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THE GENERALIZED PERPETUAL AMERICAN EXCHANGE-OPTION PROBLEM

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Abstract

This paper revisits a general optimal stopping problem that often appears as a special case in some finance applications. The problem is essentially of the same form as the investment-timing problem of McDonald and Siegel (1986) in which the underlying processes are two correlated geometric Brownian motions (GBMs) with drifts less than the discount rate. By contrast, we attempt to analyze the underlying optimal stopping problem to its full generality without imposing any restriction on the drifts of the GBMs. By extending the first passage time approach of Xia and Zhou (2007) to the current context, we manage to obtain a complete and explicit characterization of the solution to the problem on all possible drift domains. Our analysis leads to a new and interesting observation that the underlying optimal stopping problem admits a two-sided optimal continuation region on some certain parameter domains.

Keywords: Optimal stopping problem; free boundary approach; first passage time approach

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1. Introduction

The optimal stopping problem under consideration can be described as follows. Suppose that $V_1(t)$ and $V_2(t)$ are two correlated geometric Brownian motions (GBMs) that satisfy the following stochastic differential equations (SDEs):

$$\frac{\mathrm{d}V_1(t)}{V_1(t)} = \mu_1\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_1(t), \qquad V_1(0) = x \geq 0, \tag{1}$$

$$\frac{\mathrm{d}V_2(t)}{V_2(t)} = \mu_2\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_2(t), \qquad V_2(0) = y > 0, \tag{2}$$

where $W_1(t)$ and $W_2(t)$ are two correlated Wiener processes with $\langle \mathrm{d}W_1(t), \mathrm{d}W_2(t)\rangle = \rho\,\mathrm{d}t$. We denote by $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathrm{P})$ the probability space where the two Wiener processes $W_1(t)$ and $W_2(t)$ live in, where P and $(\mathcal{F}_t)_{t\geq 0}$ respectively denote the probability measure and the natural filtration generated by $W_1(t)$ and $W_2(t)$. Given any real number r, the problem amounts to seeking an *optimal stopping time* τ^* with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, if it exists, such that

$$\begin{aligned} f^*(x, y) &:= \sup_{\tau\in\mathcal{T}_0} \mathrm{E}_0^{x,y}[\mathrm{e}^{-r\tau}(V_1(\tau) - V_2(\tau))_+ \mathbf{1}_{\{\tau<\infty\}}] \\ &= \mathrm{E}_0^{x,y}[\mathrm{e}^{-r\tau^*}(V_1(\tau^*) - V_2(\tau^*))_+ \mathbf{1}_{\{\tau^*<\infty\}}], \end{aligned} \tag{3}$$

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where $\mathrm{E}_t^{x,y}[\cdot]$ denotes the conditional expectation operator associated with P and $\mathcal{F}_t$ with initial process values $V_1(0) = x$ and $V_2(0) = y$. Here, $\mathcal{T}_t$ denotes the collection of all admissible stopping times τ with $\tau \geq t$. The function f^* is often referred to as the *value function*.

The optimal stopping problem formulated above is not new in the finance literature. In their seminal paper McDonald and Siegel (1986) considered an investment-timing problem in which the project revenue and cost take the roles of $V_1(t)$ and $V_2(t)$, respectively, with the additional drift assumption $r > \mu_{1,2}$. Within their context, the problem is translated to maximizing the expected discounted net project value by choosing an optimal investment time τ^*. Turning to the field of option pricing, the valuation of a perpetual American exchange option written on two traded assets $V_1(t)$ and $V_2(t)$ can also be considered as a special case of the problem above where we have $\mu_1, \mu_2 \leq r$, owing to the no-arbitrage pricing principle, and the strict inequality holds when the assets are dividend paying. In the existing literature the two specific applications given above are usually solved using a free-boundary approach, which gives a unique solution to the problem when $r > \mu_{1,2}$ holds. Little, however, has been discussed in the literature on the problem outside the domain $r > \mu_{1,2}$.

In a recent article by Xia and Zhou (2007), however, the authors presented an interesting case of the optimal stopping problem (1)–(3) outside the domain $r > \mu_{1,2}$ and an innovative approach for solving their problem. The problem concerned the valuation of a perpetual stock loan which, in simple words, is a perpetual American call option with a deterministic and exponentially increasing strike price. Embedded in the formulation (1)–(3), the stock loan problem becomes a special case with $\mu_1 \leq r < \mu_2$ and $\sigma_2 = 0$. The authors demonstrated that, under this particular setting, the conventional free boundary approach failed to give a conclusive solution to the problem. Subsequently, they proposed a first passage time approach for solving the stock loan problem, which led to an explicit value function and an explicit optimal exercising strategy for the stock loan.

Motivated by the stock loan analysis of Xia and Zhou (2007), we are tempted to extend their approach to analyze the optimal stopping problem (1)–(3) to its full generality. In particular, we are interested in exploring the extent to which we can relax the drift assumption while still having a well-defined solution to the problem. In fact, it has appeared to be widely believed in the real options literature that the drift condition $r > \mu_{1,2}$ is required to guarantee a well-defined solution to the underlying problem. The results of Xia and Zhou (2007) have already shown that this belief is not true in general. The key is that the drift condition is only sufficient, but not necessary. In this paper we shall show that there is more to tell about the optimal stopping problem (1)–(3). Our results not only echo the results in the existing literature, including those of Xia and Zhou (2007), but also lead to a new and quite surprising observation which, to the author's knowledge, has not yet been documented in the literature. Throughout this paper, we shall assume that r is fixed and analyze the problem over the domains of $\delta_{1,2} := r - \mu_{1,2}$ instead of the domains of $\mu_{1,2}$. The drift assumption, $\mu_{1,2} < r$, is then equivalent to $\delta_{1,2} > 0$.

The remainder of the paper is organized as follows. In Section 2 we review the conventional free boundary approach for solving the optimal stopping problem (1)–(3). Some well-known results are derived under this approach. We then discuss circumstances where the free boundary approach breaks down. This leads us to Section 3, where an alternative approach along the lines of Xia and Zhou (2007) is proposed for solving the underlying problem. A complete and explicit characterization of the solution to the problem is given over all possible drift domains. In Section 4 we summarize the results.

2. Review of the conventional free boundary approach

The connection between optimal stopping problems and free boundary problems has been well studied in the literature; see, e.g. McKean (1965) and Moerbeke (1976). By formulating an optimal stopping problem as a free boundary problem, we may obtain a solution by solving the associated partial or ordinary differential equation (PDE or ODE) with appropriate *value-matching* and *smooth-pasting* conditions. Previous studies have often solved the optimal stopping problem (1)–(3) using a free boundary approach under the assumption that $\delta_{1,2} > 0$. In what follows we give a brief review of this approach and discuss the difficulty in applying this approach when the assumption $\delta_{1,2} > 0$ is relaxed. For simplicity, we introduce the following notation throughout this paper:

$$U(t) := \frac{V_1(t)}{V_2(t)}, \qquad \sigma := \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}. \tag{4}$$

It is clear that the payoff function $\mathrm{e}^{-rt}(x-y)_+$ is homogeneous of degree 1 in x and y, i.e. $\mathrm{e}^{-rt}(kx-ky)_+ = k\mathrm{e}^{-rt}(x-y)_+$ for all $k > 0$. This hints that the stopping decision at time t should only depend on the ratio $U(t)$. In the real options literature we often start with the assumption that the optimal stopping time τ^* exists and takes the form

$$\tau^* = \inf\{t \geq 0 \mid U(t) \geq c^*\} \tag{5}$$

for some fixed threshold $c^* > 0$. This states that we should continue to observe as long as $U(t)$ stays below the threshold c^*, while stopping becomes optimal only when $U(t)$ first hits or exceeds the threshold c^*. The regions $[0, c^*)$ and $[c^*, \infty)$ are referred to as the *continuation region* and the *stopping region* of the optimal stopping problem, respectively.

