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CENTRAL DOUBLE CENTRALIZERS ON QUASI-CENTRAL BANACH ALGEBRAS WITH BOUNDED APPROXIMATE IDENTITY

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1. Introduction. We assume throughout this paper that A is a semisimple, quasi-central, complex Banach algebra with a bounded approximate identity $\{e_\alpha\}$. The author [**6**] has shown that every central double centralizer T on A can be, under suitable conditions, represented as a bounded continuous complex-valued function Φ_T on Prim A, the structure space of A with the hull-kernel topology, such that

$$Tx + P = \Phi_T(P)(x + P) \quad \text{for all } x \in A \quad \text{and} \quad P \in \text{Prim } A.$$

Here $x + P$ for $P \in$ Prim A denotes the canonical image of x in A/P. This map Φ is called Dixmier's representation of $Z(M(A))$, the central double centralizer algebra of A. We denote by τ the canonical isomorphism of A into the Banach algebra $D(A)$ with the restricted Arens product as defined in [**6**]. Also denote by μ Davenport's representation of $Z(M(A))$. In fact, this map μ is given by

$$\mu T = \text{weak*-}\lim_\alpha \tau(Te_\alpha)$$

for each $T \in Z(M(A))$. Then μ is a continuous algebraic isomorphism of $Z(M(A))$ onto $Z(D(A))$, the ideal center of A (see [**6**] or [**7**]). In [**7**], we have shown that if $Z(D(A))$ has a Hausdorff structure space, then

$$\mu^{-1}(\widetilde{\tau(Z(A))}) = \Phi^{-1}(C_0(\text{Prim } A)).$$

Here $Z(A)$ denotes the center of A and $\widetilde{\tau(Z(A))}$ denotes the kernel of the hull of $\tau(Z(A))$ in the structure space of $Z(D(A))$. Also $C_0(\text{Prim } A)$ denotes the commutative Banach algebra, with the supremum norm, consisting of all bounded continuous complex-valued functions on Prim A which vanish at infinity. If A is a C^*-algebra, then the ideal center $Z(D(A))$ becomes a commutative C^*-algebra and hence it always has a Hausdorff structure space. However, if G is a non-discrete locally compact Abelian group and if $A = L^1(G)$, the group algebra of G, then A is completely regular but $Z(M(A))$ is not regular and so $Z(D(A))$ does not have a Hausdorff structure space (see [**4**, p. 42]). We will therefore discuss what can be said about the above result when the Hausdorff

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condition on the structure space of $Z(D(A))$ is replaced by the weaker condition that $Z(A)$ is completely regular. Actually, in the next section, it will be shown that

$$\mu^{-1}(\widetilde{\tau(Z(A))}) = \Phi^{-1}(C_0(\text{Prim } A)) \cap Z_{\text{low}}(M(A)),$$

whenever $Z(A)$ is completely regular. Here $Z_{\text{low}}(M(A))$ denotes the set of all central double centralizers T on A such that the map : $M \rightarrow |\chi_M(\mu T)|$ is lower semi-continuous on Prim $Z(D(A))$, where χ_M for $M \in$ Prim $Z(D(A))$ is the non-zero homomorphism of $Z(D(A))$ onto the complex field induced by M.

In the final section, it will be shown that if T is a central double centralizer on A such that the support of Φ_T is quasi-compact (i.e., it satisfies the Borel-Lebesgue axiom without necessarily being Hausdorff) and if I is a closed two-sided ideal of A such that the hull of I disjoints from the support of Φ_T, then there exists a unique element z of $Z(A) \cap I$ with $Lz = T$ whenever $Z(A)$ is completely regular. Here Lz for $z \in Z(A)$ is the central double centralizer on A defined by $Lz(x) = zx$ for each $x \in A$. Moreover, the following Tauber type theorem is shown as an application of the above result. If $Z(A)$ is completely regular and if the two-sided ideal $Z_{00}(A)$ of $Z(A)$ consisting of all $z \in Z(A)$ such that the support of Φ_{Lz} is quasi-compact is norm dense in $Z(A)$, then every closed two-sided ideal of A which does not contain $Z(A)$ is contained in some primitive ideal of A. In particular, if A is a quasi-central C^*-algebra, then $Z_{00}(A)$ is always norm dense in $Z(A)$ from the density theorem of Archbold [**1**].

