\, \mathbf{B}$ be the polynomial defined by $\hat{g}\big((x,y)\big) = \big(g(x),g(y)\big)$. Since $(a,b) \in B$, this yields $\hat{g}\big((a,b)\big) = (1,0) \in B$, contradicting Claim 1. Thus, $g(b) = g(d) = 1$. Hence, using the fact that $\hat{g}$ is also a polynomial of $\mathbf{C}$,

$$\langle (1,1),(0,1)\rangle = \langle \hat{g}\big((a,b)\big), \hat{g}\big((c,d)\big)\rangle \in Cg^{\mathbf{C}}(\langle (0,0),(1,0)\rangle).$$

Thus we conclude that $Cg^{\mathbf{C}}(\langle (1,1),(0,1)\rangle) \leq Cg^{\mathbf{C}}(\langle (0,0),(1,0)\rangle)$. Looking again at Figure 1, we see that this conclusion is false. This contradiction proves Claim 2.

Of course, $\alpha = Cg^{\mathbf{C}}(\langle (1,1),(0,1)\rangle)|_B > 0_B$. By Claim 2, Lemma 2.3 and the fact that $Cg^{\mathbf{C}}(\langle (0,0),(1,0)\rangle) \prec Cg^{\mathbf{C}}(\langle (1,1),(0,1)\rangle)$ we must have $0_B \prec \alpha$. Similarly, since $0_B < Cg^{\mathbf{C}}\big(\langle (0,0),(0,1)\rangle\big)|_B = \beta$ we must have $0_B \prec \beta$.

Now, let $\mathbf{D} \in \mathcal{V}$ be a maximal essential extension of $\mathbf{B}$. $\mathbf{D}$ has no proper essential extensions, so $\mathbf{D}$ is injective in $\mathcal{V}$. We have a diagram:

$$\begin{array}{ccc} \mathbf{D} & & \\ {\scriptstyle h}\uparrow & & \\ \mathbf{B} & \xrightarrow{\ i\ } & \mathbf{A}^2 \end{array}$$

where i is the inclusion map and h is some essential embedding of $\mathbf{B}$ into $\mathbf{D}$. Since $\mathbf{A}$ is injective, and the class of injectives is closed under direct products, there is an $\hat{\imath}\colon \mathbf{D} \longrightarrow \mathbf{A}^2$ such that $\hat{\imath} \circ h = i$. If $\ker \hat{\imath} > 0_D$ then, since h is essential, $\ker \hat{\imath} \circ h = \ker i > 0_B$, a contradiction. Therefore $\hat{\imath}$ is one-to-one. We will identify $\mathbf{D}$ with $\hat{\imath}(\mathbf{D})$ and consider h and $\hat{\imath}$ to be inclusion maps.

CLAIM 3. $\mathbf{D} \neq \mathbf{A}^2$.

PROOF OF CLAIM 3. It is enough to show that $\mathbf{A}^2$ is not an essential extension of $\mathbf{B}$, since $\mathbf{D}$ is.

$\mathcal{V}$ has the CEP, so

$$Cg^{\mathbf{A}^2}\big(\langle (0,0),(1,0)\rangle\big)|_B = \Big(Cg^{\mathbf{A}^2}\big(\langle (0,0),(1,0)\rangle\big)|_C\Big)|_B = Cg^{\mathbf{C}}\big(\langle (0,0),(1,0)\rangle\big)|_B = 0_B.$$

Thus, $B \subseteq D \subset A \times A$.

Since $\mathbf{D}$ is injective, $\mathbf{D} \cong \mathbf{D}_0 \times \cdots \mathbf{D}_{k-1}$ where each $\mathbf{D}_i$ is a subdirectly irreducible algebra. Let δ_i denote the kernel of the projection of $\mathbf{D}$ onto $\mathbf{D}_i$. Let η_0 and η_1 denote the restriction of the coordinate projection congruences on $\mathbf{A}^2$ to $\mathbf{D}$. $\mathbf{D}$ contains the diagonal, so $\mathbf{D}/\eta_j \cong \mathbf{A}$, which proves that η_0 and η_1 have unique upper covers in $\mathrm{Con}\,\mathbf{D}$; we denote these upper covers by μ_0 and μ_1, respectively. We focus our attention on the two congruences $\bar{\alpha} = Cg^{\mathbf{D}}(\alpha)$ and $\bar{\beta} = Cg^{\mathbf{D}}(\beta)$. Since $\mathcal{V}$ has the CEP, $\bar{\alpha}|_B = \alpha$ and $\bar{\beta}|_B =$