Proposition 1. *Assume that* $\delta_{1,2} > 0$. *Then, we have*

$$f^*(x,y) = \begin{cases} c^{*-\phi_+}(c^*-1)\left(\dfrac{x}{y}\right)^{\phi_+} y, & \dfrac{x}{y} \in [0, c^*), \\ x - y, & \dfrac{x}{y} \in [c^*, \infty), \end{cases}$$

$$c^* = \frac{\phi_+}{\phi_+ - 1},$$

where ϕ_+ *is the larger root of the quadratic equation*

$$\frac{\sigma^2}{2}\phi^2 + \left(\delta_2 - \delta_1 - \frac{\sigma^2}{2}\right)\phi - \delta_2 = 0. \tag{6}$$

Proof. Let τ^* be defined as in (5). Define

$$\tilde{f}(x,y) := \mathrm{E}_0^{x,y}[\mathrm{e}^{-r\tau^*}(V_1(\tau^*) - V_2(\tau^*))_+].$$

The results of Exercise 9.12 of Øksendal (2003) imply that $\tilde{f}$ satisfies the following boundary problem:

$$\mu_1 x \tilde{f}_x + \mu_2 y \tilde{f}_y + \frac{1}{2}\sigma_1^2 x^2 \tilde{f}_{xx} + \frac{1}{2}\sigma_2^2 y^2 \tilde{f}_{yy} + \rho\sigma_1\sigma_2 xy \tilde{f}_{xy} = r\tilde{f}, \qquad \frac{x}{y} \in (0, c^*),$$

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with the boundary conditions

$$\tilde{f}(x,y) = x - y \quad \text{for all } x/y = c^*,$$
$$\tilde{f}(0,y) = 0 \quad \text{for all } y > 0.$$

Define $c := x/y$, and write $\tilde{f}$ in the form $\tilde{f}(x,y) = y\tilde{g}(c)$ for some function $\tilde{g}$. Then, the PDE above can be transformed into the following ODE in $\tilde{g}$:

$$\frac{\sigma^2 c^2}{2}\tilde{g}_{cc} + (\delta_2 - \delta_1)c\tilde{g}_c - \delta_2\tilde{g} = 0, \qquad c \in (0, c^*), \tag{7}$$

where the boundary conditions are now given by

$$\tilde{g}(c^*) = c^* - 1, \qquad \tilde{g}(0) = 0.$$

Equation (7) is a standard second-order linear ODE. Let $\phi_\pm$ denote the two roots of the quadratic equation (6):

$$\phi_\pm = \left(\frac{1}{2} - \frac{\delta_2 - \delta_1}{\sigma^2}\right) \pm \sqrt{\left(\frac{\delta_2 - \delta_1}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\delta_2}{\sigma^2}}.$$

It is easy to verify that $\phi_- < 0 < 1 < \phi_+$ when $\delta_1, \delta_2 > 0$. Standard ODE theory states that if the two roots $\phi_\pm$ are real and distinct, the solution to (7) takes the general form

$$\tilde{g}(c) = K_1 c^{\phi_+} + K_2 c^{\phi_-},$$

where K_1 and K_2 are constants. The boundary condition $\tilde{g}(0) = 0$ implies that $K_2 = 0$, while the boundary condition $\tilde{g}(c^*) = c^* - 1$ gives $K_1 = (c^* - 1)/c^{*\phi_+}$. Hence, the function $\tilde{g}(c)$ also depends on c^*. Using some simple algebra, it can be shown that, for all $c \geq 0$, the function $\tilde{g}(c)$ is maximized when $c^* = \phi_+/(\phi_+ - 1)$. This also implies that, for all $x \geq 0$ and $y > 0$, the function $\tilde{f}(x,y)$ is maximized when $c^* = \phi_+/(\phi_+ - 1)$. Note that, by definition, we have

$$\tilde{f}(x,y) = x - y \quad \text{for all } \frac{x}{y} > c^* = \frac{\phi_+}{\phi_+ - 1}.$$

To complete the proof, we still need to verify that $f^* = \tilde{f}$ for $c^* = \phi_+/(\phi_+ - 1)$, which will also show that the stopping time τ^* defined in (5) with $c^* = \phi_+/(\phi_+ - 1)$ is optimal. This can be done by applying the verification theorem in Øksendal (2003, Theorem 10.4.1) or by following the arguments given in the proof of Theorem 6.7 of Karatzas and Shreve (1998, pp. 64–66). Hence, we omit the details here.

The existing literature has relied exclusively on the assumption that $\delta_{1,2} > 0$ when using the free boundary approach to solve problem (1)–(3). Proposition 1 confirms that if this assumption holds, this approach does give a well-defined solution to the problem. It is natural to ask whether or not this approach remains applicable when the assumption $\delta_1, \delta_2 > 0$ is relaxed. From the proof of Proposition 1, it is easy to see that when $\phi_\pm > 0$, the free boundary approach breaks down as the positivity of the roots $\phi_\pm$ implies that the boundary condition $\tilde{g}(0) = 0$ becomes uninformative. The stock loan problem of Xia and Zhou (2007) has provided such a case, where $\delta_2 < 0 < \delta_1$ implies that $0 < \phi_- < 1 < \phi_+$. Consequently, we are left with one condition, $\tilde{g}(c^*) = c^* - 1$, but two unknowns, K_1 and K_2. This explains the major difficulty in applying the free boundary approach to problem (1)–(3) when the assumption $\delta_{1,2} > 0$ is absent. This may have also led many to believe (wrongly) that the problem has a well-defined solution only when this drift assumption holds.

In general, the free boundary approach can be applied to an optimal stopping problem only when an optimal stopping time is known to exist and the formulation correctly reflects the form of the stopping region. This has been the case with the arguments provided in the proof of Proposition 1 under the condition that $\delta_{1,2} > 0$, which are used quite often in the existing literature. However, what may seem striking, as we shall show in the next section, is that on certain domains of (δ_2, δ_1) an optimal stopping time exists, but the form of the stopping region is different from what has been assumed in the free boundary approach described above, i.e. $[c^*, \infty)$. For a general analysis of the problem, we believe that it is necessary to go back to the basic theory and look at the problem in a more rigorous manner. The first passage time approach of Xia and Zhou (2007) appears to be promising to us.

3. A first passage time approach

The derivation of Proposition 1 has shed some light on the connection between the roots $\phi_\pm$ and the solution to problem (1)–(3). To this end, we may ask how the behavior of the solution to the problem is connected to the values of $\phi_\pm$. In this section we shall make clear the answer to this question by extending the approach of Xia and Zhou (2007) to the current problem setting. Unless otherwise stated, we assume that $\sigma > 0$ throughout our analysis.

In what follows we consider the underlying problem under five mutually exclusive cases according to the values of $\phi_\pm$:

(A1) $\phi_+ = 1$ *and* $\phi_- \le 1$,

(A2) $\phi_+ > 1$ *and* $\phi_- \le 1$,

(A3) $\phi_+ > 1$ *and* $\phi_- > 1$,

(A4) $\phi_\pm$ *are complex*,

(A5) $\phi_+ < 1$ *and* $\phi_- < 1$.

Using some simple algebra, it is easy to check that the five cases above divide the (δ_2, δ_1)-space into five disjoint domains:

$$
\begin{aligned}
&(\mathrm{A1}) \quad \Longleftrightarrow \quad \mathrm{D1} = \left\{\delta_1 = 0,\ \delta_2 \ge -\frac{\sigma^2}{2}\right\},\\
&(\mathrm{A2}) \quad \Longleftrightarrow \quad \mathrm{D2} = \{\delta_1 \ge 0\} - \mathrm{D1},\\
&(\mathrm{A3}) \quad \Longleftrightarrow \quad \mathrm{D3} = \left\{\delta_2 < -\frac{\sigma^2}{2},\ \delta_2 - \frac{\sigma^2}{2} + \sqrt{-2\delta_2\sigma^2} \le \delta_1 < 0\right\},\\
&(\mathrm{A4}) \quad \Longleftrightarrow \quad \mathrm{D4} = \left\{\delta_2 < 0,\ \delta_2 - \frac{\sigma^2}{2} - \sqrt{-2\delta_2\sigma^2} < \delta_1 < \delta_2 - \frac{\sigma^2}{2} + \sqrt{-2\delta_2\sigma^2}\right\},\\
&(\mathrm{A5}) \quad \Longleftrightarrow \quad \mathrm{D5} = \mathbb{R}^2 - \mathrm{D1} \cup \mathrm{D2} \cup \mathrm{D3} \cup \mathrm{D4}.
\end{aligned}
$$

Note that most of the relevant real options applications and the stock loan problem of Xia and Zhou (2007) are embedded in (A2).