In the remainder of this paper, we denote by $\phi_{I,B}$ the natural homeomorphism of Prim I into Prim B when B is an algebra and I is a two-sided ideal of B. In this case, we notice that $\phi_{I,B}(P) \cap I = P$ for all $P \in$ Prim I (see [**5**, Theorem 2.6.6]).

2. Dixmier's representation of $\mu^{-1}(\widetilde{\tau(Z(A))})$. The purpose of this section is, as promised, to prove the following result which is an extension of [**7**, Theorem 3.3].

THEOREM 2.1. *Let A be a semi-simple, quasi-central, complex Banach algebra with a bounded approximate identity. If the center $Z(A)$ of A is completely regular, then*

$$\mu^{-1}(\widetilde{\tau(Z(A))}) = \Phi^{-1}(C_0(\text{Prim } A)) \cap Z_{\text{low}}(M(A)).$$

In order to prove this theorem, we have to prepare some lemmas. Denote by $\text{Ann}_l(E)$ and $\text{Ann}_r(E)$ the left annihilater and the right annihilater of E, respectively, provided E is an arbitrary subset of an algebra B.

LEMMA 2.2. *Let B be an algebra and let I be a two-sided ideal of B. If I is semi-simple and*

$$\mathrm{Ann}_l(I) \cap \mathrm{Ann}_r(I) = \{0\},$$

then $\phi_{I,B}(\mathrm{Prim}\ I)$ is dense in Prim B.

Proof. Set $\phi = \phi_{I,B}$. Then in order to get the lemma, it is sufficient to show that

$$\ker(\phi(\mathrm{Prim}\ I)) = \{0\}.$$

For this, let x be a fixed element of $\ker(\phi(\mathrm{Prim}\ I))$. Then for any element P of Prim I, we have $x \in \phi(P)$, so that

$$xy \in \phi(P) \cap I = P \quad \text{and} \quad yx \in \phi(P) \cap I = P$$

for all $y \in I$. Therefore $xI \cup Ix$ is contained in the radical of I. However since I is semi-simple, it follows that $xI = \{0\}$ and $Ix = \{0\}$. So the assumption

$$\mathrm{Ann}_l(I) \cap \mathrm{Ann}_r(I) = \{0\}$$

implies that $x = 0$, and hence

$$\ker(\phi(\mathrm{Prim}\ I)) = \{0\}.$$

Following [7], we define $U(A)$ to be the set

$$U(A) = \tau(A) + Z(D(A)).$$

Then $U(A)$ is a subalgebra of $D(A)$ since $\tau(A)$ is a two-sided ideal of the Banach algebra $D(A)$.

COROLLARY 2.3. *Let A be a semi-simple Banach algebra with a bounded approximate identity. Then*

$$\phi_{\tau(A),U(A)}(\mathrm{Prim}\ \tau(A))$$

is dense in Prim $U(A)$.

Proof. Notice that

$$\mathrm{Ann}_l(\tau(A)) = \mathrm{Ann}_r(\tau(A)) = \{0\}$$

from [2, Lemma 2.6, 2.6.3]. Note also that $\tau(A)$ is semi-simple since τ is an isomorphim. Therefore our corollary follows immediately from the preceding lemma.

The following lemma plays an essential role in the proof of our main theorem.

LEMMA 2.4. *Let B be a commutative Banach algebra with an identity element and let I be a closed ideal of B. Assume that K is a (hull-kernel)*

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closed subset of Prim I. *If* K *is compact in the Gelfand topology, then* $\phi_{I,B}(K)$ *is also closed in* Prim B. *In particular, if* I *is completely regular, then* $\phi_{I,B}(K)$ *is closed in* Prim B *for each compact subset* K *of* Prim I.