β. **D** is an essential extension of **B**, so $\bar{\alpha}$ and $\bar{\beta}$ are minimal, nonzero congruences of **D**. Now $\langle 0_D, \bar{\alpha}\rangle$ and $\langle \eta_0, \mu_0\rangle$ are perspective prime quotients in Con **D**, so typ$(0_D, \bar{\alpha}) =$ typ$(\eta_0, \mu_0) =$ typ$(0_A, \mu) =$ **5**. Similarly, typ$(0_D, \bar{\beta}) =$ **5**. Because the type is **5**, we can talk about the *pseudo-complement* of $\bar{\alpha}$ over 0_D or of $\bar{\beta}$ over 0_D. The pseudo-complement of $\bar{\alpha}$ over 0_D is the largest $\gamma \in$ Con **D** such that $\gamma \wedge \bar{\alpha} = 0_D$. Since $\eta_0 \wedge \bar{\alpha} = 0_D$ and $\mu_0 \wedge \bar{\alpha} = \bar{\alpha}$, η_0 is the pseudo-complement of $\bar{\alpha}$ over 0_D. Similarly, η_1 is the pseudo-complement of $\bar{\beta}$ over 0_D. It is impossible that $\delta_i \wedge \bar{\alpha} > 0_D$ for all values of i, so there is an i_0 such that $\delta_{i_0} \wedge \bar{\alpha} = 0_D$. Equivalently, there is an i_0 such that $\delta_{i_0} \leq \eta_0$ (since η_0 is the pseudo-complement of $\bar{\alpha}$ over 0_D). There must also be an i_1 such that $\delta_{i_1} \leq \eta_1$. Of course, $i_0 \neq i_1$, for otherwise $\delta_{i_0} \leq \eta_0 \wedge \eta_1 = 0_D$. This would lead to $\mathbf{D} = \mathbf{D}_{i_0}$ which contradicts the fact that **D** has at least two distinct minimal congruences while $\mathbf{D}_{i_0}$ is subdirectly irreducible.

We can finish the proof by observing that

$$|A|^2 > |D| \geq |D_{i_0}| \times |D_{i_1}| \geq |A|^2.$$

The last inequality is a consequence of the fact that $\mathbf{A}(\cong \mathbf{D}/\eta_0 \cong \mathbf{D}/\eta_1)$ is a homomorphic image of $\mathbf{D}_{i_0} = \mathbf{D}/\delta_{i_0}$ and of $\mathbf{D}_{i_1}$.

Thus, the assumption that $\mathbf{5} \in \text{typ}\{\mathcal{V}\}$ leads to a contradiction. ∎

The following result is an extension of Theorem 10.4 of [5].

THEOREM 4.4. *Assume that $\mathcal{V}$ is locally finite, omits type **5** and is residually small. If $\mathbf{A} \in \mathcal{V}$, then* Con $\mathbf{A}/\overset{ss}{\sim}$ *is a modular lattice.*

PROOF. Suppose instead that there is an $\mathbf{A} \in \mathcal{V}$ such that Con $\mathbf{A}/\overset{ss}{\sim}$ is not modular. Then there exist $\alpha, \beta, \gamma \in$ Con **A** such that $\alpha/\overset{ss}{\sim} < \beta/\overset{ss}{\sim}$, $(\alpha \vee \gamma)/\overset{ss}{\sim} = (\beta \vee \gamma)/\overset{ss}{\sim}$ and $(\alpha \wedge \gamma)/\overset{ss}{\sim} = (\beta \wedge \gamma)/\overset{ss}{\sim}$. Choose a finitely generated subalgebra, **F**, for which $(\alpha|_F)/\overset{ss}{\sim} < \big((\alpha|_F \vee \gamma|_F) \wedge (\beta|_F)\big)/\overset{ss}{\sim}$. **F** is finite. Now, let θ denote the least congruence on **F** in the $\overset{ss}{\sim}$ - equivalence class, $\big((\alpha|_F \vee \gamma|_F) \wedge (\beta|_F)\big)/\overset{ss}{\sim}$. Let δ denote the largest congruence in $(\alpha|_F)/\overset{ss}{\sim}$ which lies below θ. Let $\psi = \gamma|_F$. Then,

$$\delta \vee \psi \overset{ss}{\sim} \alpha|_F \vee \gamma|_F \geq \theta,$$

so $\delta \vee \psi \geq \theta$. Further,

$$\theta \wedge \psi \leq (\alpha|_F \vee \gamma|_F) \wedge \beta|_F \wedge \gamma|_F = (\beta \wedge \gamma)|_F \overset{ss}{\sim} (\alpha \wedge \gamma)|_F \leq \alpha|_F \overset{ss}{\sim} \delta,$$

so $(\theta \wedge \psi) \vee \delta \overset{ss}{\sim} \delta$ which implies that $\theta \wedge \psi \leq \delta$. Hence, $\delta < \theta$, $\delta \vee \psi = \theta \vee \psi$ and $\delta \wedge \psi = \theta \wedge \psi$. By our choice of **F** we also get that $\delta \overset{ss}{\not\sim} \theta$. This means that there are congruences $\delta', \theta' \in$ Con **F** such that $\delta \leq \delta' \prec \theta' \leq \theta$ and typ$(\delta', \theta') \neq$ **1**. By relabelling we may assume that $\delta \prec \theta$ and typ$(\delta, \theta) \neq$ **1**.