3.1. Some technical results

The first passage time approach to be developed critically hinges on Theorem 1.

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Theorem 1. *The following assertions hold.*

(a) *Suppose that $\phi_\pm$ are real and that $\phi_+ > 1$. Then*

$$\mathrm{E}_0^{x,y}\Big[\sup_{t\geq 0} \mathrm{e}^{-rt}(V_1(t) - V_2(t))_+\Big] < \infty. \tag{8}$$

Moreover, the value function associated with problem (1)–(3) is bounded and an optimal stopping time τ^* exists which takes the form

$$\tau^* = \inf\{t \geq 0 \mid (V_1(t) - V_2(t))_+ \geq f_t^*(V_1(t), V_2(t))\}, \tag{9}$$

where

$$f_t^*(V_1(t), V_2(t)) := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathrm{E}_t^{x,y}[\mathrm{e}^{-r(\tau-t)}(V_1(\tau) - V_2(\tau))_+ \mathbf{1}_{\{\tau<\infty\}}].$$

(b) *Suppose that (8) holds. Then*

$$f_t^*(V_1(t), V_2(t)) = f^*(V_1(t), V_2(t)) = \frac{V_2(t)}{y} f^*(U(t)y, y). \tag{10}$$

Proof. The proof of (8), below, is an extended version of the proof of Lemma 3.1 of Xia and Zhou (2007). For any sufficiently large $z > 0$, and $\lambda, \eta > 1$ with $1/\lambda + 1/\eta = 1$, we have

$$\begin{aligned}
&\mathrm{P}\Big\{\sup_{t\geq 0} \mathrm{e}^{-rt}(V_1(t) - V_2(t))_+ > z\Big\}\\
&\quad = \mathrm{P}\{\text{there exists } t \geq 0, \mathrm{e}^{-rt}(V_1(t) - V_2(t)) > z\}\\
&\quad = \mathrm{P}\bigg\{\text{there exists } t \geq 0, \Big(-\delta_1 - \frac{\sigma_1^2}{2}\Big)t + \sigma_1 W_1(t)\\
&\qquad\qquad > \ln\Big[\frac{z}{x} + \frac{y}{x}\exp\Big(\Big(-\delta_2 - \frac{\sigma_2^2}{2}\Big)t + \sigma_2 W_2(t)\Big)\Big]\bigg\}\\
&\quad \leq \mathrm{P}\bigg\{\text{there exists } t \geq 0, \Big(-\delta_1 - \frac{\sigma_1^2}{2}\Big)t + \sigma_1 W_1(t)\\
&\qquad\qquad > \frac{1}{\lambda}\ln\Big(\frac{\lambda z}{x}\Big) + \frac{1}{\eta}\ln\Big[\frac{\eta y}{x}\exp\Big(\Big(-\delta_2 - \frac{\sigma_2^2}{2}\Big)t + \sigma_2 W_2(t)\Big)\Big]\bigg\}\\
&\quad = \mathrm{P}\bigg\{\text{there exists } t \geq 0, \Big(-\delta_1 + \frac{\delta_2}{\eta} + \frac{\sigma_2^2}{2\eta} - \frac{\sigma_1^2}{2}\Big)t + \sigma_1 W_1(t) - \frac{\sigma_2}{\eta} W_2(t)\\
&\qquad\qquad > \frac{1}{\lambda}\ln\Big(\frac{\lambda z}{V}\Big) + \frac{1}{\eta}\ln\Big(\frac{\eta y}{x}\Big)\bigg\}\\
&\quad = \mathrm{P}\bigg\{\sup_{t\geq 0}\Big[W^*(t) + \frac{1}{\tilde\sigma}\Big(-\delta_1 + \frac{\delta_2}{\eta} + \frac{\sigma_2^2}{2\eta} - \frac{\sigma_1^2}{2}\Big)t\Big] > \frac{1}{\lambda\tilde\sigma}\ln\Big(\frac{\lambda z}{x}\Big) + \frac{1}{\eta\tilde\sigma}\ln\Big(\frac{\eta y}{x}\Big)\bigg\}\\
&\quad = \exp\Big(\frac{2}{\tilde\sigma}\Big(-\delta_1 + \frac{\delta_2}{\eta} + \frac{\sigma_2^2}{2\eta} - \frac{\sigma_1^2}{2}\Big)\Big(\frac{1}{\lambda\tilde\sigma}\ln\Big(\frac{\lambda z}{x}\Big) + \frac{1}{\eta\tilde\sigma}\ln\Big(\frac{\eta y}{x}\Big)\Big)\Big)\\
&\quad = K\Big(\frac{x}{\lambda z}\Big)^{\omega},
\end{aligned} \tag{11}$$

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where $W^*(t)$ is a standard Wiener process,

$$\tilde{\sigma} = \sqrt{\sigma_1^2 + \frac{\sigma_2^2}{\eta^2} - \frac{2\rho\sigma_1\sigma_2}{\eta}},$$

$$K = \left(\frac{x}{\eta y}\right)^{(2/\eta\tilde{\sigma}^2)(\delta_1 - \delta_2/\eta - \sigma_2^2/2\eta + \sigma_1^2/2)},$$

$$\omega = \frac{2}{\lambda\tilde{\sigma}^2}\left(\delta_1 - \frac{\delta_2}{\eta} - \frac{\sigma_2^2}{2\eta} + \frac{\sigma_1^2}{2}\right) = 1 + \frac{\delta_1\eta^2 - (\delta_1 + \delta_2 + \sigma^2/2)\eta + \delta_2}{2\eta^2\tilde{\sigma}^2},$$

and σ is as given in (4). Equality (11) follows from a well-known result in Borodin and Salminen (2002) on the running maximum of a Wiener process. To show (8), it suffices to show that there exists an $\eta > 1$ such that $\omega > 1$ or

$$p(\eta) := \delta_1\eta^2 - \left(\delta_1 + \delta_2 + \frac{\sigma^2}{2}\right)\eta + \delta_2 > 0.$$

We claim that $\phi_+ > 1$ is sufficient. We shall show this by looking into the two domains D2 and D3 where $\phi_+ > 1$ holds.

- *Domain D2*. On D2 where $\delta_1 > 0$, the parabola $p(\eta)$ is concave upward and the claim follows trivially. On D2 where $\delta_1 = 0$ and $\delta_2 < -\sigma^2/2$,

$$p(\eta) = -\left(\delta_2 + \frac{\sigma^2}{2}\right)\eta + \delta_2$$

  is a linear function of η with $-(\delta_2 + \sigma^2/2) > 0$, and the claim follows easily.

- *Domain D3*. On D3, it is easy to check that $\delta_2 - \delta_1 + \sigma^2/2 \le 0$. Denote by η_+ the smaller root of the quadratic equation $p(\eta) = 0$. We have

$$\begin{aligned}
\eta_+ &= \frac{(\delta_1 + \delta_2 + \sigma^2/2) + \sqrt{(\delta_1 + \delta_2 + \sigma^2/2)^2 - 4\delta_1\delta_2}}{2\delta_1} \\
&= \frac{(\delta_1 + \delta_2 + \sigma^2/2) + \sqrt{(\delta_2 - \delta_1 + \sigma^2/2)^2 + 2\delta_1\sigma^2}}{2\delta_1} \\
&> \frac{(\delta_1 + \delta_2 + \sigma^2/2) + |\delta_2 - \delta_1 + \sigma^2/2|}{2\delta_1} \qquad \text{(therefore } \delta_1 < 0 \text{ on D3)} \\
&= \frac{(\delta_1 + \delta_2 + \sigma^2/2) - (\delta_2 - \delta_1 + \sigma^2/2)}{2\delta_1} \\
&= \frac{2\delta_1}{2\delta_1} \\
&= 1.
\end{aligned}$$

  Since the parabola $p(\eta)$ is concave downward on D3 and the two roots of $p(\eta) = 0$ are distinct, the claim follows easily.