Proof. Since K is compact in the Gelfand topology, ker K is a modular ideal of I from [**5**, Theorem 3.6.7] and hence there exists an element e of I such that $\chi_M(e) = 1$ for all $M \in K$. Set $e' = 1 - e$ and $\phi = \phi_{I,B}$. Note that

$$\chi_{\phi(M)}|I = \chi_M \quad \text{for each } M \in \text{Prim } I.$$

We then have

$$\chi_{\phi(M)}(e') = 1 - \chi_{\phi(M)}(e) = 1 - \chi_M(e) = 0$$

for all $M \in K$ and

$$\chi_R(e') = 1 - \chi_R(e) = 1$$

for all $R \in$ hull I. In other words, $e' \in \ker \phi(K)$ and $e' \notin R$ for all $R \in$ hull I. Now in order to show that $\phi(K)$ is closed in Prim B, let R be any element of Prim B with $\ker \phi(K) \subset R$. Then R belongs to Prim B $-$ hull I from the above arguments, so that there exists an element M of Prim I with $R = \phi(M)$. We therefore have

$$\begin{aligned} M = I \cap R \supset I \cap \ker \phi(K) &= \cap \{I \cap \phi(N) : N \in K\} \\ &= \cap \{N : N \in K\} = \ker K. \end{aligned}$$

Hence M must be in K since K is closed in Prim I, so that R is in $\phi(K)$. Thus $\phi(K)$ is closed in Prim B as required.

In particular, if I is completely regular and K is a compact subset of Prim I, then since the hull-kernel topology in the carrier space of I is equivalent to the Gelfand topology, $\phi(K)$ is closed in Prim B by the above arguments.

For each central double centralizer T on A, let $\Phi_T{}^U$ be the bounded complex-valued function on Prim $U(A)$ as defined in Section 5 of [**7**].

LEMMA 2.5. *If* $Z(A)$ *is completely regular, then*

$$\Phi_T(P) = \Phi_T{}^U(\phi_{\tau(A),U(A)}(\tau(P)))$$

for all $P \in$ Prim A *and* $T \in Z(M(A))$.

Proof. This lemma follows directly from the second half of the proof in [**7**, Theorem 5.1] plus [**6**, Theorem 3.6].

LEMMA 2.6. *If* $Z(A)$ *is completely regular, then each element* T *of* $Z(M(A))$ *with* $\mu T \in \widetilde{\tau(Z(A))}$ *belongs to* $Z_{\text{low}}(M(A))$.

Proof. Let T be any element of $Z(M(A))$ with $\mu T \in \widetilde{\tau(Z(A))}$. By [**7**, Lemma 4.4, (i)], $\tau(Z(A))$ is an ideal of $Z(D(A))$ and so is $\widetilde{\tau(Z(A))}$.

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Set

$$\phi = \phi_{\widetilde{\tau(Z(A))}, Z(D(A))}.$$

Then ϕ^{-1} is a continuous map of Prim $Z(D(A))$ − hull $\widetilde{\tau(Z(A))}$ onto Prim $\widetilde{\tau(Z(A))}$. Now the complete regularity of $Z(A)$ also implies the complete regularity of $\widetilde{\tau(Z(A))}$ from [7, Lemma 4.3, (ii)]. Hence the map : $N \to \chi_N(\mu T)$ is continuous on Prim $\widetilde{\tau(Z(A))}$. Observe that

$$\chi_{\phi^{-1}(M)}(\mu T) = \chi_M(\mu T)$$

for each $M \in$ Prim $Z(D(A))$ − hull $\widetilde{\tau(Z(A))}$. Therefore the map : $M \to \chi_M(\mu T)$ is continuous on Prim $Z(D(A))$ − hull $\widetilde{\tau(Z(A))}$. Let ϵ be an arbitrary positive number and set

$$G = \{M \in \text{Prim } Z(D(A)) : |\chi_M(\mu T)| > \epsilon\}.$$

Since $\chi_M(\mu T) = 0$ for all $M \in$ hull $\widetilde{\tau(Z(A))}$, it follows that

$$G \subset \text{Prim } Z(D(A)) - \text{hull } \widetilde{\tau(Z(A))}.$$

Therefore the openness of Prim $Z(D(A))$ − hull $\widetilde{\tau(Z(A))}$ in Prim $Z(D(A))$ also implies the openness of G in Prim $Z(D(A))$. In other words, T belongs to $Z_{\text{low}}(M(A))$.