By further juggling our choices for δ, θ and ψ we may assume that the interval $I[\delta \wedge \psi, \theta \vee \psi]$ is minimal under inclusion in Con **F** for which $\delta \prec \theta$ and typ$(\delta, \theta) \neq$ **1**, $\delta \vee \psi = \theta \vee \psi$ and $\delta \wedge \psi = \theta \wedge \psi$. We may also assume that ψ is a minimal element of $I[\delta \wedge \psi, \theta \vee \psi]$ for these properties.

If $\delta \wedge \psi \not\prec \psi$, then there is a congruence $\xi \in \mathrm{Con}\,\mathbf{F}$ such that $\delta \wedge \psi < \xi \prec \psi$ and the minimality assumptions of the last paragraph insure that Con **F** has the sublattice pictured in Figure 2.

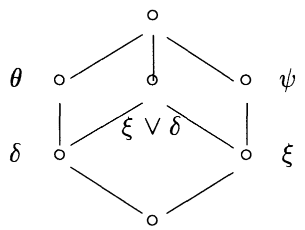

FIGURE 2

The argument that proves this is not difficult; it is essentially the proof of Lemma 10.1 of [5]. Now, for any $\rho \in \mathrm{Con}\,\mathbf{F}$ for which $\xi \vee \delta \leq \rho \prec \delta \vee \psi$ we have, by Lemma 6.3 of [5], that $\mathrm{typ}(\rho, \delta \vee \psi) \in \mathrm{typ}\{\mathcal{V}\} \cap \{\mathbf{1}, \mathbf{5}\} \subseteq \{\mathbf{1}\}$. But now $\langle \delta, \theta \rangle$ and $\langle \rho, \delta \vee \psi \rangle$ are perspective prime quotients of different types; this contradicts Lemma 6.2 of [5]. This contradiction proves that we have $\delta \wedge \psi \prec \psi$.

Now we can use Lemmas 10.2 and 10.3 of [5] which prove that, since $\mathcal{V}$ is residually small and $\mathbf{5} \notin \mathrm{typ}\{\mathcal{V}\}$, $\mathrm{typ}(\delta, \theta) \notin \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$. Of course, $\mathrm{typ}(\delta, \theta) \neq \mathbf{1}$ by assumption, and $\mathcal{V}$ omits type **5**. Thus any choice for $\mathrm{typ}(\delta, \theta)$ leads to a contradiction. This establishes the theorem. ∎

COROLLARY 4.5. *Assume that $\mathcal{V}$ is a locally finite variety satisfying a nontrivial, idempotent Mal'cev condition and that $\mathcal{V}$ has enough injectives. $\mathcal{V}$ is congruence modular iff every finite injective factors as a direct product of subdirectly irreducible algebras. $\mathcal{V}$ is congruence distributive iff every finite injective factors uniquely as a direct product of subdirectly irreducible algebras.*

PROOF. As we mentioned after the statement of Conjecture 2, the second statement of this Corollary follows from the first, so we will prove only the first statement. Since $\mathcal{V}$ satisfies a nontrivial, idempotent Mal'cev condition, $\mathbf{1} \notin \mathrm{typ}\{\mathcal{V}\}$, by Lemma 9.3 of [5]. By Theorem 5.6 and Corollary 7.5 of [5], this means precisely that for any $\mathbf{A} \in \mathcal{V}$ the congruence $\overset{ss}{\sim}$ on Con **A** is the trivial congruence. Now Theorems 4.3 and 4.4 prove that $\mathcal{V}$ is congruence modular. ∎

In Corollary 4.5 we do not assume that the subdirectly irreducible factors in a direct decomposition of a finite injective algebra are injective in order to prove that the variety is congruence modular. It is enough that there exists a direct decomposition into subdirectly irreducibles. In this way Corollary 4.5 is a stronger result than Conjecture 1 for varieties omitting type **1**. Still, whether we assume that the factors are injective or not, Proposition 3.10 of [6] proves that, in a congruence modular variety, a direct product is injective iff each factor is.

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