For the proof of the remaining part of (a), see Appendix D of Karatzas and Shreve (1998).

The first equality in (10) follows from the joint Markov property of $(V_1(t), V_2(t))$ and (8). Its derivation is not trivial and can be found in Fakeev (1971). Finally, the second equality in (10) can be obtained as follows:

$$\begin{aligned}
f^*(V_1(t), V_2(t)) &= \Big[\sup_{\tau\in\mathcal{T}_0} \mathrm{E}_0^{x,y}[\mathrm{e}^{-r\tau}(V_1(\tau) - V_2(\tau))_+ \mathbf{1}_{\{\tau<\infty\}}]\Big]\Big|_{x=V_1(t),\, y=V_2(t)} \\
&= \Big[y \sup_{\tau\in\mathcal{T}_0} \mathrm{E}_0^{x/y,1}[\mathrm{e}^{-r\tau}(V_1(\tau) - V_2(\tau))_+ \mathbf{1}_{\{\tau<\infty\}}]\Big]\Big|_{x=V_1(t),\, y=V_2(t)} \\
&= yf^*\Big(\frac{x}{y}, 1\Big)\Big|_{x=V_1(t),\, y=V_2(t)} \\
&= V_2(t) f^*(U(t), 1) \\
&= V_2(t)\Big[\sup_{\tau\in\mathcal{T}_0} \mathrm{E}_0^{a,b}[\mathrm{e}^{-r\tau}(V_1(\tau) - V_2(\tau))_+ \mathbf{1}_{\{\tau<\infty\}}]\Big]\Big|_{a=U(t),\, b=1} \\
&= \frac{V_2(t)}{y}\Big[\sup_{\tau\in\mathcal{T}_0} \mathrm{E}_0^{a,b}[\mathrm{e}^{-r\tau}(yV_1(\tau) - yV_2(\tau))_+ \mathbf{1}_{\{\tau<\infty\}}]\Big]\Big|_{a=U(t),\, b=1} \\
&= \frac{V_2(t)}{y} \sup_{\tau\in\mathcal{T}_0} \mathrm{E}_0^{a,b}[\mathrm{e}^{-r\tau}(V_1(\tau) - V_2(\tau))_+ \mathbf{1}_{\{\tau<\infty\}}]|_{a=U(t)y,\, b=y} \\
&= \frac{V_2(t)}{y} f^*(U(t)y, y).
\end{aligned}$$

This completes our proof of part (b).

By definition we have $f^* = f_0^*$. The process $\mathrm{e}^{-rt} f_t^*$ is known as the Snell envelope for the process $\mathrm{e}^{-rt}(V_1(t) - V_2(t))_+$, i.e. the smallest supermartingale majorant to the process $\mathrm{e}^{-rt}(V_1(t) - V_2(t))_+$. As we shall see later, Theorem 1(b) is crucial in the development of the optimal stopping regions for the underlying problem.

Suppose that (8) holds. We claim that $\tilde{\tau} = \tau^*$ almost surely (a.s.), where τ^* is given in (9) and

$$\tilde{\tau} := \inf\{t \ge 0 \mid V_1(t) - V_2(t) \ge f_t^*(V_1(t), V_2(t))\}. \tag{12}$$

That is, $\tilde{\tau}$ also solves the optimal stopping problem (1)–(3). To show this, we can simply verify the following:

$$\mathrm{E}_0^{x,y}[\mathrm{e}^{-r\tilde{\tau}}(V_1(\tilde{\tau}) - V_2(\tilde{\tau}))_+ \mathbf{1}_{\{\tilde{\tau}<\infty\}}] = \mathrm{E}_0^{x,y}[\mathrm{e}^{-r\tau^*}(V_1(\tau^*) - V_2(\tau^*))_+ \mathbf{1}_{\{\tau^*<\infty\}}].$$

Hence, it suffices to verify that (i) $\mathbf{1}_{\{\tilde{\tau}<\infty\}} = \mathbf{1}_{\{\tau^*<\infty\}}$ a.s., and (ii) $\tilde{\tau}\,\mathbf{1}_{\{\tilde{\tau}<\infty\}} = \tau^*\,\mathbf{1}_{\{\tau^*<\infty\}}$ a.s. Since $f_t^*(V_1(t), V_2(t)) > 0$ a.s. for all $t \ge 0$, both (i) and (ii) follow readily. The characterization of the optimal stopping time $\tilde{\tau}$ plays a key role in solving problem (3). As we shall see later, $\tilde{\tau}$ can be expressed in the form of a first passage time of a Wiener process. To proceed, we need Lemma 1.

Lemma 1. *Let B_t be a standard Wiener process and let μ be a real number. Let*

$$t^* = \inf\{t \ge 0 \mid \mu t + B_t = m\}, \qquad m \ne 0.$$

Then, the Laplace transform of the hitting time t^* is given by

$$\mathrm{E}[\mathrm{e}^{\lambda t^*}\mathbf{1}_{\{t^*<\infty\}}] = \begin{cases} \exp(\mu m - |m|\sqrt{\mu^2 - 2\lambda}), & \mu^2 - 2\lambda \ge 0, \\ \infty, & \mu^2 - 2\lambda < 0. \end{cases}$$

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Proof. It is well known that the probability density function (PDF) of t^* is given by

$$p_m(t) = \frac{|m|}{\sqrt{2\pi t^3}} \exp\biggl(-\frac{(m-\mu t)^2}{2t}\biggr), \qquad t > 0.$$

We then have

$$\mathrm{E}[\mathrm{e}^{\lambda t^*} \mathbf{1}_{\{t^*<\infty\}}] = \int_0^\infty \mathrm{e}^{\lambda t} p_m(t)\,\mathrm{d}t.$$

When $\mu^2 - 2\lambda < 0$, it is easy to verify that $\lim_{t\to\infty} \mathrm{e}^{\lambda t} p_m(t) = \infty$. Hence, the result for the case in which $\mu^2 - 2\lambda < 0$ follows. For the case in which $\mu^2 - 2\lambda \geq 0$, see Lemma 3.2 of Xia and Zhou (2007).

Proposition 2. *For any $c > 0$, define*

$$G(c; x, y) := \mathrm{E}_0^{x,y}[\exp(-r\tau_c)(V_1(\tau_c) - V_2(\tau_c))_+ \mathbf{1}_{\{\tau_c<\infty\}}], \qquad (13)$$
$$\tau_c := \inf\{t \geq 0 \mid U(t) = c\}.$$

Then,

(a) *when $\phi_\pm$ are real,*

$$G(c; x, y) = \begin{cases} c^{-\phi_+}(c-1)\biggl(\dfrac{x}{y}\biggr)^{\phi_+} y, & c \geq \dfrac{x}{y}, \\ c^{-\phi_-}(c-1)\biggl(\dfrac{x}{y}\biggr)^{\phi_-} y, & 0 < c < \dfrac{x}{y}, \end{cases}$$

(b) *when $\phi_\pm$ are complex,*

$$G(c; x, y) = \begin{cases} (x-y)_+, & c = x/y, \\ \infty, & c \neq x/y. \end{cases}$$