We are now in a position to prove our main theorem.

Proof of Theorem 2.1. The fact that $\mu^{-1}(\widetilde{\tau(Z(A))})$ is contained in $\Phi^{-1}(C_0(\text{Prim } A)) \cap Z_{\text{low}}(M(A))$ is a consequence of [7, Theorem 3.2] and Lemma 2.6.

Now we have to show that an arbitrary element T of $Z_{\text{low}}(M(A))$ with $\Phi_T \in C_0(\text{Prim } A)$ belongs to $\mu^{-1}(\widetilde{\tau(Z(A))})$. Suppose, on the contrary, that there exists an element $T_0 \in Z_{\text{low}}(M(A))$ such that

$$\Phi_{T_0} \in C_0(\text{Prim } A)$$

but

$$\mu T_0 \notin \widetilde{\tau(Z(A))}.$$

Then, by the definition of $\widetilde{\tau(Z(A))}$, there exists a primitive ideal M_0 of $Z(D(A))$ such that $\tau(Z(A)) \subset M_0$ but $\mu T_0 \notin M_0$. Since M_0 belongs to the hull of $\tau(Z(A))$ in Prim $Z(D(A))$, it follows from [7, Lemma 4.5] that there exists an element R_0 of the hull of $\tau(A)$ in Prim $U(A)$ such that

$$M_0 = R_0 \cap Z(D(A)).$$

Set

$$\epsilon_0 = |\chi_{M_0}(\mu T_0)|.$$

Then $\epsilon_0 > 0$. We next set

$$K_0 = \{P \in \text{Prim } A : |\Phi_{T_0}(P)| \geqq \epsilon_0/2\}.$$

Since Φ_{T_0} vanishes at infinity, K_0 is a quasi-compact subset of Prim A. Moreover set

$$\phi(P) = \phi_{\tau(A),U(A)}(\tau(P))$$

for each $P \in$ Prim A. Then ϕ is a homeomorphism of Prim A onto Prim $U(A)$ − hull $\tau(A)$ and hence $\phi(K_0)$ is also quasi-compact in Prim $U(A)$. We now write

$$\Psi(R) = R \cap Z(D(A))$$

for each $R \in$ Prim $U(A)$. Then Ψ is a continuous map of Prim $U(A)$ into Prim $Z(D(A))$ from [**7**, Lemma 4.10]. Therefore $\Psi(\phi(K_0))$ is also quasi-compact in Prim $Z(D(A))$. Note that

$$\Psi(\phi(\text{Prim } A)) \subset \text{Prim } Z(D(A)) - \text{hull } \tau(Z(A)).$$

Suppose actually that there exists an element Q_0 of Prim A such that

$$\tau(Z(A)) \subset \Psi(\phi(Q_0)).$$

Then $\tau(Z(A)) \subset \phi(Q_0)$ and so

$$\tau(Z(A)) \subset \phi(Q_0) \cap \tau(A) = \tau(Q_0).$$

Since τ is one-to-one, $Z(A) \subset Q_0$. This contradicts the quasi-centrality of A. We thus obtain that $\Psi(\phi(K_0))$ is contained in Prim $Z(D(A))$ − hull $\tau(Z(A))$. Therefore $\Psi(\phi(K_0))$ is also quasi-compact in Prim $Z(D(A))$ − hull $\tau(Z(A))$. Hence the set

$$\phi^{-1}_{\tau(Z(A)),Z(D(A))}(\Psi(\phi(K_0)))$$

is compact in Prim $\tau(Z(A))$. Since $\tau(Z(A))$ is completely regular, it follows from Lemma 2.4 that $\Psi(\phi(K_0))$ is closed in Prim $Z(D(A))$. Hence $\Psi^{-1}(\Psi(\phi(K_0)))$ is also closed in Prim $U(A)$. Moreover, we see from [**7**, Lemma 4.5] that

$$\Psi^{-1}(\Psi(\phi(K_0))) \subset \text{Prim } U(A) - \text{hull } \tau(A).$$

Since R_0 belongs to the hull of $\tau(A)$ in Prim $U(A)$, setting,

$$G_1 = \text{Prim } U(A) - \Psi^{-1}(\Psi(\phi(K_0))),$$

G_1 is an open neighbourhood of R_0. We further set

$$W = \{M \in \text{Prim } Z(D(A)) : |\chi_M(\mu T_0)| > \epsilon_0/2\}.$$

Since T_0 belongs to $Z_{\text{low}}(M(A))$, W is open in Prim $Z(D(A))$. Then setting,

$$G_2 = \Psi^{-1}(W),$$

G_2 is also open in Prim $U(A)$. Note that

$$|\chi_{\Psi(R_0)}(\mu T_0)| = |\chi_{R_0 \cap Z(D(A))}(\mu T_0)| = |\chi_{M_0}(\mu T_0)| > \epsilon_0/2.$$

Then $\Psi(R_0) \in W$, that is $R_0 \in G_2$. Thus G_2 is an open neighbourhood of R_0. Set $G_0 = G_1 \cap G_2$. Then G_0 is an open neighbourhood of R_0. Notice that $\phi(\text{Prim } A)$ is dense in Prim $U(A)$ from Corollary 2.3 and so we can find an element P_0 of Prim A such that $\phi(P_0) \in G_0$. Then $\Psi(\phi(P_0))$ is in W and hence

(2.1.1) $|\chi_{\Psi(\phi(P_0))}(\mu T_0)| > \epsilon_0/2.$

Also since

$$\phi(K_0) \subset \Psi^{-1}(\Psi(\phi(K_0))) = \text{Prim } U(A) - G_1$$
$$\subset \text{Prim } U(A) - G_0,$$

it follows that $\phi(P_0) \notin \phi(K_0)$ and hence $P_0 \notin K_0$. We thus obtain that

(2.1.2) $|\Phi_{T_0}(P_0)| < \epsilon_0/2.$

Recall, from the definition of $\Phi_T{}^U$, $T \in Z(M(A))$, that

$$\Phi_{T_0}{}^U(\phi(P_0))(J + \phi(P_0)) = \mu T_0 + \phi(P_0)$$

and so

$$\Phi_{T_0}{}^U(\phi(P_0))J - \mu T_0 \in \phi(P_0) \cap Z(D(A)) = \Psi(\phi(P_0)).$$

Here J denotes the identity element of $D(A)$. We therefore have

$$\chi_{\Psi(\phi(P_0))}(\mu T_0) = \Phi_{T_0}{}^U(\phi(P_0))\chi_{\Psi(\phi(P_0))}(J) = \Phi_{T_0}{}^U(\phi(P_0)).$$

Hence it follows from Lemma 2.5 that

$$\chi_{\Psi(\phi(P_0))}(\mu T_0) = \Phi_{T_0}(P_0)$$

and so (2.1.1) and (2.1.2) are not compatible. This completes the proof.

COROLLARY 2.7 (cf. [**7**, Theorem 3.3]). *Let A be a semi-simple, quasi-central, complex Banach algebra with a bounded approximate identity. If the ideal center $Z(D(A))$ of A has a Hausdorff structure space, then*

$$\mu^{-1}(\widetilde{\tau(Z(A))}) = \Phi^{-1}(C_0(\text{Prim } A)).$$

Proof. Since $Z(D(A))$ has the identity element and its structure space is Hausdorff, $Z(D(A))$ is completely regular. Then $\tau(Z(A))$ is also completely regular from [**5**, Theorem 2.7.2] and so is $Z(A)$ since τ is an isomorphism. Note also that the complete regularity of $Z(D(A))$ implies that the map : $M \to \chi_M(\mu T)$ is continuous on Prim $Z(D(A))$ for each $T \in Z(M(A))$ and hence

$$Z_{\text{low}}(M(A)) = Z(M(A)).$$

Therefore the corollary follows from the preceding theorem.

Remark. In the preceding theorem, $\mu^{-1}(\widetilde{\tau(Z(A))})$ is still contained in $\Phi^{-1}(C_0(\text{Prim } A)) \cap Z_{\text{low}}(M(A))$ without necessarily assuming semi-simplicity of A because [**7**, Theorem 3.2] and Lemma 2.6 is true even if A is not semi-simple.