Proof. Assume that $c > 0$. Define an equivalent probability measure $\tilde{\mathrm{P}}$ on $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F})$ via the measure transformation $\mathrm{d}\tilde{\mathrm{P}}/\mathrm{d}\,\mathrm{P}|_{\mathcal{F}_t} = \exp(-\mu_2 t)V_2(t)/y$. Using Girsanov's theorem, it is easy to verify that the process $U(t)$ satisfies the following dynamics:

$$\frac{\mathrm{d}U(t)}{U(t)} = (\delta_2 - \delta_1)\,\mathrm{d}t + \sigma\,\mathrm{d}\widetilde{W}_t,$$

where $\widetilde{W}_t$ is a Wiener process under $\tilde{\mathrm{P}}$. Then, we have

$$G(c; x, y) = (c-1)\,\mathrm{E}_0^{x,y}[V_2(\tau_c)\exp(-r\tau_c)\,\mathbf{1}_{\{\tau_c<\infty\}}] = (cy-y)\,\widetilde{\mathrm{E}}_0^{x,y}[\exp(-\delta_2\tau_c)\,\mathbf{1}_{\{\tau_c<\infty\}}], \qquad (14)$$

where $\widetilde{\mathrm{E}}_0^{x,y}[\cdot]$ denotes the expectation taken with respect to $\tilde{\mathrm{P}}$. (See Shiryaev (1999, p. 762) for a detailed justification of the last equality in (14).) Define

$$\tilde{\mu} := \frac{\delta_2 - \delta_1}{\sigma} - \frac{\sigma}{2}.$$

Then, we can rewrite τ_c as

$$\tau_c = \inf\biggl\{t \geq 0 \;\bigg|\; \tilde{\mu}t + \widetilde{W}_t = \frac{1}{\sigma}\ln\frac{cy}{x}\biggr\}.$$

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It is easy to check that

$$\begin{aligned} \phi_\pm \text{ are real} &\iff \tilde{\mu} + 2\delta_2 \geq 0, \\ \phi_\pm \text{ are complex} &\iff \tilde{\mu} + 2\delta_2 < 0. \end{aligned}$$

Applying Lemma 1 to the expectation $\widetilde{\mathrm{E}}_0[\exp(-\delta_2\tau_c)\,\mathbf{1}_{\{\tau_c<\infty\}}]$ in (14), we obtain

$$\widetilde{\mathrm{E}}_0^{x,y}[\exp(-\delta_2\tau_c)\,\mathbf{1}_{\{\tau_c<\infty\}}] = \begin{cases} \left(\dfrac{x}{cy}\right)^{\phi_+}, & c \geq \dfrac{x}{y}, \\ \left(\dfrac{x}{cy}\right)^{\phi_-}, & 0 < c < \dfrac{x}{y}, \end{cases}$$

when $\phi_\pm$ are real, and we obtain

$$\widetilde{\mathrm{E}}_0^{x,y}[\exp(-\delta_2\tau_c)\,\mathbf{1}_{\{\tau_c<\infty\}}] = \begin{cases} (x-y)_+, & c = x/y, \\ \infty, & c \neq x/y, \end{cases}$$

when $\phi_\pm$ are complex. The proposition follows.

3.2. Solution under (A1) and (A2)

The (A1) and (A2) cases cover most of the existing applications utilizing the formulation (1)–(3). Figure 1 illustrates the domains D1 and D2 on the (δ_2, δ_1)-space that correspond to the (A1) and (A2) cases, respectively.

Lemma 2. *Under (A1) and (A2), the following assertions hold:*

(a) $(x-y)_+ \leq f^*(x,y) \leq x$ *for all* $x, y > 0$,

(b) *the function* $f^*(x,y)$ *is convex, nondecreasing, and continuous in* x.

Proof. Note that we have

$$\max(x-y, 0) = \mathrm{E}_0^{x,y}[\mathrm{e}^{-r\tau}(V_1(\tau) - V_2(\tau))_+\,\mathbf{1}_{\{\tau<\infty\}}]|_{\tau=0} \leq f^*(x,y).$$

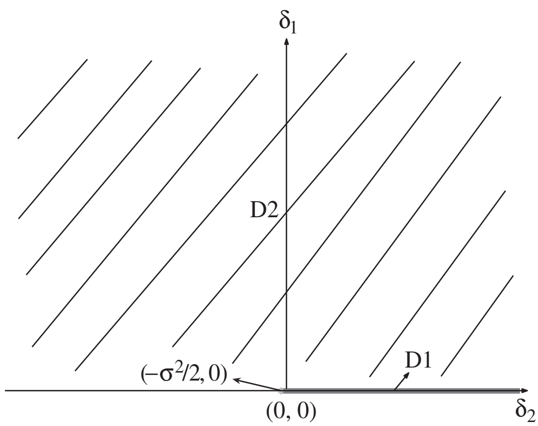

FIGURE 1: Domains D1 and D2 on the (δ_2, δ_1)-space.

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This yields the first inequality in part (a). Since $\delta_1 \geq 0$, we have

$$\begin{aligned} f^*(x, y) &\leq \sup_{\tau \in \mathcal{T}_0} \mathrm{E}_0^{x,y}[\mathrm{e}^{-r\tau} V_1(\tau)] \\ &= x \sup_{\tau \in \mathcal{T}_0} \mathrm{E}_0^{x,y}\bigg[\exp\bigg(\bigg(-\delta_1 - \frac{\sigma_1^2}{2}\bigg)\tau + \sigma_1 W_1(\tau)\bigg) \mathbf{1}_{\{\tau<\infty\}}\bigg] \\ &\leq x \sup_{\tau \in \mathcal{T}_0} \mathrm{E}_0^{x,y}\bigg[\exp\bigg(-\frac{\sigma_1^2}{2}\tau + \sigma_1 W_1(\tau)\bigg) \mathbf{1}_{\{\tau<\infty\}}\bigg] \\ &\leq x. \end{aligned} \tag{15}$$

This gives the second inequality in part (a). The last step in (15) follows from Fatou's lemma and Doob's optional sampling theorem. By definition, $f^*(x, y)$ is nondecreasing in x. Convexity follows since

$$(\lambda x_1 + (1-\lambda)x_2 - y)_+ \leq \lambda(x_1 - y)_+ + (1-\lambda)(x_2 - y)_+$$

for any arbitrary $x_1, x_2, y > 0$ and $0 \leq \lambda \leq 1$. Finally, continuity follows from convexity and the fact that $f(x, y)$ is finite on its domain. This completes the proof of part (b).

Proposition 3. *For any $y > 0$, define*

$$c^* := \inf\{c > 0 \mid cy - y = f^*(cy, y)\},$$

where $\inf \varnothing := \infty$. *Under (A1) and (A2), we have $c^* \geq 1$ and*

$$\{c > 0 \mid cy - y = f^*(cy, y)\} = [c^*, \infty).$$

Proof. We omit the trivial case in which $c^* = \infty$. Suppose that $c^* \in [1, \infty)$ (c^* cannot be less than 1 since f^* is nonnegative). Part (b) of Lemma 2 implies that $f^*(c^*y, y) = c^*y - y$. We claim that $f^*(cy, y) = cy - y$ for all $c \geq c^*$. Suppose that this does not hold. Then there exists a $\tilde{c} > c^*$ such that $f^*(\tilde{c}y, y) > \tilde{c}y - y$. The convexity of $f^*(x, y)$ in x implies that

$$\frac{f^*(cy, y) - f^*(c^*y, y)}{c - c^*} \geq \frac{f^*(\tilde{c}y, y) - f^*(c^*y, y)}{\tilde{c} - c^*} =: \beta > y$$

for all $c \geq \tilde{c}$ and, consequently, we have

$$f^*(cy, y) \geq \beta(c - c^*) + c^*y - y \quad \text{for all } c \geq \tilde{c}.$$

In other words, it holds that $f^*(cy, y) > cy$ for all sufficiently large c since $\beta > y$. This contradicts part (a) of Lemma 2. The proposition follows.

Proposition 4. *Assume that (A2) holds. Then an optimal stopping time exists. In particular, the stopping time $\tilde{\tau}$ given in (12) is optimal and can be written in the following form:*

$$\tilde{\tau} = \inf\{t \geq 0 \mid U(t) \geq c^*\}, \tag{16}$$

where $1 \leq c^* < \infty$ is as defined in Proposition 3.

Proof. The existence of an optimal stopping time follows from the fact that $\phi_+ > 1$ under (A2) and Theorem 1. Our previous discussion shows that the stopping time $\tilde{\tau}$ given in (12) is one such optimal time. Using (10), (12), and Proposition 3, we then obtain

$$\begin{aligned}\tilde{\tau} &= \inf\{t \geq 0 \mid U(t)V_2(t) - V_2(t) \geq f_t^*(U(t)V_2(t), V_2(t))\} \\ &= \inf\left\{t \geq 0 \;\middle|\; U(t)V_2(t) - V_2(t) \geq \frac{V_2(t)}{y} f^*(U(t)y, y)\right\} \\ &= \inf\{t \geq 0 \mid U(t)y - y \geq f^*(U(t)y, y)\} \\ &= \inf\{t \geq 0 \mid U(t) \geq c^*\}.\end{aligned}$$

We now state the main results under (A1) and (A2).

Theorem 2. *Let c^* be defined as in Proposition 3. Then the following assertions hold.*

(a) *Under (A1), there exists no optimal stopping time to the optimal stopping problem (1)–(3) and we have $f^*(x, y) = x$. More precisely, there exists a sequence of admissible stopping times $\{\tau^{(n)}\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \tau^{(n)} = \infty$ a.s. and*

$$f^*(x, y) = \lim_{n\to\infty} \mathrm{E}_0^{x,y}[\exp(-r\tau^{(n)})(V_1(\tau^{(n)}) - V_2(\tau^{(n)}))_+ \mathbf{1}_{\{\tau^{(n)}<\infty\}}] = x.$$

(b) *Under (A2), the optimal stopping problem (1)–(3) is solved by the optimal stopping time $\tilde{\tau}$ given in (16) with*

$$c^* = \frac{\phi_+}{\phi_+ - 1}.$$

Moreover, the value function is given by

$$f^*(x, y) = \begin{cases} x - y, & \dfrac{x}{y} \geq c^*, \\ c^{*-\phi_+}(c^* - 1)\left(\dfrac{x}{y}\right)^{\phi_+} y, & \dfrac{x}{y} < c^*. \end{cases}$$

Proof. When (A1) holds, we have $f^*(x, y) \leq x$ for all $x, y > 0$ by Lemma 2. Moreover, it is easy to verify that

$$\lim_{n\to\infty} \mathrm{E}_0[\mathrm{e}^{-rn}(V_1(n) - V_2(n))_+] = x.$$

Hence, part (a) follows. Under (A2), Proposition 4 yields

$$\begin{aligned} f^*(x, y) &= \begin{cases} x - y, & x/y \geq c^*, \\ G(c^*; x, y), & x/y < c^*, \end{cases} \\ &= \sup_{c \geq \max(1, x/y)} G(c; x, y), \end{aligned}$$

where $G(\cdot)$ is as defined in (13). Using Proposition 2 and some simple calculus, it is easy to verify that $c^* = \phi_+/(\phi_+ - 1)$. The value function $f^*(x, y)$ is then obtained by applying Proposition 2 to $G(c^*; x, y)$. This gives part (b).

It is now clear that under (A1) it is suboptimal to stop at any finite time. This result echoes the well-known result on perpetual American calls without dividends. On the other hand, under (A2), it is optimal to stop once the ratio $U(t)$ hits the region $[c^*, \infty)$. Figure 2 illustrates the typical shapes of the continuation region, the stopping region, and the optimal stopping boundary under (A1) and (A2).

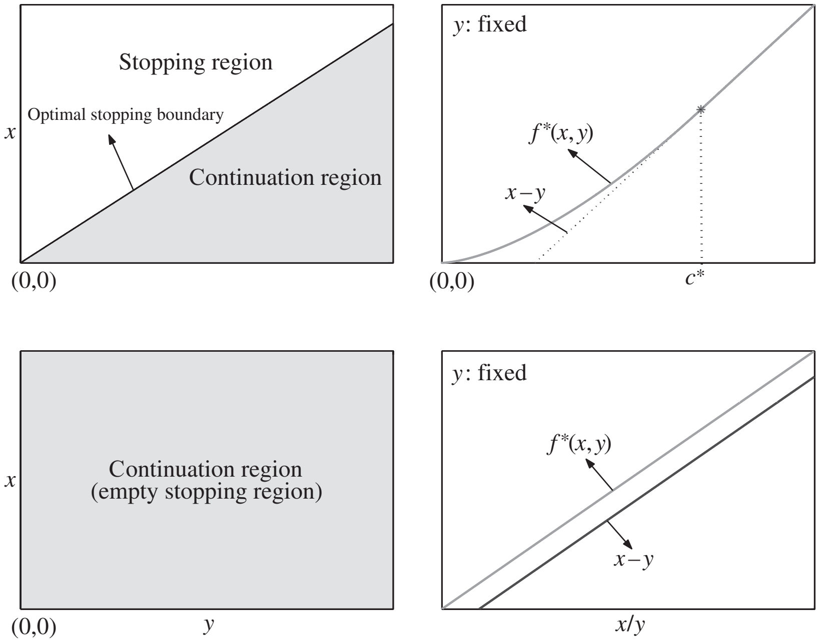

FIGURE 2: The plots depict the continuation region, the stopping region, the optimal stopping boundary, and the value function under (A2) (*top row*) and (A1) (*bottom row*).

Corollary 1, below, gives the probability that the optimal stopping occurs at a finite time under (A2).

Corollary 1. *When (A2) holds, we have*

$$\mathrm{P}\{\tilde{\tau} < \infty\} = \begin{cases} 1, & \dfrac{x}{y} \geq c^*,\ \text{or}\ \dfrac{x}{y} < c^* \text{ and } \delta_2 - \delta_1 \geq \dfrac{\sigma^2}{2}, \\ \left(\dfrac{c^* y}{x}\right)^{2(\delta_2-\delta_1)/\sigma^2-1}, & \dfrac{x}{y} < c^* \text{ and } \delta_2 - \delta_1 < \dfrac{\sigma^2}{2}. \end{cases}$$

Proof. Note that

$$\mathrm{P}\{\tilde{\tau} < \infty\} = \lim_{\lambda \uparrow 0} \mathrm{E}_0[\mathrm{e}^{\lambda\tilde{\tau}}\, \mathbf{1}_{\{\tilde{\tau}<\infty\}}],$$

provided that the Laplace transform of $\tilde{\tau}$ exists on some interval $(-\varepsilon, 0)$ with $\varepsilon > 0$. The result then follows by applying Lemma 1.

3.3. Solution under (A3)

As we shall see, the (A3) case turns out to be the most interesting and unexpected case. Figure 3 depicts the domain D3 on the (δ_2, δ_1)-space corresponding to (A3).

Lemma 3. *Under (A3), the following assertions hold:*

(a) $(x - y)_+ \leq f^*(x, y) < \infty$ *for all* $x, y > 0$,

(b) *the function* $f^*(x, y)$ *is convex, nondecreasing, and continuous in* x.

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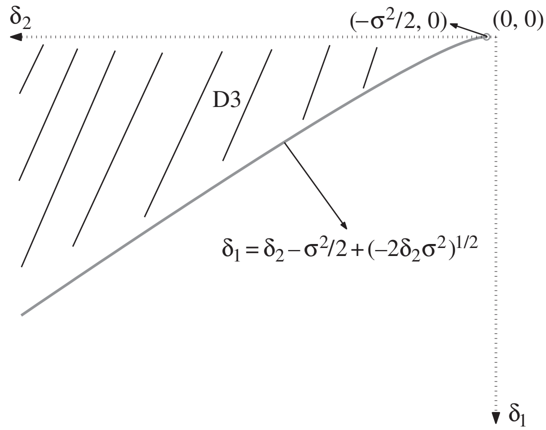

FIGURE 3: Domain D3 on the (δ_2, δ_1)-space. The δ_2-axis is excluded from the domain.