3. Tauber type theorem depending on $Z(A)$. In this section, we will consider some spectral synthesis problems depending on the center $Z(A)$ of A. The following lemma plays an essential role in our considerations and its proof can be observed in the proofs of [**5**, Theorem 2.7.9 and 2.7.10].

LEMMA 3.1. *Let A be a quasi-central Banach algebra with a completely regular center and let F be any closed subset of* Prim A. *Then F is quasi-compact if and only if* ker F *is modular.*

If f is a complex-valued function on Prim A, we denote by supp (f) the hull-kernel closure of the set of all $P \in$ Prim A such that $f(P) \neq 0$ and it is called the support of f.

THEOREM 3.2. *Suppose that the center $Z(A)$ of A is completely regular. If T is a central double centralizer on A such that Φ_T has quasi-compact support and if I is a closed two-sided ideal of A such that*

$$\text{supp}\ (\Phi_T) \cap \text{hull}\ I = \emptyset,$$

then there exists a unique element z of $Z(A) \cap I$ with $Lz = T$.

Proof. By the preceding lemma, ker (supp (Φ_T)) is modular and so is $I +$ ker (supp (Φ_T)). Then

$$\text{supp}\ (\Phi_T) \cap \text{hull}\ I = \emptyset$$

implies that

$$A = I + \text{ker}\ (\text{supp}\ (\Phi_T)).$$

Let e be an identity for modulo ker (supp (Φ_T)). Hence we can write $e = u + v$, where $u \in I$ and $v \in$ ker (supp (Φ_T)). Set $z = Tu$. We first show that z is in $Z(A)$. In fact, let x be an arbitrary element of A. If $P \in$ supp (Φ_T), then

$$(3.2.1) \quad xu + P = xe + P = x + P = ex + P = ux + P$$

and hence

$$(3.2.2) \quad \begin{aligned} xz + P &= x(Tu) + P = T(xu) + P = \Phi_T(P)(xu + P) \\ &= \Phi_T(P)(ux + P) = T(ux) + P = (Tu)x + P \\ &= zx + P. \end{aligned}$$

If $P \notin$ supp (Φ_T), then $\Phi_T(P) = 0$ and hence

$$z + P = Tu + P = \Phi_T(P)(u + P) = 0,$$

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so that

(3.2.3) $zx + P = 0 = xz + P.$

We thus observe that $zx - xz$ belongs to the radical of A. Hence the semi-simplicity of A implies that $zx = xz$ for all $x \in A$, that is $z \in Z(A)$. Also since

$$z = Tu = \lim_\alpha T(ue_\alpha) = \lim_\alpha u(Te_\alpha),$$

the element z belongs to the norm closure of I and hence I. Finally we show that $Lz = T$. In fact, if $P \in \operatorname{supp}(\Phi_T)$, then (3.2.1) and (3.2.2) imply that

(3.2.4) $zx + P = \Phi_T(P)(ux + P) = \Phi_T(P)(x + P)$

for all $x \in A$. Also if $P \notin \operatorname{supp}(\Phi_T)$, then (3.2.3) implies that

(3.2.5) $zx + P = 0 = \Phi_T(P)(x + P)$

for all $x \in A$. Note that for each $x \in A$ and $P \in \operatorname{Prim} A$,

$$zx + P = \Phi_{Lz}(P)(x + P).$$

Therefore (3.2.4) and (3.2.5) imply that

$$\Phi_T(P)(x + P) = \Phi_{Lz}(P)(x + P)$$

for all $x \in A$ and $P \in \operatorname{Prim} A$. We thus obtain that $\Phi_T = \Phi_{Lz}$ and hence $T - Lz$ must be in $ZM_R(A)$ from [**6**, Theorem 3.6]. Here $ZM_R(A)$ denotes the set of all $T \in Z(M(A))$ such that $T(A)$ is contained in the radical of A. However since A is semi-simple, $ZM_R(A) = \{0\}$. Then $Lz = T$ as wanted. Also the uniqueness of z is clear because A has the approximate identity.

Remark. Theorem 3.6 in [**6**] states that if the center $Z(A)$ of A is completely regular then the map : $T \to \Phi_T$ is a continuous homomorphism of $Z(M(A))$ into C^b (Prim A), the Banach algebra of all bounded continuous complex-valued functions on Prim A. But the kernel of this homomorphism is, of course, equal to $ZM_R(A)$ as can be seen in the proof of [**6**, Theorem 3.2].

COROLLARY 3.3. *If T is an element of $Z(M(A))$ such that Φ_T has a quasi-compact support, then there exists a unique element z of $Z(A)$ with $Lz = T$.*

Proof. By taking A instead of I in the preceding theorem, our corollary follows immediately from the theorem.

If x is an arbitrary element of an algebra B, we denote by $\operatorname{supp}_B(x)$, or simply by $\operatorname{supp}(x)$, the hull-kernel closure of the set of all $P \in \operatorname{Prim} B$ with $x \notin P$ and it is called the support of x. We also denote by B_{00} the set of all $x \in B$ such that $\operatorname{supp}(x)$ is quasi-compact. Note that B_{00} is a two-sided ideal of B. Let $Z_{00}(A)$ be the set of all $z \in Z(A)$ such that Φ_{Lz} has quasi-compact support. Then $Z_{00}(A)$ is also an ideal of $Z(A)$ since the map : $z \rightarrow \Phi_{Lz}$ is homomorphic on $Z(A)$.

LEMMA 3.4. *If the center $Z(A)$ of A is completely regular, then* hull $A_{00} = \emptyset$ *and* $Z_{00}(A) = A_{00} \cap Z(A)$.

Proof. By [7, Theorem 3.1], Prim A is a locally quasi-compact space. Therefore if there exists an element P_0 of hull A_{00}, we can find an open neighbourhood U_0 of P_0 with quasi-compact closure. We then have

$$\ker(\text{Prim } A - \overline{U_0}) \subset A_{00} \subset P_0,$$

and hence

$$P_0 \in \text{hull}(\ker(\text{Prim } A - \overline{U_0})) = \overline{\text{Prim } A - \overline{U_0}} \subset \overline{\text{Prim } A - U_0} = \text{Prim } A - U_0,$$

where the bar denotes the hull-kernel closure. This contradicts that P_0 is in U_0. We thus obtain that hull $A_{00} = \emptyset$. Also by [7, (6.1.2)],

$$\Phi_{Lz}(P) = \chi_{P \cap Z(A)}(z)$$

for all $z \in Z(A)$ and $P \in$ Prim A. Therefore

$$\text{supp}_A(z) = \text{supp}(\Phi_{Lz}) \quad \text{for all } z \in Z(A)$$

and hence $Z_{00}(A) = A_{00} \cap Z(A)$.

The following result is a Tauber type theorem depending on $Z(A)$.

THEOREM 3.5. *Suppose that $Z(A)$ is completely regular. If $Z_{00}(A)$ is norm dense in $Z(A)$, then every closed two-sided ideal of A which does not contain $Z(A)$ is contained in some primitive ideal of A. Conversely, if every closed two-sided ideal of A which does not contain $Z(A)$ is contained in some primitive ideal of A, then $Z(A)$ is contained in the norm closure of A_{00}.*

Proof. Assume first that $Z_{00}(A)$ is norm dense in $Z(A)$. Let I be any closed two-sided ideal of A with $Z(A) \not\subset I$. We want to show that I is contained in some primitive ideal of A. Suppose, on the contrary, that hull $I = \emptyset$. If z is any element of $Z_{00}(A)$, then Φ_{Lz} has a quasi-compact support and so z must be in I from Theorem 3.2. In other words, $Z_{00}(A) \subset I$. Also since $Z_{00}(A)$ is norm dense in $Z(A)$ and I is norm closed, we have $Z(A) \subset I$. This is a contradiction and hence we obtain the first assertion. Assume conversely that every closed two-sided ideal of A which does not contain $Z(A)$ is contained in some primitive ideal of A. Note that the norm closure $\overline{A_{00}}$ of A_{00} is a closed two-sided ideal of A. If $Z(A)$ is not contained in $\overline{A_{00}}$, then hull $\overline{A_{00}} \neq \emptyset$ from the assumption. But this is impossible since hull A_{00} = hull $\overline{A_{00}}$ and hull $A_{00} = \emptyset$ from Lemma 3.4. We thus obtain the second assertion.