Proof. The first inequality in part (a) is trivial while the second inequality is a direct result of (8) in Theorem 1. Part (b) follows from the same arguments used in the proof of Lemma 2(b).

Unlike (A1) and (A2), we do not know whether or not the bound $f^*(x, y) \le x$ holds under (A3). On the other hand, from the proof of Proposition 3 we see that the bound $f^*(x, y) \le x$ is crucial in concluding that there exists *at most* one stopping threshold c^* under (A1) and (A2). In other words, the bound $f^*(x, y) \le x$ confirms that under (A1) and (A2), the stopping region is either empty or takes the form $[c^*, \infty)$ with $c^* < \infty$. Surprisingly, it turns out that this result does not hold under (A3). Proposition 5, below, serves as our first step to derive the stopping region under the current setting.

Proposition 5. *For any* $y > 0$, *define*

$$c^* := \inf\{c > 0 \mid cy - y = f^*(cy, y)\}$$

and

$$k^* := \inf\{c > c^* \mid cy - y < f^*(cy, y)\},$$

where $\inf \varnothing := \infty$. *Then, we have* $1 \le c^* \le k^*$ *and*

$$\begin{aligned} f^*(cy, y) &> cy - y, \qquad 1 < c \le c^* \text{ or } c > k^*, \\ f^*(cy, y) &= cy - y, \qquad c \in [c^*, k^*]. \end{aligned}$$

Proof. A trivial exercise.

Note that Proposition 5 applies to all five cases (A1)–(A5). For example, we have $c^* = k^* = \infty$ under (A1), and $c^* = \phi_+/(\phi_+ - 1)$ and $k^* = \infty$ under (A2). More importantly, the proposition implies that there can exist at most *two* stopping thresholds to the underlying optimal stopping problem, namely c^* and k^*. Since $\phi_+ > 1$ under (A3), the existence of an optimal stopping time is guaranteed by Theorem 1. This in fact implies that $c^* < \infty$ under (A3).

Proposition 6. *Assume that (A3) holds. Then an optimal stopping time exists. In particular, the stopping time $\tilde{\tau}$ given in (12) is optimal and can be written in the following form:*

$$\tilde{\tau} = \inf\{t \geq 0 \mid U(t) \in [c^*, k^*]\}, \tag{17}$$

where $c^*, k^* \geq 1$ are defined as in Proposition 5.

Proof. The proposition follows from similar arguments used in the proof of Proposition 4.

Theorem 3, below, gives the main results under (A3).

Theorem 3. *Assume that (A3) holds. Then the optimal stopping problem (1)–(3) is solved by the optimal stopping time $\tilde{\tau}$ given in (17) with*

$$c^* = \frac{\phi_+}{\phi_+ - 1} < \infty, \qquad k^* = \frac{\phi_-}{\phi_- - 1} < \infty.$$

Moreover, the value function $f^*(x, y)$ is given by

$$f^*(x, y) = \begin{cases} c^{*-\phi_+}(c^* - 1)\left(\dfrac{x}{y}\right)^{\phi_+} y, & \dfrac{x}{y} < c^*, \\ x - y, & \dfrac{x}{y} \in [c^*, k^*], \\ k^{*-\phi_-}(k^* - 1)\left(\dfrac{x}{y}\right)^{\phi_-} y, & \dfrac{x}{y} > k^*. \end{cases} \tag{18}$$

Proof. Using Proposition 6, we have

$$\begin{aligned} f^*(x, y) &= \begin{cases} G(c^*; x, y), & x/y < c^*, \\ x - y, & x/y \in [c^*, k^*], \\ G(k^*; x, y), & x/y > k^*, \end{cases} \\ &= \sup_{c \geq 1} G(c; x, y). \end{aligned} \tag{19}$$

Using Proposition 2 and some simple calculus, it is easy to verify that

$$c^* = \frac{\phi_+}{\phi_+ - 1} \quad \text{and} \quad k^* = \frac{\phi_-}{\phi_- - 1}.$$

Finally, (18) follows from (19) and Proposition 2.

Figure 4 illustrates the typical shapes of the continuation region, the stopping region, and the optimal stopping boundary under (A3). It is interesting to note that when $\phi_+ = \phi_- > 1$, the stopping region reduces to a singleton $\{c^* = k^*\}$ and the value function is strictly convex. Corollary 2, below, gives the probability that optimal stopping occurs at a finite time under (A3).

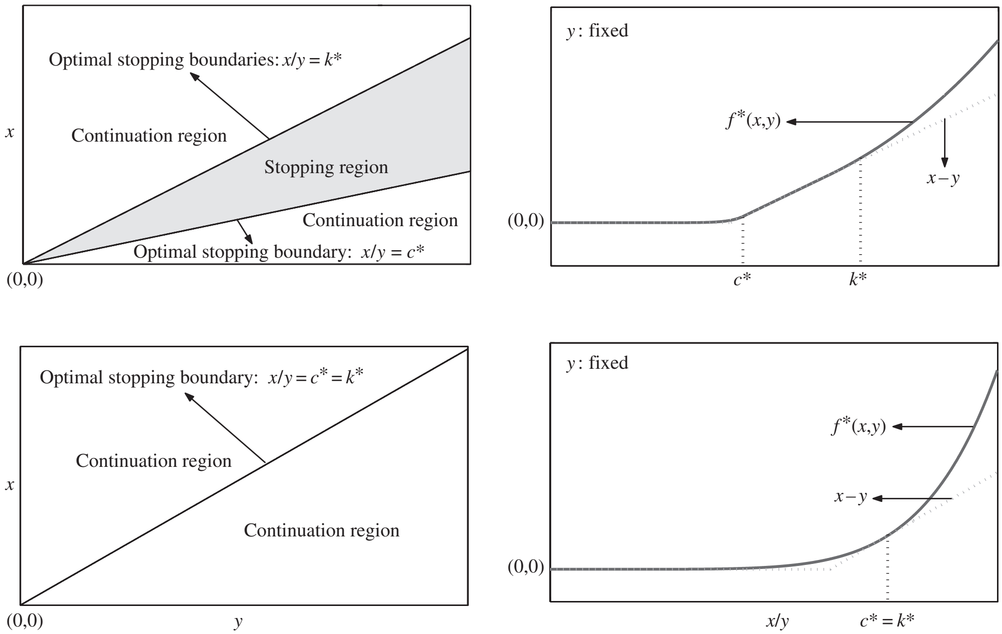

FIGURE 4: The plots depict the continuation region, the stopping region, the optimal stopping boundary, and the value function under (A3). The plots correspond to the case in which $c^* < k^*$ (*top row*) and the case in which $c^* = k^*$ (*bottom row*).

Corollary 2. *Under (A3), we have*

$$\mathrm{P}\{\tilde{\tau} < \infty\} = \begin{cases} \left(\dfrac{c^* y}{x}\right)^{2(\delta_2-\delta_1)/\sigma^2-1}, & \dfrac{x}{y} < c^* \textit{ and } \delta_2-\delta_1 < \dfrac{\sigma^2}{2}, \\ 1, & \dfrac{x}{y} \in [c^*, k^*], \textit{ or} \\ & \dfrac{x}{y} < c^* \textit{ and } \delta_2-\delta_1 \geq \dfrac{\sigma^2}{2}, \textit{ or} \\ & \dfrac{x}{y} > k^* \textit{ and } \delta_2-\delta_1 \leq \dfrac{\sigma^2}{2}, \\ \left(\dfrac{k^* y}{x}\right)^{2(\delta_2-\delta_1)/\sigma^2-1}, & \dfrac{x}{y} > k^* \textit{ and } \delta_2-\delta_1 > \dfrac{\sigma^2}{2}. \end{cases}$$

Proof. Similar to the proof of Corollary 1.

A counterintuitive implication of Theorem 3 is that under (A3), there exists two optimal stopping thresholds to the problem (1)–(3), c^* for x/y from below and k^* for x/y from above. To the author's knowledge, this is the first time such an interesting phenomenon has been documented in the literature. Many believe that (possibly from the American call option or exchange option pricing literature) if it is optimal to stop at a certain threshold c^*, it should also be optimal to stop at any threshold $c > c^*$. In fact, McDonald and Siegel (1986) claimed that this must hold true. Their arguments were as follows. Suppose that the claim does not hold. Then there exists some numbers c_0 and c_1 with $1 < c_0 < c_1$ such that it is optimal to stop immediately if $x/y = c_0$ while it is not optimal to stop if $x/y = c_1 > c_0$. This implies that

$$f^*(c_0 y, y) = c_0 y - y$$

and

$$f^*(c_1 y, y) > c_1 y - y.$$

Hence, if there are two identical investment projects with the same installation cost y, it could be optimal to invest immediately in the project with the lower revenue $c_0 y$ while leaving the project with the higher revenue $c_1 y$ unexercised. Based on this reasoning, they concluded that the stopping region for the optimal stopping problem must be in the form $[c^*, \infty)$.

The rather informal arguments of McDonald and Siegel (1986) may have led one to believe that the stopping region for the problem, provided it is well defined, is always in the form $[c^*, \infty)$. Also note that although McDonald and Siegel (1986) imposed the drift assumption $\mu_{1,2} < r$, their arguments in deriving the optimal stopping region did not make use of such a drift assumption. On the other hand, Theorem 3 shows that their conclusion regarding the optimal stopping region does not always hold should the drift assumption be relaxed. This point has often been ignored in the literature.

3.4. Solution under (A4) and (A5)

Under (A4) and (A5), the roots $\phi_\pm$ of the quadratic equation (6) are complex and both less than 1. Figures 5 and 6 respectively illustrate the domains D4 and D5 on the (δ_2, δ_1)-space that correspond to (A4) and (A5), respectively.

Theorem 4. *The following assertions hold.*

(a) *Under (A4), we have*

$$f^*(x, y) = G(c; x, y) = \infty$$

for all $c > 1$ *and* $c \neq x/y$.

(b) *Under (A5), we have*

$$f^*(x, y) = \lim_{c \uparrow \infty} G(c; x, y) = \infty.$$

Proof. Both (a) and (b) follow readily from Proposition 2.

From Theorem 4, it is clear that an infinite value function is achieved under both (A4) and (A5). However, the stopping strategies to achieve an infinite value function under the two cases are different. In particular, any hitting time τ_c with $c \neq x/y$ and $c > 1$ is optimal under (A4). On the other hand, it is suboptimal to stop at any finite time under (A5). This result can be made rigorous using martingale theory. Fix $T > 0$ and define

$$Z_s := \mathrm{e}^{-rs}(V_1(s) - V_2(s))$$

for $s \geq 0$. Under (A5), it is easy to verify the following:

$$\mathrm{E}_s^{x,y}[Z_T] \geq \mathrm{e}^{-rs}(V_1(s) - V_2(s)) = Z_s \quad \text{a.s.} \quad \text{and} \quad \lim_{T \to \infty} \mathrm{E}_s^{x,y}[Z_T] = \infty \quad \text{a.s.,}$$

for any $s \leq T$. In other words, Z_s is a submartingale. Now, using the convexity of the function

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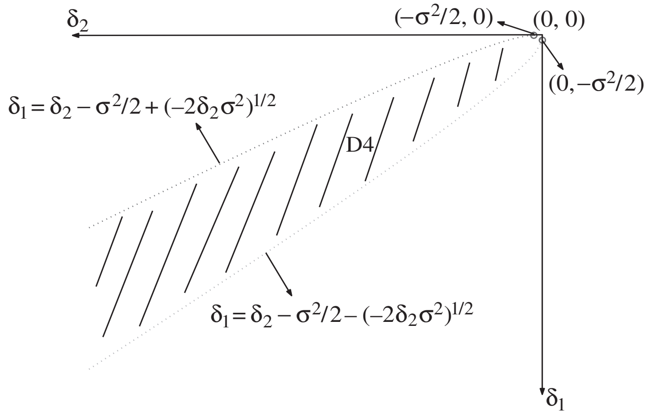

FIGURE 5: Domain D4 on the (δ_2, δ_1)-space. The dotted boundary is excluded from the domain.

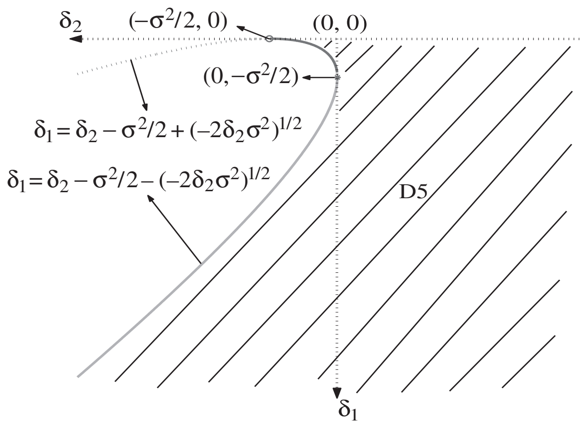

FIGURE 6: Domain D5 on the (δ_2, δ_1)-space. The dotted boundary is excluded from the domain.

$(\cdot)_+$ and Jensen's inequality, we obtain

$$\begin{aligned}
\mathrm{E}_0^{x,y}[\mathrm{e}^{-rT}(V_1(T) - V_2(T))_+] &= \mathrm{E}_0^{x,y}[\mathrm{E}_0^{x,y}[\mathrm{e}^{-rT}(V_1(T) - V_2(T))_+ \mid \mathcal{F}_\tau]] \\
&\geq \mathrm{E}_0^{x,y}[(\mathrm{E}_0^{x,y}[\mathrm{e}^{-rT}(V_1(T) - V_2(T)) \mid \mathcal{F}_\tau])_+] \\
&\geq \mathrm{E}_0^{x,y}[\mathrm{e}^{-r\tau}(V_1(\tau) - V_2(\tau))_+]
\end{aligned}$$

for any stopping time $\tau \leq T$. The last inequality above follows from Doob's optional sampling theorem for submartingales. The conclusion then follows by taking $T \to \infty$.

4. Summary

The analysis in the previous section allows us to deduce an explicit relationship between the roots $\phi_\pm$ of the quadratic equation (6) and the solution to the optimal stopping problem (1)–(3). Interestingly, we find that under certain conditions the smaller root ϕ_- also plays a role in the solution to the problem. We now give a summary of the complete solution to the optimal stopping problem (1)–(3) in Table 1.

TABLE 1: Relationship between $\phi_\pm$ and the solution to problem (3).

Case	$\phi_\pm$	Optimal stopping time as a hitting time	Stopping region	Value function $f^*(x, y)$
A1	$\phi_+ = 1, \phi_- \le 1$	Does not exist	$\varnothing$	x
A2	$\phi_+ > 1,\ \phi_- \le 1$	Exists	$[c^*, \infty),\ c^* = \dfrac{\phi_+}{\phi_+ - 1}$	$\begin{cases} x - y, & \dfrac{x}{y} \ge c^*, \\ c^{*-\phi_+}(c^* - 1)\left(\dfrac{x}{y}\right)^{\phi_+} y, & \dfrac{x}{y} < c^*. \end{cases}$
A3	$\phi_\pm > 1$	Exists	$[c^*, k^*],\ \begin{cases} c^* = \dfrac{\phi_+}{\phi_+ - 1}, \\ k^* = \dfrac{\phi_-}{\phi_- - 1}. \end{cases}$	$\begin{cases} k^{*-\phi_-}(k^* - 1)\left(\dfrac{x}{y}\right)^{\phi_-} y, & \dfrac{x}{y} > k^*, \\ x - y, & \dfrac{x}{y} \in [c^*, k^*], \\ c^{*-\phi_+}(c^* - 1)\left(\dfrac{x}{y}\right)^{\phi_+} y, & \dfrac{x}{y} < c^*. \end{cases}$
A4	$\phi_\pm$ are complex	Exists	Can be chosen in many ways	∞
A5	$\phi_\pm < 1$	Does not exist	$\varnothing$	∞

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