COROLLARY 3.6. *Let A be a quasi-central C^*-algebra. Then $Z_{00}(A)$ is norm dense in $Z(A)$.*

Proof. Note that every proper closed two-sided ideal of A is always contained in some primitive ideal of A from [**3**, Théorème 2.9.7]. Also $Z(A)$ is of course completely regular. Furthermore since A_{00} is a two-sided ideal of A, it follows from the density theorem of Archbold [**1**] that

$$\overline{A_{00} \cap Z(A)} = \overline{A_{00}} \cap Z(A).$$

Therefore the corollary follows immediately from Lemma 3.4 and the preceding theorem.

LEMMA 3.7. *If $Z(A)$ is completely regular, then*

$$Z_{00}(A) \subset (Z(A))_{00}.$$

Proof. Set $\sigma(P) = P \cap Z(A)$ for each $P \in \text{Prim } A$. Then σ is a continuous map of Prim A onto Prim $Z(A)$ from [**5**, Theorem 2.7.5]. Since $\Phi_{Lz}(P) = \chi_{\sigma(P)}(z)$ for all $z \in Z(A)$ and $P \in \text{Prim } A$ from [**7**, (6.1.2)], it follows that

$$(3.7.1) \quad \sigma(\{P \in \text{Prim } A : \Phi_{Lz}(P) \neq 0\}) = \{M \in \text{Prim } Z(A) : z \notin M\}$$

for all $z \in Z(A)$. Hence the continuity of σ implies that

$$\sigma(\text{supp } (\Phi_{Lz})) \subset \text{supp}_{Z(A)} (z)$$

for all $z \in Z(A)$. We next assert that

$$\text{supp}_{Z(A)} (z) \subset \sigma(\text{supp } (\Phi_{Lz}))$$

for all $z \in Z_{00}(A)$. In fact, let $z \in Z_{00}(A)$ and $M \in \text{supp}_{Z(A)} (z)$. Then there exists a net $\{M_\lambda\}$ in Prim $Z(A)$ such that $\lim_\lambda M_\lambda = M$ and $z \notin M_\lambda$ for each λ. By (3.7.1), there exists a primitive ideal P_λ of A such that $\sigma(P_\lambda) = M_\lambda$ and $\Phi_{Lz}(P_\lambda) \neq 0$ for each λ. Hence every P_λ belongs to supp (Φ_{Lz}). Then since supp (Φ_{Lz}) is quasi-compact, there exists a subnet $\{P_{\lambda'}\}$ of $\{P_\lambda\}$ and an element P of supp (Φ_{Lz}) such that

$$\lim_{\lambda'} P_{\lambda'} = P.$$

Therefore

$$\lim_{\lambda'} M_{\lambda'} = \lim_{\lambda'} \sigma(P_{\lambda'}) = \sigma(P).$$

Since Prim $Z(A)$ is Hausdorff, $\sigma(P) = M$ and we thus obtain the assertion. Now by the above arguments, if $z \in Z_{00}(A)$ then

$$\sigma \text{ (supp } (\Phi_{Lz})) = \text{supp}_{Z(A)} (z)$$

and hence $\text{supp}_{Z(A)} (z)$ must be compact since supp (Φ_{Lz}) is quasi-compact and σ is continuous on Prim A. In other words, $Z_{00}(A) \subset (Z(A))_{00}$.

Remark. If B is a completely regular Banach algebra and B_{00} is norm dense in B, then B is said to be *Tauberian* (cf. [**5**, p. 92] or [**8**]). Then, by the preceding lemma, if $Z_{00}(A)$ is norm dense in $Z(A)$ then $Z(A)$ is Tauberian. But we don't know conditions under which $Z_{00}(A) = (Z(A))_{00}$.

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